Available online at http://vonneumann-publishing.com/cna
Commun. Nonlinear Anal. 3 (2017), 68–86
Research Article

Communications in Nonlinear Analysis

Journal Homepage: www.vonneumann-publishing.com/cna

Some remarks on tripled fixed point theorems for a sequence of mappings satisfying Geraghty contraction with applications

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Abstract

The purpose of this paper is three fold. Firstly, we establish a tripled coincidence fixed point theorem for a sequence of mappings involving Geraghty contraction using compatibility and weakly reciprocally continuous maps in the structure of partially ordered metric spaces. The technique used in A. Roldan et al. [9] and in S. Radenovic [10] are not applicable on the presented theorems, we show that our results can not be obtained from the existing results in this field of study and thus our results are completely new and give rise a new dimension. Secondly, the notable works due to V. Berinde [3], V. Lakshmikantam and L. Ciric [8] and Babu and Subhashini [1] are generalized and extended. Finally, some sufficient conditions are given for the uniqueness of a tripled common fixed point. Consequently, we point out some slip-ups in the main results of R. Vats et al.[12] and present a furnished version of the same. Some illustrative examples to highlight the realized improvements are also furnished. Moreover, existence and uniqueness for the solution of an initial-boundary-value problem is discussed. On the other hand, as an application to establish existence and uniqueness for the system of integral equations our results are utilized.

Keywords: Partially ordered metric spaces, compatible mappings, weakly reciprocally maps, tripled coincidence point, tripled fixed point.
2010 MSC: 47H10, 54H25.

1. Introduction

In this section, some elementary definitions and fundamental results are discussed, which are essential in our subsequent discussion.

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Received 2017-06-04

Definition 1.1 ([2])**.** An element $(x,y)\in X\times X$ is called a coupled fixed point of the mapping $F:X\times X\to X$ if

$$F(x,y)=x,\qquad F(y,x)=y.$$

Definition 1.2 ([8])**.** An element $(x,y)\in X\times X$ is called a coupled coincidence point of the mappings $F:X\times X\to X$ and $g:X\to X$ if

$$F(x,y)=gx,\qquad F(y,x)=gy.$$

Definition 1.3 ([4])**.** Let $(X,\preceq)$ be a partially ordered set and $F:X\times X\times X\to X$ be a mapping. The mapping F is said to have the mixed monotone property if F is monotone non decreasing in x and z and is monotone non increasing in y, that is, for any $x,y,z\in X$,

$$x_1,x_2\in X, x_1\preceq x_2\Rightarrow F(x_1,y,z)\preceq F(x_2,y,z),$$

$$y_1,y_2\in X, y_1\preceq y_2\Rightarrow F(x,y_1,z)\succeq F(x,y_2,z)$$

and

$$z_1,z_2\in X, z_1\preceq z_2\Rightarrow F(x,y,z_1)\preceq F(x,y,z_2).$$

Definition 1.4 ([5])**.** Let $(X,\preceq)$ be a partially ordered set and $F:X\times X\times X\to X$ and $g:X\to X$ are two mappings. The mapping F is said to have the mixed g-monotone property if F is monotone g-non decreasing in x and z and is monotone g-non increasing in y, that is, for any $x,y,z\in X$,

$$x_1,x_2\in X, gx_1\preceq gx_2\Rightarrow F(x_1,y,z)\preceq F(x_2,y,z),$$

$$y_1,y_2\in X, gy_1\preceq gy_2\Rightarrow F(x,y_1,z)\succeq F(x,y_2,z)$$

and

$$z_1,z_2\in X, gz_1\preceq gz_2\Rightarrow F(x,y,z_1)\preceq F(x,y,z_2).$$

Definition 1.5 ([4])**.** An element $(x,y,z)\in X\times X\times X$ is called a tripled fixed point of the mapping $F:X\times X\times X\to X$ if

$$F(x,y,z)=x,\qquad F(y,x,y)=y\quad and\quad F(z,y,x)=z.$$

Definition 1.6. Let $(X,d,\preceq)$ be a partially ordered metric space. We say that X is regular if the following conditions hold:

(i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n\to x$, then $x_n\preceq x$ for all $n\geq 0$.

(ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n\to y$, then $y\preceq y_n$ for all $n\geq 0$.

Definition 1.7 ([12])**.** Let (X,d) be a metric space. $\{F_i\}_{i\in N}$ and g are compatible if

$$\lim_{n\to+\infty} d\Big(g\Big(F_n(x_n,y_n,z_n)\Big),F_n\Big(gx_n,gy_n,gz_n\Big)\Big)=0,$$

$$\lim_{n\to+\infty} d\Big(g\Big(F_n(y_n,x_n,y_n)\Big),F_n\Big(gy_n,gx_n,gy_n\Big)\Big)=0$$

and

$$\lim_{n\to+\infty} d\Big(g\Big(F_n(z_n,y_n,x_n)\Big),F_n\Big(gz_n,gy_n,gx_n\Big)\Big)=0,$$

whenever $\{x_n\},\{y_n\}$ and $\{z_n\}$ are sequences in X, such that

$$\lim_{n\to+\infty} F_n(x_n,y_n,z_n)=\lim_{n\to+\infty} gx_{n+1}=x,$$

$$\lim_{n\to+\infty} F_n(y_n,x_n,y_n)=\lim_{n\to+\infty} gy_{n+1}=y$$

and

$$\lim_{n\to+\infty} F_n(z_n,y_n,x_n)=\lim_{n\to+\infty} gz_{n+1}=z,$$

for some $x,y,z\in X$.

Definition 1.8 ([12]). $\{F_i\}_{i\in\mathbb{N}}$ and g are called weakly reciprocally continuous if

$$\lim_{n\to+\infty} g\Big(F_n(x_n,y_n,z_n)\Big) = gx \;\; or \;\; \lim_{n\to+\infty} F_n\Big(gx_n,gy_n,gz_n\Big)\Big) = F_n(x,y,z),$$

$$\lim_{n\to+\infty} g\Big(F_n(y_n,x_n,y_n)\Big) = gy \;\; or \;\; \lim_{n\to+\infty} F_n\Big(gy_n,gx_n,gy_n\Big)\Big) = F_n(y,x,y)$$

and

$$\lim_{n\to+\infty} g\Big(F_n(z_n,y_n,x_n)\Big) = gz \;\; or \;\; \lim_{n\to+\infty} F_n\Big(gz_n,gy_n,gx_n\Big)\Big) = F_n(z,y,x),$$

whenever $\{x_n\},\{y_n\}$ and $\{z_n\}$ are sequences in X, such that

$$\lim_{n\to+\infty} F_n(x_n,y_n,z_n) = \lim_{n\to+\infty} gx_{n+1} = x,$$

$$\lim_{n\to+\infty} F_n(y_n,x_n,y_n) = \lim_{n\to+\infty} gy_{n+1} = y$$

and

$$\lim_{n\to+\infty} F_n(z_n,y_n,x_n) = \lim_{n\to+\infty} gz_{n+1} = z,$$

for some $x,y,z\in X$.

