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Existence of Positive Solutions for $2n^{\text{th}}$ Order Lidstone Boundary Value Problems with p-Laplacian Operator

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Abstract

In this paper, we establish the existence of positive solutions for $2n^{\text{th}}$ order Lidstone boundary value problems with p-Laplacian of the form

$$(-1)^n[\phi_p(y^{(2n-2)}(t) - k^2 y^{(2n-4)}(t))]'' = f(t, y(t)), \quad t \in [0, 1],$$

$$y^{(2i)}(0) = 0 = y^{(2i)}(1),$$

for $0 \le i \le n-1$, where $n \ge 2$ and $k > 0$ is a constant, by applying Guo–Krasnosel'skii fixed point theorem.

Keywords: Green's function, p-Laplacian, boundary value problem, positive solution, cone, fixed point theorem.
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1. Introduction

The theory of differential equations is useful in providing a mathematical basis to interpret various real world problems of present society, which are complex and multidisciplinary in nature. In this theory, one of the most important operators is the classical one dimensional p-Laplacian operator and is defined by $\phi_p(s) = |s|^{p-2}s$, where $p > 1$, $\phi_p^{-1} = \phi_q$ and $\frac{1}{p} + \frac{1}{q} = 1$. There has been a surge of interest to study p-Laplacian boundary value problems, which arise in various contexts such as viscoelastic flows, image processing, turbulent filtration in porous media, biophysics, plasma physics, rheology, glaciology, radiation

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of heat, plastic molding etc. In particular, the concept of viscosity solutions is quite suitable for large values of p. For more details on applications, we refer [9].

In recent years, the existence of positive solutions for nonlinear boundary value problems with p-Laplacian operator have received great attention due to its wide applicability. To mention a few paper along these lines are Wang [33], Lian and Wong [24], Agarwal et al. [2], Li and Ge [21], Liu and Ge [25], Avery and Henderson [3], Li and Shen [23] and for further development in the topic, see [41, 12, 14, 39, 42, 36, 37].

In this paper, we establish the existence of positive solutions for $2n^{\text{th}}$ order p-Laplacian boundary value problem of the form

$$(-1)^n[\phi_p(y^{(2n-2)}(t) - k^2 y^{(2n-4)}(t))]'' = f(t, y(t)), \quad t \in [0,1], \tag{1.1}$$

$$y^{(2i)}(0) = 0 = y^{(2i)}(1), \tag{1.2}$$

for $0 \le i \le n-1$, where $n \ge 2$, $k > 0$ is a constant and $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, by applying Guo–Krasnosel'skii fixed point theorem. In the past few decades for $k = 0$ and $p = 2$, a lot of works have been done on the existence of positive solutions of $2n^{\text{th}}$ order boundary value problems associated with ordinary differential equations by using various methods, see [6, 38, 11, 7, 4, 40, 19, 20] and for $k \neq 0$ and $p = 2$, most of the researchers focussed and established the existence of positive solutions of second order differential equations satisfying Neumann and Sturm-Liouville boundary conditions, see [17, 31, 32, 22, 5, 30, 34, 35, 16, 26, 28]. However, few works have been carried out in establishing the existence of positive solutions of $2n^{\text{th}}$ order p-Laplacian boundary value problems, see [13, 29, 10, 27].

The rest of the paper is organized as follows. In Section 2, we express the solution of the boundary value problem (1.1)-(1.2) as a solution of an equivalent integral equation involving Green functions and establish some inequalities for these Green functions. In Section 3, we develop criteria for the existence of at least one positive solution of the boundary value problem (1.1)-(1.2) by an application of Guo–Krasnosel'skii fixed point theorem. Finally as an application, we give an example to demonstrate our results.

2. Green's Function and Bounds

In this section, we express the solution of the boundary value problem (1.1)-(1.2) as a solution of an equivalent integral equation involving Green functions and then establish some inequalities for these Green functions.