Definition 1.9. Let Φ denote the class of all functions $\phi:[0,+\infty)\to[0,+\infty)$ satisfying $\phi(t)<t$ for $t>0$ and $\phi(t)=0$ if and only if $t=0$.

Definition 1.10 ([6]). Let S denotes the class of the functions $\beta:[0,+\infty)\to[0,1)$ which satisfy the condition $\beta(t_n)\to 1 \Rightarrow t_n\to 0$.

2. Main Result

Our essential result is given as follows.

Theorem 2.1. *Let $(X,\preceq)$ be a complete partially ordered metric space. Let g be a self-mapping on X and $\{F_i\}_{i\in\mathbb{N}}$ be a sequence of mappings from $X\times X\times X$ into X such that*
$F_i(x,y,z)\preceq F_{i+1}(u,v,w)$, $F_{i+1}(v,u,v)\preceq F_i(y,x,y)$ and $F_i(z,y,x)\preceq F_{i+1}(w,v,u)$,
(where $i=r-1; r\in\mathbb{N}$) for $x,y,z,u,v,w\in X$ with $gx\preceq gu, gv\preceq gy$ and $gz\preceq gw$, or $gx\succeq gu, gv\succeq gy$ and $gz\succeq gw$. Suppose that the following hold:

- *(i) g is continuous;*
- *(ii) $F_i(X\times X\times X)\subseteq g(X)$;*
- *(iii) $\{F_i\}_{i\in\mathbb{N}}$ and g are compatible and weakly reciprocally continuous;*
- *(iv) there exists $(x_0,y_0,z_0)\in X\times X\times X$ such that $gx_0\preceq F_0(x_0,y_0,z_0)$, $gy_0\succeq F_0(y_0,x_0,y_0)$ and $gz_0\preceq F_0(z_0,y_0,x_0)$;*
- *(v) there exist $\phi,\psi\in\Phi$, $\beta\in S$ and $L\geq 0$ such that*

$$d\Big(F_i(x,y,z),F_j(u,v,w)\Big) \leq \beta\Big(M_{i,j}(x,y,z,u,v,w)\Big)\phi\Big(M_{i,j}(x,y,z,u,v,w)\Big)+L\psi\Big(N_{i,j}(x,y,z,u,v,w)\Big), \tag{2.1}$$

where

$$\begin{aligned} M_{i,j}(x,y,z,u,v,w) = \max\Big\{ &\frac{d(gx,gu)+d(gy,gv)+d(gz,gw)}{3}, \\ &\frac{d(F_i(x,y,z),F_j(u,v,w))+d(F_i(y,x,y),F_j(v,u,v))+d(F_i(z,y,x),F_j(w,v,u))}{3}, \\ &\frac{d(gu,F_j(u,v,w))+d(gv,F_j(v,u,v))+d(gw,F_j(w,v,u))}{3}\Big\}, \end{aligned}$$

and

$$N_{i,j}(x,y,z,u,v,w) = \min\Big\{d(gx,F_i(x,y,z)), d(gx,F_j(u,v,w)), d(gu,F_i(x,y,z)), d(gu,F_j(u,v,w))\Big\},$$

where $i=r-1, j=s-1; r,s\in\mathbb{N}$.

(vi) (a) F_i *is continuous for each* i *or (b)* X *is regular.*
Then $\{F_i\}_{i\in\mathbb{N}}$ *and* g *have tripled coincidence point. That is, there exists* $(x,y,z)\in X\times X\times X$ *such that* $gx=F_i(x,y,z)$, $gy=F_i(y,x,y)$ *and* $gz=F_i(z,y,x)$ *for some* $i\in\mathbb{N}$.

Proof. Let $x_0,y_0,z_0\in X$ such that $gx_0\preceq F_0(x_0,y_0,z_0)$, $gy_0\succeq F_0(y_0,x_0,y_0)$ and $gz_0\preceq F_0(z_0,y_0,x_0)$. Since it is given that, $F_0(X\times X\times X)\subseteq g(X)$, we can choose $x_1,y_1,z_1\in X$ such that $gx_1=F_0(x_0,y_0,z_0)$, $gy_1=F_0(y_0,x_0,y_0)$ and $gz_1=F_0(z_0,y_0,x_0)$. Again, we can choose $x_2,y_2,z_2\in X$ such that $gx_2=F_1(x_1,y_1,z_1)$, $gy_2=F_1(y_2,x_2,y_2)$ and $gz_2=F_1(z_1,y_1,x_1)$. Continuing this process, we can construct three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$gx_{n+1}=F_n(x_n,y_n,z_n),\ gy_{n+1}=F_n(y_n,x_n,y_n) \text{ and } gz_{n+1}=F_n(z_n,y_n,x_n) \text{ for all } n\geq 0. \tag{2.2}$$

If for some $n_0\in\mathbb{N}$, we have $gx_{n_0}=F_{n_0}(x_{n_0},y_{n_0},z_{n_0})$, $gy_{n_0}=F_{n_0}(y_{n_0},x_{n_0},y_{n_0})$ and $gz_{n_0}=F_{n_0}(z_{n_0},y_{n_0},x_{n_0})$ then $\{F_i\}_{i\in\mathbb{N}}$ and g have a tripled coincidence point. Therefore, in what follows, we suppose that for each $n\geq 0$, $gx_{n+1}=F_n(x_n,y_n,z_n)\neq gx_n$ or $gy_{n+1}=F_n(y_n,x_n,y_n)\neq gy_n$ or $gz_{n+1}=F_n(z_n,y_n,x_n)\neq gz_n$ holds.
Now, we shall show that

$$gx_n\preceq gx_{n+1},\ gy_n\succeq gy_{n+1} \text{ and } gz_n\preceq gz_{n+1}, \tag{2.3}$$

for all $n\geq 0$. For this purpose, we use the mathematical induction. By (iv) and in view of $gx_1=F_0(x_0,y_0,z_0)$, $gy_1=F_0(y_0,x_0,y_0)$ and $gz_1=F_0(z_0,y_0,x_0)$, we arrive at $gx_0\preceq gx_1$, $gy_0\succeq gy_1$ and $gz_0\preceq gz_1$, thus (2.3) is true for $n=0$. We presume that (2.3) is true for some $n>0$. Now, utilizing inequality (2.2) and (2.3), which yields that

$$gx_{n+1}=F_n(x_n,y_n,z_n)\preceq F_{n+1}(x_{n+1},y_{n+1},z_{n+1})=gx_{n+2},$$

$$gy_{n+2}=F_{n+1}(y_{n+1},x_{n+1},y_{n+1})\preceq F_n(y_n,x_n,y_n)=gy_{n+1}$$

and

$$gz_{n+1}=F_n(z_n,y_n,x_n)\preceq F_{n+1}(z_{n+1},y_{n+1},x_{n+1})=gz_{n+2}.$$

Hence, (2.3) holds for $n+1$. Proceeding by mathematical induction, (2.3) follows. Thus, we get