The Green's function for the second order homogeneous problem of the form

$$-y''(t) + k^2 y(t) = 0, \quad t \in [0,1], \tag{2.1}$$

$$y(0) = 0 = y(1), \tag{2.2}$$

is constructed and denoted by $G(t,s)$. Let $u(t) = (-1)^{n-2}[\phi_p(x(t))^{(2n-4)}]$ and $x(t) = -y''(t) + k^2 y(t)$. Then the Green's function for the second order homogeneous boundary value problem

$$-u''(t) = 0, \quad t \in [0,1], \tag{2.3}$$

$$u(0) = 0 = u(1), \tag{2.4}$$

is constructed and denoted by $H_1(t,s)$. Using this Green's function $H_1(t,s)$, we obtain the Green's function $H_{n-2}(t,s), n \ge 3$, recursively for the $(2n-4)^{\text{th}}$ order homogeneous boundary value problem

$$(-1)^{n-2} x^{(2n-4)}(t) = 0, \quad t \in [0,1], \tag{2.5}$$

$$x^{(2i)}(0) = 0 = x^{(2i)}(1), \tag{2.6}$$

for $0 \le i \le n-3$.

Lemma 2.1. *The Green's function $G(t,s)$ for the homogeneous boundary value problem (2.1)-(2.2) is given by*

$$G(t,s)=\begin{cases}\dfrac{\sin h(kt)\sin h(k(1-s))}{k\sin h(k)}, & t\le s,\\[2ex] \dfrac{\sin h(ks)\sin h(k(1-t))}{k\sin h(k)}, & s\le t.\end{cases}\tag{2.7}$$

Proof. By algebraic calculations, we can establish the result. □

Lemma 2.2. *[1] The Green's function $H_1(t,s)$ for the homogeneous boundary value problem (2.3)-(2.4) is given by*

$$H_1(t,s)=\begin{cases}t(1-s), & t\le s,\\ s(1-t), & s\le t.\end{cases}\tag{2.8}$$

Lemma 2.3. *[1, 38] The Green's function for the homogeneous boundary value problem (2.5)-(2.6) is $H_{n-2}(t,s)$, where $H_{n-2}(t,s)$ is defined recursively as*

$$H_j(t,s)=\int_0^1 H_{j-1}(t,\tau)H_1(\tau,s)d\tau,\ \text{for } 2\le j\le n-2,\tag{2.9}$$

and $H_1(t,s)$ is given in (2.8).

Therefore, the solution of the boundary value problem (1.1)-(1.2) is given by

$$y(t)=\int_0^1 H(t,s)\phi_q\left[\int_0^1 H_1(s,r)f(r,y(r))dr\right]ds,\tag{2.10}$$

where

$$H(t,s)=\int_0^1 G(t,\xi)H_{n-2}(\xi,s)d\xi.\tag{2.11}$$

Lemma 2.4. *The Green's function $G(t,s)$ in (2.7) satisfies the following inequalities:*

(i) $G(t,s)\ge 0$, *for all* $t,s\in[0,1]$,

(ii) $G(t,s)\le G(s,s)$, *for all* $t,s\in[0,1]$,

(iii) $G(t,s)\ge mG(s,s)$, *for all* $t\in I$ *and* $s\in[0,1]$,

where $m=\dfrac{\sinh(\frac{k}{4})}{\sinh(k)}$ *and* $I=\left[\frac{1}{4},\frac{3}{4}\right]$.

Proof. By algebraic calculations, one can establish the inequalities. □

Lemma 2.5. *[38] The Green's function $H_1(t,s)$ in (2.8) satisfies the following inequalities:*

(i) $H_1(t,s)\ge 0$, *for all* $t,s\in[0,1]$,

(ii) $H_1(t,s)\le H_1(s,s)$, *for all* $t,s\in[0,1]$,

(iii) $H_1(t,s)\ge \dfrac{1}{4}H_1(s,s)$, *for all* $t\in I$ *and* $s\in[0,1]$,

where $I=\left[\frac{1}{4},\frac{3}{4}\right]$.

Lemma 2.6. *[38] The Green's function $H_{n-2}(t,s)$ in (2.9) satisfies the following inequalities:*

(i) $H_{n-2}(t,s) \geq 0$, *for all* $t, s \in [0,1]$,

(ii) $H_{n-2}(t,s) \leq \dfrac{1}{6^{n-3}} H_1(s,s)$, *for all* $t, s \in [0,1]$,

(iii) $H_{n-2}(t,s) \geq \dfrac{1}{4^{n-2}} \left(\dfrac{11}{96}\right)^{n-3} H_1(s,s)$, *for all* $t \in I$ *and* $s \in [0,1]$,

where $I = \left[\frac{1}{4}, \frac{3}{4}\right]$.