$$gx_0\preceq gx_1\preceq gx_2\preceq\ldots\preceq gx_{n+1}\preceq\ldots,$$

$$gy_0\succeq gy_1\succeq gy_2\succeq\ldots\succeq gy_{n+1}\succeq\ldots$$

and

$$gz_0\preceq gz_1\preceq gz_2\preceq\ldots\preceq gz_{n+1}\preceq\ldots.$$

On using inequality (2.1) and (2.2), we acquire that

$$\begin{aligned} d(gx_1, gx_2) &= d(F_0(x_0, y_0, z_0), F_1(x_1, y_1, z_1)) && (2.4)\\ &\leq \beta\Big(M_{0,1}(x_0, y_0, z_0, x_1, y_1, z_1)\Big)\phi\Big(M_{0,1}(x_0, y_0, z_0, x_1, y_1, z_1)\Big) && (2.5)\\ &\quad + L\psi\Big(N_{0,1}(x_0, y_0, z_0, x_1, y_1, z_1)\Big), && (2.6) \end{aligned}$$

where

$$\begin{aligned} M_{0,1}(x_0, y_0, z_0, x_1, y_1, z_1) = \max\Big\{ &\frac{d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)}{3},\\ &\frac{d(gx_1, gx_2) + d(gy_1, gy_2) + d(gz_1, gz_2)}{3},\\ &\frac{d(gx_1, gx_2) + d(gy_1, gy_2) + d(gz_1, gz_2)}{3}\Big\} \end{aligned}$$

and

$$N_{0,1}(x_0, y_0, z_0, x_1, y_1, z_1) = \min\Big\{d(gx_0, gx_1), d(gx_0, gx_2), d(gx_1, gx_1), d(gx_1, gx_2)\Big\} = 0.$$

For simplicity, let us denote $\delta_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1})$, with this substitution above leads to following cases.
Case (i):

$$M_{0,1}(x_0, y_0, z_0, x_1, y_1, z_1) = \max\Big\{\frac{\delta_0}{3}, \frac{\delta_1}{3}, \frac{\delta_1}{3}\Big\} = \frac{\delta_1}{3}.$$

With this valued and by the properties of β, ϕ and ψ functions, (2.4) turns into

$$d(gx_1, gx_2) \leq \beta(\frac{\delta_1}{3})\phi(\frac{\delta_1}{3}) < \frac{\delta_1}{3}. \tag{2.7}$$

Similarly,

$$d(gy_1, gy_2) < \frac{\delta_1}{3} \tag{2.8}$$

and

$$d(gz_1, gz_2) < \frac{\delta_1}{3}. \tag{2.9}$$

Adding (2.7), (2.8) and (2.9), we acquire that

$$d(gx_1, gx_2) + d(gy_1, gy_2) + d(gz_1, gz_2) = \delta_1 < \delta_1.$$

Leads to a contradiction. The similar conclusion holds for all $n \geq 0$.
Case (ii):

$$M_{0,1}(x_0, y_0, z_0, x_1, y_1, z_1) = \max\Big\{\frac{\delta_0}{3}, \frac{\delta_1}{3}, \frac{\delta_1}{3}\Big\} = \frac{\delta_0}{3}.$$

Which in turn yields

$$d(gx_1, gx_2) \leq \beta(\frac{\delta_0}{3})\phi(\frac{\delta_0}{3}) < \frac{\delta_0}{3}.$$

Analogously, we derive

$$d(gx_2, gx_3) \leq \beta(\frac{\delta_1}{3})\phi(\frac{\delta_1}{3}) < \frac{\delta_1}{3}.$$

Repeating the above procedure, we get

$$d(gx_n, gx_{n+1}) \leq \beta(\frac{\delta_{n-1}}{3})\phi(\frac{\delta_{n-1}}{3}) < \frac{\delta_{n-1}}{3}. \tag{2.10}$$

Using similar arguments as above, one can easily show that

$$d(gy_n, gy_{n+1}) \leq \beta(\frac{\delta_{n-1}}{3})\phi(\frac{\delta_{n-1}}{3}) < \frac{\delta_{n-1}}{3} \tag{2.11}$$

and

$$d(gz_n, gz_{n+1}) \leq \beta(\frac{\delta_{n-1}}{3})\phi(\frac{\delta_{n-1}}{3}) < \frac{\delta_{n-1}}{3}. \tag{2.12}$$

Adding (2.10), (2.11) and (2.12), which implies

$$d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1}) = \delta_n < \delta_{n-1}.$$

It follows that the sequence $\{\delta_n\}$ is monotone decreasing sequence of non-negative real numbers. Hence, there exists an $\delta \geq 0$ such that $\lim_{n\to\infty} \delta_{n+1} = \delta$. We shall show that $\delta = 0$. Assume to the contrary that $\delta > 0$, then from (2.10)-(2.12) and by the property of ϕ, we have

$$d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1}) = \delta_n \leq 3\beta(\frac{\delta_{n-1}}{3})\phi(\frac{\delta_{n-1}}{3}).$$

So,

$$\delta_n \leq \beta(\frac{\delta_{n-1}}{3})\delta_{n-1}.$$

Which on making $n \to \infty$, give rise $\delta \leq \lim_{n\to\infty} \beta(\frac{\delta_{n-1}}{3})\delta < \delta$. Which yields $\lim_{n\to\infty} \beta(\frac{\delta_{n-1}}{3}) = 1$.
Since, $\beta \in S$, therefore $d(gx_n, gx_{n+1}) \to 0$, $d(gy_n, gy_{n+1}) \to 0$ and $d(gz_n, gz_{n+1}) \to 0$. Which gives a contradiction, yielding thereby $\delta = 0$, as $n \to \infty$, that is

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} (d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1})) = 0. \tag{2.13}$$

Next, we show that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences. On the contrary, suppose that at least one of $\{gx_n\}$ or $\{gy_n\}$ or $\{gz_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ for which we can search sub-sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integer k, $n(k) > m(k) \geq k$ and let

$$\delta_k = d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)}) + d(gz_{m(k)}, gz_{n(k)}) \geq \epsilon. \tag{2.14}$$

Moreover, corresponding to $m(k)$, we may choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (2.14). Then

$$d(gx_{m(k)}, gx_{n(k)-1}) + d(gy_{m(k)}, gy_{n(k)-1}) + d(gz_{m(k)}, gz_{n(k)-1}) < \epsilon. \tag{2.15}$$

Due to triangle inequality and (2.15), it follows that

$$\epsilon \leq \delta_k \leq d(gx_{m(k)}, gx_{n(k)-1}) + d(gy_{m(k)}, gy_{n(k)-1}) + d(gz_{m(k)}, gz_{n(k)-1}) \tag{2.16}$$

$$+ d(gx_{n(k)-1}, gx_{n(k)}) + d(gy_{n(k)-1}, gy_{n(k)}) + d(gz_{n(k)-1}, gz_{n(k)}) \tag{2.17}$$

$$< \epsilon + d(gx_{n(k)-1}, gx_{n(k)}) + d(gy_{n(k)-1}, gy_{n(k)}) + d(gz_{n(k)-1}, gz_{n(k)}). \tag{2.18}$$

Now, we prove that $\lim_{k\to\infty} \delta_k = \epsilon$. By taking the limit of supremum in (2.16) as $k \to \infty$ and using (2.13), one obtains $\epsilon \leq \lim_{k\to\infty} \sup \delta_k \leq \epsilon$, which get that

$$\limsup_{k\to\infty} \delta_k = \epsilon. \tag{2.19}$$

Again, on taking the inferior limit as $k \to \infty$ in (2.16) and from (2.19), we arrive at
$\epsilon \leq \liminf_{k\to\infty} \delta_k \leq \limsup_{k\to\infty} \delta_k = \epsilon$, which gives

$$\liminf_{k\to\infty} \delta_k = \epsilon. \tag{2.20}$$

Hence from (2.19) and (2.20), one deduce that $\liminf_{k\to\infty} \delta_k$ exists and

$$\liminf_{k\to\infty} \delta_k = \liminf_{k\to\infty} \left(d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)}) + d(gz_{m(k)}, gz_{n(k)})\right) = \epsilon. \tag{2.21}$$

Then from (2.20) and (2.21), $\lim_{k\to\infty} \delta_k$ exists and

$$\lim_{k\to\infty} \delta_k = \epsilon. \tag{2.22}$$

Also, by the triangle inequality, we acquire

$$\delta_k = d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)}) + d(gz_{m(k)}, gz_{n(k)}) \tag{2.23}$$
$$\leq \delta_{m(k)} + \delta_{n(k)} + d(gx_{m(k)+1}, gx_{n(k)+1}) + d(gy_{m(k)+1}, gy_{n(k)+1}) + d(gz_{m(k)+1}, gz_{n(k)+1}). \tag{2.24}$$

As, $n(k) > m(k)$ and $gx_{m(k)} \preceq gx_{n(k)}$, $gy_{m(k)} \succeq gy_{n(k)}$ and $gz_{m(k)} \preceq gz_{n(k)}$. Thus, from inequality (2.1) and (2.2), we have

$$\begin{aligned}
&d(gx_{m(k)+1}, gx_{n(k)+1}) + d(gy_{m(k)+1}, gy_{n(k)+1}) + d(gz_{m(k)+1}, gz_{n(k)+1})\\
&= d(F_{m(k)}(x_{m(k)}, y_{m(k)}, z_{m(k)}), F_{n(k)}(x_{n(k)}, y_{n(k)}, z_{n(k)}))\\
&\quad + d(F_{m(k)}(y_{m(k)}, x_{m(k)}, y_{m(k)}), F_{n(k)}(y_{n(k)}, x_{n(k)}, y_{n(k)}))\\
&\quad + d(F_{m(k)}(z_{m(k)}, y_{m(k)}, x_{m(k)}), F_{n(k)}(z_{n(k)}, y_{n(k)}, x_{n(k)}))\\
&\leq 3\beta\Big(M_{m(k),n(k)}(x_{m(k)}, y_{m(k)}, z_{m(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)})\Big)\phi\Big(M_{m(k),n(k)}(x_{m(k)}, y_{m(k)}, z_{m(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)})\Big)\\
&\quad + 3L\psi\Big(N_{m(k),n(k)}(x_{m(k)}, y_{m(k)}, z_{m(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)})\Big).
\end{aligned} \tag{2.25}$$

Using the definition of $M_{i,j}(x, y, z, u, v, w)$, $N_{i,j}(x, y, z, u, v, w)$ and keeping the inequalities (2.13) and (2.22) in mind, we get

$$\begin{aligned}
\lim_{k\to\infty} M_{m(k),n(k)}(x_{m(k)}, y_{m(k)}, z_{m(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)}) &= \epsilon/3;\\
\lim_{k\to\infty} N_{m(k),n(k)}(x_{m(k)}, y_{m(k)}, z_{m(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)}) &= 0.
\end{aligned} \tag{2.26}$$

Indeed

$$\begin{aligned}
&M_{m(k),n(k)}(x_{m(k)}, y_{m(k)}, z_{m(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)})\\
&= \max\Big\{\frac{d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)}) + d(gz_{m(k)}, gz_{n(k)})}{3},\\
&\frac{d(gx_{m(k)+1}, gx_{n(k)+1}) + d(gy_{m(k)+1}, gy_{n(k)+1}) + d(gz_{m(k)+1}, gz_{n(k)+1})}{3},\\
&\frac{d(gx_{n(k)}, gx_{n(k)+1}) + d(gy_{n(k)}, gy_{n(k)+1}) + d(gz_{n(k)}, gz_{n(k)+1})}{3}\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&N_{m(k),n(k)}(x_{m(k)}, y_{m(k)}, z_{m(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)})\\
&= \min\Big\{d(gx_{m(k)}, gx_{m(k)+1}), d(gx_{m(k)}, gx_{n(k)+1}), d(gx_{n(k)}, gx_{m(k)+1}), d(gx_{n(k)}, gx_{n(k)+1})\Big\}.
\end{aligned}$$

Therefore, it follows from (2.23) and (2.25) that

$$\begin{aligned}
\delta_k &\leq \delta_{m(k)} + \delta_{n(k)}\\
&\quad + 3\beta\Big(M_{m(k),n(k)}(x_{m(k)}, y_{m(k)}, z_{m(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)})\Big)\phi\Big(M_{m(k),n(k)}(x_{m(k)}, y_{m(k)}, z_{m(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)})\Big)\\
&\quad + 3L\psi\Big(N_{m(k),n(k)}(x_{m(k)}, y_{m(k)}, z_{m(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)})\Big).
\end{aligned} \tag{2.27}$$

Letting the limit as $k \to \infty$ in (2.27) and using inequality (2.26), we arrive at $\lim_{k\to\infty} \delta_k = \epsilon \le \lim_{k\to\infty} \beta(\frac{\delta_k}{3})\epsilon \le \epsilon$, which deduce that $\lim_{k\to\infty} \beta(\frac{\delta_k}{3}) = 1$. Hence, we assert that $\delta_k \to 0$ as $k \to \infty$, which is a contradiction to (2.22). Thus, $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in X. Since, X is complete, then there exists $(x, y, z) \in X \times X \times X$, such that

$$\lim_{n\to+\infty} gx_{n+1} = \lim_{n\to+\infty} F_n(x_n, y_n, z_n) = x,$$

$$\lim_{n\to+\infty} gy_{n+1} = \lim_{n\to+\infty} F_n(y_n, x_n, y_n) = y$$

and

$$\lim_{n\to+\infty} gz_{n+1} = \lim_{n\to+\infty} F_n(z_n, y_n, x_n) = z.$$

Since, $\{F_i\}_{i\in\mathbb{N}}$ and g are compatible, we acquire

$$\lim_{n\to+\infty} d\Big(g\Big(F_n(x_n, y_n, z_n)\Big), F_n\Big(gx_n, gy_n, gz_n\Big)\Big) = 0,$$

$$\lim_{n\to+\infty} d\Big(g\Big(F_n(y_n, x_n, y_n)\Big), F_n\Big(gy_n, gx_n, gy_n\Big)\Big) = 0$$

and

$$\lim_{n\to+\infty} d\Big(g\Big(F_n(z_n, y_n, x_n)\Big), F_n\Big(gz_n, gy_n, gx_n\Big)\Big) = 0. \tag{2.28}$$