Lemma 2.7. *The Kernel* $H(t,s)$ *in* (2.10) *satisfies the following inequalities:*

(i) $H(t,s) \geq 0$, *for all* $t, s \in [0,1]$,

(ii) $H(t,s) \leq \dfrac{\mathcal{K}}{6^{n-3}} H_1(s,s)$, *for all* $t, s \in [0,1]$,

(iii) $H(t,s) \geq \dfrac{m\mathcal{L}}{4^{n-2}} \left(\dfrac{11}{96}\right)^{n-3} H_1(s,s)$, *for all* $t \in I$ *and* $s \in [0,1]$,

where $\mathcal{K} = \displaystyle\int_0^1 G(\tau,\tau)d\tau$ *and* $\mathcal{L} = \displaystyle\int_{\tau \in I} G(\tau,\tau)d\tau$.

Proof. By algebraic calculations, one can establish the inequalities. □

To establish the existence of positive solutions of the boundary value problem (1.1)-(1.2), we will employ the following Guo–Krasnosel'skii fixed point theorem will be the fundamental tool.

Theorem 2.8. *[8, 15, 18] Let* X *be a Banach Space,* $\kappa \subseteq X$ *be a cone and suppose that* Ω_1, Ω_2 *are open subsets of* X *with* $0 \in \Omega_1$ *and* $\overline{\Omega}_1 \subset \Omega_2$*. Suppose further that* $T : \kappa \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \kappa$ *is completely continuous operator such that either*

(i) $\|Tu\| \leq \|u\|$, $u \in \kappa \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$, $u \in \kappa \cap \partial\Omega_2$, *or*

(ii) $\|Tu\| \geq \|u\|$, $u \in \kappa \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$, $u \in \kappa \cap \partial\Omega_2$ *holds.*

Then T *has a fixed point in* $\kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Existence of Positive Solutions

In this section, we establish the existence of at least one positive solution for the nonlinear p-Laplacian boundary value problem (1.1)-(1.2) by using Guo–Krasnosel'skii fixed point theorem.

Let $B = \{y : y \in C[0,1]\}$ be a Banach space with the norm

$$\|y\| = \max_{t \in [0,1]} |y(t)|$$

and let

$$P = \{y \in B : y(t) \geq 0 \text{ on } t \in [0,1] \text{ and } \min_{t \in I} y(t) \geq \mathcal{M}\|y\|\},$$

where $\mathcal{M} = \left(\frac{m\mathcal{L}}{\mathcal{K}}\right)\left(\frac{11^{n-3}}{2^{6n-16}}\right)$. We note that P is a cone in B.

Let the operator $T : P \to B$ be defined as

$$Ty(t) = \int_0^1 H(t,s)\phi_q\left[\int_0^1 H_1(s,r)f(r,y(r))dr\right] ds. \tag{3.1}$$

To obtain a positive solution of (1.1)-(1.2), we shall seek a fixed point of the operator T in the cone P.

We assume the following conditions hold throughout this paper:

$(A1)$ $0 < \int_0^1 H_1(t,s)ds < \infty$,
$(A2)$ $f(t,y)$ is a nondecreasing function with respect to y.

Define the nonnegative extended real numbers f_0, f^0, f_∞ and f^∞ by

$$f_0 = \lim_{y\to 0^+} \min_{t\in[0,1]} \frac{f(t,y)}{\phi_p(y)}, \; f^0 = \lim_{y\to 0^+} \max_{t\in[0,1]} \frac{f(t,y)}{\phi_p(y)},$$

$$f_\infty = \lim_{y\to\infty} \min_{t\in[0,1]} \frac{f(t,y)}{\phi_p(y)} \text{ and } f^\infty = \lim_{y\to\infty} \max_{t\in[0,1]} \frac{f(t,y)}{\phi_p(y)},$$

and assume that they will exist. The case $f^0 = 0$ and $f_\infty = \infty$ represents superlinear and the case $f_0 = \infty$ and $f^\infty = 0$ represents the sublinear.