Since, $\{F_i\}_{i\in\mathbb{N}}$ and g are weakly reciprocally continuous, we acquire

$$\lim_{n\to+\infty} g\Big(F_n(x_n, y_n, z_n)\Big) = gx,$$

$$\lim_{n\to+\infty} g\Big(F_n(y_n, x_n, y_n)\Big) = gy$$

and

$$\lim_{n\to+\infty} g\Big(F_n(z_n, y_n, x_n)\Big) = gz.$$

It follows

$$\lim_{n\to+\infty} F_n\Big(gx_n, gy_n, gz_n\Big) = gx, \tag{2.29}$$

$$\lim_{n\to+\infty} F_n\Big(gy_n, gx_n, gy_n\Big) = gy \tag{2.30}$$

and

$$\lim_{n\to+\infty} F_n\Big(gz_n, gy_n, gx_n\Big) = gz. \tag{2.31}$$

Consider the two possibilities given in condition (vi).
(a) Assume that F_i is continuous. Due to inequality (2.1), we arrive at

$$\begin{aligned} d\Big(F_i(x, y, z), F_n\Big(gx_n, gy_n, gz_n\Big)\Big) \le \beta\Big(M_{i,n}(x, y, z, gx_n, gy_n, gz_n)\Big)\phi\Big(M_{i,n}(x, y, z, gx_n, gy_n, gz_n)\Big) \\ + L\psi\Big(N_{i,n}(x, y, z, gx_n, gy_n, gz_n)\Big), \end{aligned}$$

where

$$M_{i,n}(x,y,z,gx_n,gy_n,gz_n) = \max\Big\{\frac{d(gx,g(gx_n))+d(gy,g(gy_n))+d(gz,g(gz_n))}{3},$$
$$\frac{d(F_i(x,y,z),F_n(gx_n,gy_n,gz_n))+d(F_i(y,x,y),F_n(gy_n,gx_n,gy_n))+d(F_i(z,y,x),F_n(gz_n,}{3}$$
$$\frac{d(g(gx_n),g(gx_{n+1}))+d(g(gy_n),g(gy_{n+1}))+d(g(gz_n),g(gz_{n+1}))}{3}\Big\}$$

and

$$N_{i,n}(x,y,z,gx_n,gy_n,gz_n) = \min\Big\{d(gx,gx_{i+1}),d(gx,gx_{n+1}),d(g(gx_n),gx_{i+1}),d(g(gx_n),g(gx_{n+1}))\Big\}.$$

Passing to the limit as $n\to\infty$ and from (2.28), (2.29) and utilizing the continuity of F_i and g, one can conclude that

$$d\Big(F_i(x,y,z),gx\Big) \le \lim_{n\to+\infty}\beta\Big(d\Big(F_i(x,y,z),F_n\Big(gx_n,gy_n,gz_n\Big)\Big)\Big)d\Big(F_i(x,y,z),gx\Big)$$

Which yields, $d(F_i(x,y,z),gx)=0$, i.e., $F_i(x,y,z)=gx$, as $\beta\in S$. Using the same as mentioned above, one can obtained $F_i(y,x,y)=gy$ and $F_i(z,y,x)=gz$.

(b) Suppose that X is regular. As $\{gx_n\}$ and $\{gz_n\}$ are non-decreasing and $\{gy_n\}$ is non-increasing, utilizing the regularity of X, we get $gx_n\preceq x,\ y\preceq gy_n$ and $gz_n\preceq z$ for all $n\ge 0$.

Therefore, using inequality (2.1), we have

$$\begin{aligned}&d\Big(F_i(x,y,z),F_n\Big(gx_n,gy_n,gz_n\Big)\Big)\\&\le\beta\Big(M_{i,n}(x,y,z,gx_n,gy_n,gz_n)\Big)\phi\Big(M_{i,n}(x,y,z,gx_n,gy_n,gz_n)\Big)\\&\quad+L\psi\Big(N_{i,n}(x,y,z,gx_n,gy_n,gz_n)\Big),\end{aligned}$$

Letting the limit as $n\to\infty$ in previous inequality and applying the same treatment as above, one can easily arrive at $d\Big(F_i(x,y,z),gx\Big)=0$, which gives $F_i(x,y,z)=gx$. Analogously, it can be derived that $F_i(y,x,y)=gy$ and $F_i(z,y,x)=gz$. This concludes the Theorem.

Remark 2.2. If we relax the conditions (i), (iii) and completeness of X in Theorem 2.1 by assuming $g(X)$ to be a complete subspace of X and X is regular then the conclusion of Theorem 2.1 remains true.
Proceeding exactly as in Theorem 2.1, $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are cauchy sequences in $g(X)$. Since, $g(X)$ is complete, then there exists $(x,y,z)\in X\times X\times X$, such that

$$\lim_{n\to+\infty} gx_n = gx,$$
$$\lim_{n\to+\infty} gy_n = gy$$

and

$$\lim_{n\to+\infty} gz_n = gz.$$

As $\{gx_n\}$ and $\{gz_n\}$ are non-decreasing and $\{gy_n\}$ is non-increasing, utilizing the regularity of X, we get $gx_n\preceq gx,\ gy\preceq gy_n$ and $gz_n\preceq gz$ for all $n\ge 0$. Using (2.1) we obtain

$$\begin{aligned}d(F_i(x,y,z),gx_{n+1}) &= d(F_i(x,y,z),F_n(x_n,y_n,z_n))\\&\le\beta\Big(M_{i,n}(x,y,z,x_n,y_n,z_n)\Big)\phi\Big(M_{i,n}(x,y,z,x_n,y_n,z_n)\Big)\\&\quad+L\psi\Big(N_{i,n}(x,y,z,x_n,y_n,z_n)\Big),\end{aligned}$$

where

$$
\begin{aligned}
M_{i,n}(x,y,z,x_n,y_n,z_n) = \max\Big\{ & \frac{d(gx,gx_n)+d(gy,gy_n)+d(gz,gz_n)}{3},\\
& \frac{d(F_i(x,y,z),F_n(x_n,y_n,z_n))+d(F_i(y,x,y),F_n(y_n,x_n,y_n))+d(F_i(z,y,x),F_n(z_n,y_n,x_n))}{3},\\
& \frac{d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})+d(gz_n,gz_{n+1})}{3}\Big\}
\end{aligned}
$$

and

$$
N_{i,n}(x,y,z,x_n,y_n,z_n) = \min\Big\{d(gx,gx_{i+1}),d(gx,gx_{n+1}),d(gx_n,gx_{i+1}),d(gx_n,gx_{n+1})\Big\}.
$$

Passing to the limit as $n\to\infty$ in previous inequality and applying the same treatment as mentioned above, we can easily observe that $d\Big(F_i(x,y,z),gx\Big)=0$, which gives $F_i(x,y,z)=gx$. In a similar way, $F_i(y,x,y)=gy$ and $F_i(z,y,x)=gz$.

Remark 2.3. On replacing the weakly reciprocal continuity of $\{F_i\}_{i\in\mathbb{N}}$ and g by assuming g to be non-decreasing in Theorem 2.1, one can derive another version of our main results.

Remark 2.4. In Theorem 2.1, restrict $F: X^3\to X,\ L=0$ and $\beta(t)=k$, where $k\in(0,1)$, we obtain

$$
\begin{aligned}
& d\Big(F(x,y,z),F(u,v,w)\Big)+d\Big(F(y,x,y),F(v,u,v)\Big)+d\Big(F(z,y,x),F(w,v,u)\Big)\\
&\quad \le 3k\phi\Big(\max\Big\{\frac{d(gx,gu)+d(gy,gv)+d(gz,gw)}{3},\\
&\quad \frac{d(F(x,y,z),F(u,v,w))+d(F(y,x,y),F(v,u,v))+d(F(z,y,x),F(w,v,u))}{3},\\
&\quad \frac{d(gu,F(u,v,w))+d(gv,F(v,u,v))+d(gw,F(w,v,u))}{3}\Big\}\Big),\\
&\quad \le 3\phi\Big(\max\Big\{\frac{d(gx,gu)+d(gy,gv)+d(gz,gw)}{3},\\
&\quad \frac{d(F(x,y,z),F(u,v,w))+d(F(y,x,y),F(v,u,v))+d(F(z,y,x),F(w,v,u))}{3},\\
&\quad \frac{d(gu,F(u,v,w))+d(gv,F(v,u,v))+d(gw,F(w,v,u))}{3}\Big\}\Big).
\end{aligned}
$$

Thus, we get the another version of V. Berinde [3] for tripled coincidence points.

Remark 2.5. By restricting $F: X^3\to X,\ L=0$ and $\beta(t)=k$, where $k\in(0,1)$, one can easily get

$$
\begin{aligned}
d\Big(F(x,y,z),F(u,v,w)\Big) \le \phi\Big(\max\Big\{ & \frac{d(gx,gu)+d(gy,gv)+d(gz,gw)}{3},\\
& \frac{d(F(x,y,z),F(u,v,w))+d(F(y,x,y),F(v,u,v))+d(F(z,y,x),F(w,v,u))}{3},\\
& \frac{d(gu,F(u,v,w))+d(gv,F(v,u,v))+d(gw,F(w,v,u))}{3}\Big\}\Big).
\end{aligned}
$$

Thus, our results extend the coupled coincidence point theorems contained in V. Lakshmikantam and L. Ciric [8] for tripled coincidence points.

Remark 2.6. Restricting $F: X^3\to X,\ L=0$ Theorem 2.1 can be viewed as a generalization of Theorem 2.1 of Babu and Subhashini [1] for tripled fixed points, so it extends and generalizes the related result of Babu et al. [1], but we omit the details due to repetition.

Remark 2.7. The following observation are worth noting in the perspective of Theorem 2.1 in [12].

(i) The non-decreasing requirement of g is superfluous. Notice that, in the context of Theorem 2.1[12], authors used the sequence of mappings $\{T_i\}_{i\in\mathbb{N}}$ and in the statement and in the proof of the Theorem 2.1[12], they used $T_0(x_0, y_0, z_0)$, which is unsound as for this to hold one needs to supplant $i = r-1; r \in \mathbb{N}$;

(ii) In Example 2.3 author observed that (1,1,1) is also a tripled coincidence point of g and T_i, where $gx = x$ and $T_i(x, y, z) = \frac{x+y+z}{3i}; i \in \mathbb{N}$, which is invalid for $i > 1$;

(iii) In Theorem 2.1[12], authors used regularity of $g(X)$. On using this authors reported that $gx_n \preceq x$, $gy_n \succeq y$ and $gz_n \preceq z$ for all $n \geq 0$. Which is worthless, one needs to replace regularity of $g(X)$ by regularity of X, so that the given proof can work. Also, in Theorem 2.1[12], authors used compatibility and weakly reciprocal continuity together with completeness of $g(X)$. Notice that there is no necessity of compatibility and weakly reciprocal continuity of g and $\{T_i\}_{i\in\mathbb{N}}$ in Theorem 2.1[12], if $g(X)$ is complete subset of X and these properties are used only when X is complete. The proof can be completed on the lines of the proof of Theorem 2.1 and Remark 3.1.

Now, we present the example which demonstrate the validity of the hypotheses and degree of generality of our main result.

Example 2.8. Let $X = [-1, 1]$ with the usual metric and order. Then $(X, d, \preceq)$ be a partially ordered metric space. Consider the mappings $g : X \to X$ and $F_i : X \times X \times X \to X$ defined by
$gx = x$ and $F_i = \frac{x-y+z}{3i}; i \in \mathbb{N}$ such that $x + z < y$.
Note that $F_i(x, y, z) \subseteq gx$, $g(X)$ is complete subset of X, g is monotonic non-decreasing, continuous, as well as
$F_i(x, y, z) \preceq F_{i+1}(u, v, w),\ \ F_{i+1}(v, u, v) \preceq F_i(y, x, y)$ and $F_i(z, y, x) \preceq F_{i+1}(w, v, u)$
for $x < u$, $y > v$ and z < w. Also, one can easily verify that $\{F_i\}_{i\in\mathbb{N}}$ and g are compatible and weakly reciprocally continuous. Take $\phi, \psi : [0, +\infty) \to [0, +\infty)$ be given by $\phi(t) = \frac{3t}{4}$ and $\psi(t) = \frac{t}{2}$. And $\beta : [0, +\infty) \to [0, 1)$ be given by

$$\beta(t) = \begin{cases} \frac{2}{2+t} \text{ if t} > 0; \\ 0 \text{ if t} = 0. \end{cases}$$

Then by a routine calculation, it can be easily verified that F_i and g satisfy condition (2.33) for $L \geq 6$, when $\beta(t) > 0$ with $x \neq y \neq z$ and $u \neq v \neq w$. Also, $L \geq 10$, when $\beta(t) = 0$, with $x \neq y \neq z$ and $u \neq v \neq w$. Thus, all the hypotheses of Theorem 2.1 are satisfied and (0,0,0) is the tripled coincidence points of F_i and g.