Lemma 3.1. *The operator $T : P \to B$ defined by (3.1) is a self map on P.*

Proof. From $(A1)$ and the positivity of the Green's function $H(t,s)$ in Lemma 2.7 that for $y \in P$, $Ty(t) \geq 0$ on $t \in [0,1]$. Now, for $y \in P$ and by Lemma 2.7, we have

$$\begin{aligned} Ty(t) &= \int_0^1 H(t,s)\phi_q\Big(\int_0^1 H_1(s,r)f\big(r,y(r)\big)dr\Big)ds \\ &\leq \frac{\mathcal{K}}{6^{n-3}} \int_0^1 H_1(s,s)\phi_q\Big(\int_0^1 H_1(s,r)f\big(r,y(r)\big)dr\Big)ds \end{aligned}$$

so that

$$\|Ty\| \leq \frac{\mathcal{K}}{6^{n-3}} \int_0^1 H_1(s,s)\phi_q\Big(\int_0^1 H_1(s,r)f\big(r,y(r)\big)dr\Big)ds. \tag{3.2}$$

Then by Lemma 2.7, for $y \in P$ that

$$\begin{aligned} \min_{t\in I} Ty(t) &= \min_{t\in I}\Big\{\int_0^1 H(t,s)\phi_q\Big(\int_0^1 H_1(s,r)f\big(r,y(r)\big)dr\Big)ds\Big\} \\ &\geq \Big(\frac{m\mathcal{L}}{4^{n-2}}\Big)\Big(\frac{11}{96}\Big)^{n-3} \int_0^1 H_1(s,s)\phi_q\Big(\int_0^1 H_1(s,r)f\big(r,y(r)\big)dr\Big)ds \\ &\geq \Big(\frac{m\mathcal{L}}{\mathcal{K}}\Big)\Big(\frac{11^{n-3}}{2^{6n-16}}\Big)\|Ty\| \\ &= \mathcal{M}\|Ty\|. \end{aligned}$$

Therefore, $T : P \to P$, and hence the proof is complete. □

Further, the operator T is completely continuous by an application of the Arzela–Ascoli theorem.

Now, we establish the existence of at least one positive solution of the boundary value problem (1.1)-(1.2) for superlinear case.

Theorem 3.2. *Assume that the conditions $(A1)$ and $(A2)$ are satisfied. If $f^0 = 0$ and $f_\infty = \infty$ then the boundary value problem (1.1)-(1.2) has at least one positive solution that lies in P.*

Proof. Let T be the cone preserving, completely continuous operator that was defined by (3.1). From the definition of $f^0 = 0$, there exist $\eta_1 > 0$ and $\mathcal{H}_1 > 0$ such that

$$f(t,y) \leq \eta_1\phi_p(y), \text{ for } 0 < y \leq \mathcal{H}_1,$$

where η_1 satisfies

$$(\eta_1)^{q-1}\Big(\frac{\mathcal{K}}{6^{n-3}}\Big) \int_0^1 H_1(s,s)\phi_q\Big(\int_0^1 H_1(r,r)dr\Big)ds \leq 1. \tag{3.3}$$

Now, let $y \in P$ with $\|y\| = \mathcal{H}_1$. Then, by Lemma 2.7 and for $t \in [0,1]$, we have

$$\begin{aligned} Ty(t) &= \int_0^1 H(t,s)\phi_q\Big(\int_0^1 H_1(s,r)f\big(r,y(r)\big)dr\Big)ds \\ &\le \frac{\mathcal{K}}{6^{n-3}}\int_0^1 H_1(s,s)\phi_q\Big(\int_0^1 H_1(r,r)\eta_1\phi_p(y)\big)dr\Big)ds \\ &\le (\eta_1)^{q-1}\Big(\frac{\mathcal{K}}{6^{n-3}}\Big)\int_0^1 H_1(s,s)\phi_q\Big(\int_0^1 H_1(r,r)dr\Big)ds\|y\| \\ &\le \|y\|. \end{aligned}$$

Therefore, $\|Ty\| \le \|y\|$. If we set

$$\Omega_1 = \{y \in B : \|y\| < \mathcal{H}_1\}$$

then

$$\|Ty\| \le \|y\|, \text{ for } y \in P \cap \partial\Omega_1. \tag{3.4}$$

Further, since $f_\infty = \infty$, there exist $\eta_2 > 0$ and $\bar{\mathcal{H}}_2 > 0$ such that

$$f(t,y(t)) \ge \eta_2\phi_p(y), \text{ for } y \ge \bar{\mathcal{H}}_2,$$

where η_2 satisfies

$$(\eta_2)^{q-1}\Big(\frac{m\mathcal{L}\mathcal{M}}{4^{n-2}}\Big)\Big(\frac{11}{96}\Big)^{n-3}\int_{s\in I} H_1(s,s)\phi_q\Big(\frac{1}{4}\int_{r\in I} H_1(r,r)dr\Big)ds \ge 1. \tag{3.5}$$