Theorem 2.9. *In addition to the hypotheses of Theorem 2.1 assume that the set of coincidence points is comparable with respect to g, then $\{F_i\}_{i\in\mathbb{N}}$ and g have a unique tripled common fixed point, that is, there exists $(x, y, z) \in X \times X \times X$ such that $x = gx = F_i(x, y, z)$, $y = gy = F_i(y, x, y)$ and $z = gz = F_i(z, y, x)$ for some $i \in \mathbb{N}$.*

Proof. Theorem 2.1 implies that the set of tripled coincidence points is non-empty. Now, we prove that if (x, y, z) and (r, s, t) are tripled coincidence points, that is, if $F_i(x, y, z) = gx$, $F_i(y, x, y) = gy$, $F_i(z, y, x) = gz$, $F_i(r, s, t) = gr$, $F_i(s, r, s) = gs$ and $F_i(t, s, r) = gt$, then we will show that

$$gx = gr,\ gy = gs \text{ and } gz = gt. \tag{2.32}$$

On the contrary, assume that at least one of them is not equal, that is, $d(gx, gr) \neq 0$ or $d(gy, gs) \neq 0$ or $d(gz, gt) \neq 0$. Since, the set of coincidence points is comparable, applying inequality (2.1) to these points, we acquire

$$\begin{aligned} &d(gx, gr) + d(gy, gs) + d(gz, gt) \\ &= d(F_i(x, y, z), F_i(r, s, t)) + d(F_i(y, x, y), F_i(s, r, s)) + d(F_i(z, y, x), F_i(t, s, r)) \\ &\leq 3\beta\Big(M_{i,j}(x, y, z, r, s, t)\Big)\phi\Big(M_{i,j}(x, y, z, r, s, t)\Big) + 3L\psi\Big(N_{i,j}(x, y, z, r, s, t)\Big), \end{aligned}$$

where

$$M_{i,j}(x,y,z,r,s,t) = \max\Big\{\frac{d(gx,gr)+d(gy,gs)+d(gz,gt)}{3}, \frac{d(gx,gr)+d(gy,gs)+d(gz,gt)}{3}, \frac{d(gx,gr)+d(gy,gs)+d(gz,gt)}{3}\Big\},$$

and

$$N_{i,j}(x,y,z,r,s,t) = \min\Big\{d(gx,gx), d(gx,gu), d(gu,gx), d(gu,gu)\Big\}.$$

Which deduce that

$$\begin{aligned}
&d(gx,gr)+d(gy,gs)+d(gz,gt)\\
&\quad\le 3\beta\Big(\frac{d(gx,gr)+d(gy,gs)+d(gz,gt)}{3}\Big)\phi\Big(\frac{d(gx,gr)+d(gy,gs)+d(gz,gt)}{3}\Big)\\
&\quad< \beta\Big(\frac{d(gx,gr)+d(gy,gs)+d(gz,gt)}{3}\Big)(d(gx,gr)+d(gy,gs)+d(gz,gt))\\
&\quad< d(gx,gr)+d(gy,gs)+d(gz,gt),
\end{aligned}$$

which is a contradiction. So that (2.32) holds, and we have $gx = gr$, $gy = gs$ and $gz = gt$. Thus, $\{F_i\}_{i\in\mathbb{N}}$ and g have a unique tripled point of coincidence. As two compatible mappings are also weakly compatible, so they commute at their coincidence points. Hence, we conclude that, $\{F_i\}_{i\in\mathbb{N}}$ and g have a unique tripled common fixed point, whenever $\{F_i\}_{i\in\mathbb{N}}$ and g are weakly compatible. □

The following example illustrate Theorem 2.9.

Example 2.10. In the setting of Example 2.8 replace the mappings $\phi, \psi : [0,+\infty) \to [0,+\infty)$, $\beta : [0,+\infty) \to [0,1)$ and L by the followings besides retaining the rest:

$$\phi(t) = \tfrac{t}{2}; \quad \psi(t) = \tfrac{2t}{3}; \quad \beta(t) = \begin{cases} \frac{2e^{-2t}}{2+t} & \text{if t} > 0;\\ 0 & \text{if t} = 0.\end{cases} \quad \text{and} \quad L \ge 6.$$

By repeating the discussion above, one can easily observe that inequality (2.33) with $x \neq y \neq z$ and $u \neq v \neq w$, is satisfied. Hence, all the conditions of Theorem 2.9 are fulfilled, also (0,0,0) remains tripled common fixed point under F_i and g and is indeed unique.

In Theorem 2.1, if we restrict $F : X \times X \times X \to X$, $g = I$, $\phi(t) = t$ and $L = 0$, we deduce the following corollary.

Corollary 2.11. *Let $(X, \preceq)$ be a complete partially ordered metric space. Let F be a mapping from $X \times X \times X$ into X such that $F(x,y,z) \preceq F(u,v,w)$, $F(v,u,v) \preceq F(y,x,y)$ and $F(z,y,x) \preceq F(w,v,u)$, for $x,y,z,u,v,w \in X$ with $x \preceq u, v \preceq y$ and $z \preceq w$, or $x \succeq u, v \succeq y$ and $z \succeq w$. Suppose that the following hold:*

(i) there exists $(x_0,y_0,z_0) \in X \times X \times X$ such that $x_0 \preceq F_0(x_0,y_0,z_0)$, $y_0 \succeq F_0(y_0,x_0,y_0)$ and $z_0 \preceq F_0(z_0,y_0,x_0)$;

(ii) there exist $\beta \in S$ such that

$$d\Big(F(x,y,z), F(u,v,w)\Big) \le \beta\Big(M(x,y,z,u,v,w)\Big).M(x,y,z,u,v,w), \tag{2.33}$$

where

$$M(x,y,z,u,v,w) = \max\Big\{\frac{d(x,u)+d(y,v)+d(z,w)}{3},\\ \frac{d(F(x,y,z),F(u,v,w))+d(F(y,x,y),F(v,u,v))+d(F(z,y,x),F(w,v,u))}{3},\\ \frac{d(u,F(u,v,w))+d(v,F(v,u,v))+d(w,F(w,v,u))}{3}\Big\},$$

(iii) (a) F is continuous or (b) X is regular.
Then F has tripled fixed point. That is, there exists $(x,y,z)\in X\times X\times X$ such that $x=F(x,y,z)$, $y=F(y,x,y)$ and $z=F(z,y,x)$.