Let $\mathcal{H}_2 = \max\Big\{2\mathcal{H}_1, \frac{\bar{\mathcal{H}}_2}{\mathcal{M}}\Big\}$. Choose $y \in P$ and $\|y\| = \mathcal{H}_2$. Then

$$\min_{t\in I} y(t) \ge \mathcal{M}\|y\| \ge \bar{\mathcal{H}}_2.$$

From Lemmas 2.5, 2.7, and for $t \in [0,1]$, we have

$$\begin{aligned} Ty(t) &= \int_0^1 H(t,s)\phi_q\Big(\int_0^1 H_1(s,r)f(r,y(r))dr\Big)ds \\ &\ge \min_{t\in I}\Big\{\int_0^1 H(t,s)\phi_q\Big(\int_0^1 H_1(s,r)f(r,y(r))dr\Big)ds\Big\} \\ &\ge \Big(\frac{m\mathcal{L}}{4^{n-2}}\Big)\Big(\frac{11}{96}\Big)^{n-3}\int_{s\in I} H_1(s,s)\phi_q\Big(\int_0^1 H_1(s,r)f(r,y(r))dr\Big)ds \\ &\ge \Big(\frac{m\mathcal{L}}{4^{n-2}}\Big)\Big(\frac{11}{96}\Big)^{n-3}\int_{s\in I} H_1(s,s)\phi_q\Big(\frac{1}{4}\int_{r\in I} H_1(r,r)\eta_2\phi_p(y)dr\Big)ds \\ &\ge \Big(\frac{m\mathcal{L}}{4^{n-2}}\Big)\Big(\frac{11}{96}\Big)^{n-3}(\eta_2)^{q-1}\int_{s\in I} H_1(s,s)\phi_q\Big(\frac{1}{4}\int_{r\in I} H_1(r,r)dr\Big)\mathcal{M}\|y\|ds \\ &\ge \|y\|. \end{aligned}$$

Therefore, $\|Ty\| \ge \|y\|$. So, if we set

$$\Omega_2 = \{y \in B : \|y\| < \mathcal{H}_2\}$$

then

$$\|Ty\| \ge \|y\| \quad \text{for } y \in P \cap \partial\Omega_2. \tag{3.6}$$

Applying Theorem 2.8 to (3.4) and (3.6), it follows that T has a fixed point $y \in P \cap (\Omega_2 \setminus \bar{\Omega}_1)$ and that y is the positive solution of the boundary value problem (1.1)-(1.2). □

We now establish the existence of at least one positive solution of the boundary value problem (1.1), (1.2) for sub linear case.

Theorem 3.3. *Assume that the conditions* $(A1)$ *and* $(A2)$ *are satisfied. If* $f_0 = \infty$ *and* $f^\infty = 0$ *then the boundary value problem* (1.1)-(1.2) *has at least one positive solution that lies in* P.

Proof. Let T be the cone preserving, completely continuous operator defined by (3.1). Since $f_0 = \infty$ there exist $\bar{\eta}_1 > 0$ and $J_1 > 0$ such that

$$f(t, y) \geq \bar{\eta}_1 \phi_p(y), \text{ for } 0 < y \leq J_1,$$

where $\bar{\eta}_1 \geq \eta_2$ and η_2 is given in (3.5).

Let $y \in P$ and $\|y\| = J_1$. Then from Lemmas 2.5, 2.7, and for $t \in [0, 1]$, we have

$$\begin{aligned}
Ty(t) &= \int_0^1 H(t,s)\phi_q\Big(\int_0^1 H_1(s,r) f\big(r, y(r)\big)dr\Big)ds.\\
&\geq \min_{t\in I}\left\{\int_0^1 H(t,s)\phi_q\Big(\int_0^1 H_1(s,r) f\big(r, y(r)\big)dr\Big)ds\right\}\\
&\geq \left(\frac{m\mathcal{L}}{4^{n-2}}\right)\left(\frac{11}{96}\right)^{n-3}\int_{s\in I} H_1(s,s)\phi_q\Big(\int_0^1 H_1(r,r) f(r, y(r)))dr\Big)ds\\
&\geq \left(\frac{m\mathcal{L}}{4^{n-2}}\right)\left(\frac{11}{96}\right)^{n-3}\int_{s\in I} H_1(s,s)\phi_q\Big(\frac{1}{4}\int_{r\in I} H_1(r,r)\bar{\eta}_1\phi_p(y)\big)dr\Big)ds\\
&\geq \left(\frac{m\mathcal{L}}{4^{n-2}}\right)\left(\frac{11}{96}\right)^{n-3}(\bar{\eta}_1)^{q-1}\int_{s\in I} H_1(s,s)\phi_q\Big(\frac{1}{4}\int_{r\in I} H_1(r,r)\big)dr\Big)\mathcal{M}\|y\|ds\\
&\geq \|y\|.
\end{aligned}$$

Therefore, $\|Ty\| \geq \|y\|$. Now, if we set

$$\Omega_3 = \{y \in B : \|y\| < J_1\}$$

then

$$\|Ty\| \geq \|y\|, \text{ for } y \in P \cap \partial\Omega_3. \tag{3.7}$$

Next, since $f^\infty = 0$, there exist $\bar{\eta}_2 > 0$ and $\bar{J}_2 > 0$ such that

$$f(t, y(t)) \leq \bar{\eta}_2 \phi_p(y), \text{ for } y \geq \bar{J}_2,$$

where $\bar{\eta}_2 \leq \eta_1$ and η_1 is given in (3.3).

Set

$$f^*(t, y) = \sup_{0\leq s\leq y} f(t, s).$$

Then, it is straightforward that f^* is a non decreasing real-valued function, $f \leq f^*$ and

$$\lim_{y\to\infty}\frac{f^*(t, y)}{y} = 0.$$

It follows that there exists $J_2 > \max\{2J_1, \bar{J}_2\}$ such that

$$f^*(t, y) \leq f^*(t, J_2), \text{ for } 0 < y \leq J_2.$$

Choose $y \in P$ with $\|y\| = J_2$. Then

$$\begin{aligned} Ty(t) &= \int_0^1 H(t,s)\phi_q\Big(\int_0^1 H_1(s,r)f(r,y(r))dr\Big)ds \\ &\le \frac{\mathcal{K}}{6^{n-3}}\int_0^1 H_1(s,s)\phi_q\Big(\int_0^1 H_1(s,r)f(r,J_2)dr\Big)ds \\ &\le \frac{\mathcal{K}}{6^{n-3}}\int_0^1 H_1(s,s)\phi_q\Big(\int_0^1 H_1(s,r)\bar{\eta}_2\phi_p(J_2)dr\Big)ds \\ &\le \frac{\mathcal{K}(\bar{\eta}_2)^{q-1}}{6^{n-3}}\int_0^1 H_1(s,s)\phi_q\Big(\int_0^1 H_1(r,r)dr\Big)dsJ_2 \\ &\le J_2 = \|y\|. \end{aligned}$$

Hence, $\|Ty\| \le \|y\|$. So, if we set

$$\Omega_4 = \{y \in B : \|y\| < J_2\}$$

then

$$\|Ty\| \le \|y\|, \text{ for } y \in P \cap \partial\Omega_4. \tag{3.8}$$

Applying by Theorem 2.8 to (3.7) and (3.8), we obtain that T has a fixed point $y \in P \cap (\Omega_4 \setminus \bar{\Omega}_3)$ and that y is the positive solution of the boundary value problem (1.1)-(1.2). □

4. Example

As an application, the results are demonstrated with example.

Example 4.1. Consider the boundary value problem

$$(-1)^3[\phi_p\big(y^{(4)}(t) - k^2y''(t)\big)]'' = f(t,y(t)), \quad t \in [0,1], \tag{4.1}$$

$$\left.\begin{aligned} y(0) = 0 = y(1), \\ y''(0) = 0 = y''(1), \\ y^{(4)}(0) = 0 = y^{(4)}(1). \end{aligned}\right\} \tag{4.2}$$

For simplicity, we take $p = 2$ and $k = 1$. By algebraic computations, we get $m = 0.21494$, $\mathcal{K} = 0.15652$, $\mathcal{L} = 0.10655$ and $\mathcal{M} = 0.03658$.

(a) If $f(t,y(t)) = y^2(1+e^{-t})$, then $f^0 = 0$ and $f_\infty = \infty$. So, all the conditions of Theorem 3.2 are satisfied and hence, the boundary value problem (4.1)-(4.2) has at least one positive solution.

(b) If $f(t,y(t)) = (\sqrt{t^2+1})y^{3/4}$, then $f_0 = \infty$ and $f^\infty = 0$. So, all the conditions of Theorem 3.3 are satisfied and hence, the boundary value problem (4.1)-(4.2) has at least one positive solution.

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