Corollary 2.12. *In addition to the hypotheses of Corollary 2.11, suppose that the elements in the set of tripled fixed points are comparable. Then F has a unique tripled fixed point.*

3. Applications

3.1. Application to ordinary differential equation

In this section we present an application to ordinary differential equation and this is inspired by [[7], [11]]. Consider the following system of initial-value problems:

$$\begin{cases} u_t(x,t) = u_{xx} + f(x,t,u,u_x) + g(x,t,v,v_x) + h(x,t,w,w_x), & -\infty < x < \infty,\ 0 < t \le T, \\ u(x,0) = \varphi(x) & -\infty < x < \infty, \end{cases} \tag{3.1}$$

$$\begin{cases} v_t(x,t) = v_{xx} + f(x,t,v,v_x) + g(x,t,u,u_x) + h(x,t,v,v_x), & -\infty < x < \infty,\ 0 < t \le T, \\ v(x,0) = \varphi(x) & -\infty < x < \infty, \end{cases} \tag{3.2}$$

$$\begin{cases} w_t(x,t) = w_{xx} + f(x,t,w,w_x) + g(x,t,v,v_x) + h(x,t,u,u_x), & -\infty < x < \infty,\ 0 < t \le T, \\ w(x,0) = \varphi(x) & -\infty < x < \infty, \end{cases} \tag{3.3}$$

where φ is a continuously differentiable function and that φ and φ' are bounded and f, g and h are continuous functions. Consider the space

$$\Omega = \{r(x,t) : r, r_x \in C(\mathbb{R}\times I), \text{and } ||\mathrm{r}|| < \infty\},$$

where $I=[0,T]$ and

$$||r|| = \sup_{x\in\mathbb{R},t\in I} |r(x,t)| + \sup_{x\in\mathbb{R},t\in I} |r_x(x,t)|.$$

Obviously, this space with the metric given by

$$d(u,v) = \sup_{x\in\mathbb{R},t\in I} |u(x,t)-v(x,t)| + \sup_{x\in\mathbb{R},t\in I} |u_x(x,t)-v_x(x,t)|,$$

is a complete metric space. The metric space Ω can also equipped with a partial order given by

$$u,v\in\Omega,\ u\le v \Longleftrightarrow u(x,t)\le v(x,t),\ \ u_x(x,t)\le v_x(x,t), \quad x\in\mathbb{R}, t\in I.$$

Definition 3.1. An element $(u, v, w) \in \Omega \times \Omega \times \Omega$ is called a tripled lower-upper-lower solution of (3.3) if

$$\begin{cases} u_t(x,t) \leq u_{xx} + f(x,t,u,u_x) + g(x,t,v,v_x) + h(x,t,w,w_x), & -\infty < x < \infty,\ 0 < t \leq T, \\ u(x,0) \leq \varphi(x) & -\infty < x < \infty, \end{cases}$$

$$\begin{cases} v_t(x,t) \geq v_{xx} + f(x,t,v,v_x) + g(x,t,u,u_x) + h(x,t,v,v_x), & -\infty < x < \infty,\ 0 < t \leq T, \\ v(x,0) \geq \varphi(x) & -\infty < x < \infty, \end{cases}$$

and

$$\begin{cases} w_t(x,t) \leq w_{xx} + f(x,t,w,w_x) + g(x,t,v,v_x) + h(x,t,u,u_x), & -\infty < x < \infty,\ 0 < t \leq T, \\ w(x,0) \leq \varphi(x) & -\infty < x < \infty. \end{cases}$$

Definition 3.2. Let Ψ denote the class of those functions $\psi : [0,\infty) \to [0.\infty)$ which satisfies the following conditions:

(i) ψ is continuous, sub-additive and non-decreasing;

(ii) for each $t > 0,\ \psi(t) < t$;

(iii) $\beta(t) = \frac{\psi(t)}{t} \in S$;

(iv) ψ is positive in $(0,\infty)$ with $\psi(0) = 0$.

For example, $\psi(x) = \frac{x}{x+1}$ and $\psi(x) = \ln(x+1)$ are in Ψ.

Theorem 3.3. *Consider the problem (3.3) with $f, g, h : \mathbb{R} \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuous and suppose that the following conditions are satisfied:*

(i) for any $c > 0$ with $|\alpha| < c$ and $|\gamma| < c$ the function $f(x,t,\alpha,\gamma), g(x,t,\alpha,\gamma)$ and $h(x,t,\alpha,\gamma)$ are uniformly Holder continuous in x and t for each compact subset of $\mathbb{R} \times I$;

(ii) for all $(\alpha_1,\gamma_1),(\alpha_2,\gamma_2) \in \mathbb{R}\times\mathbb{R}$ with $\alpha_1 \leq \alpha_2$ and $\gamma_1 \leq \gamma_2$, there exist $\psi \in \Psi$ and three positive constants c_f, c_g and c_h such that

$$0 \leq f(x,t,\alpha_2,\gamma_2) - f(x,t,\alpha_1,\gamma_1) \leq c_f\, \psi\Big(\frac{\alpha_2 - \alpha_1 + \gamma_2 - \gamma_1}{3}\Big),$$

$$0 \leq g(x,t,\alpha_1,\gamma_1) - g(x,t,\alpha_2,\gamma_2) \leq c_g\, \psi\Big(\frac{\alpha_2 - \alpha_1 + \gamma_2 - \gamma_1}{3}\Big),$$

and

$$0 \leq h(x,t,\alpha_2,\gamma_2) - h(x,t,\alpha_1,\gamma_1) \leq c_h\, \psi\Big(\frac{\alpha_2 - \alpha_1 + \gamma_2 - \gamma_1}{3}\Big);$$

(iii) f, g and h are bounded for bounded α and γ;

(iv) $c_f, c_g, c_h \leq \frac{1}{3}(T + 2\pi^{\frac{-1}{2}} T^{\frac{1}{2}})^{-1}$.
Then the existence of tripled lower-upper-lower solution for the initial value problem (3.3) provides the existence of the unique solution of the problem (3.3).

Proof. The problem (3.3) is equivalent to the following integral equations

$$u(x,t) = \int_{-\infty}^{\infty} k(x-\alpha,t)\varphi(\alpha)d\alpha + \int_0^t \int_{-\infty}^{\infty} k(x-\alpha,t-\lambda)\Big[f(\alpha,\lambda,u(\alpha,\lambda),u_x(\alpha,\lambda)) \\ + g(\alpha,\lambda,v(\alpha,\lambda),v_x(\alpha,\lambda)) + h(\alpha,\lambda,w(\alpha,\lambda),w_x(\alpha,\lambda))\Big]d\alpha\, d\lambda$$

$$v(x,t) = \int_{-\infty}^{\infty} k(x-\alpha,t)\varphi(\alpha)d\alpha + \int_0^t \int_{-\infty}^{\infty} k(x-\alpha,t-\lambda)\Big[f(\alpha,\lambda,v(\alpha,\lambda),v_x(\alpha,\lambda)) + g(\alpha,\lambda,u(\alpha,\lambda),u_x(\alpha,\lambda)) + h(\alpha,\lambda,v(\alpha,\lambda),v_x(\alpha,\lambda))\Big]d\alpha\, d\lambda$$

$$w(x,t) = \int_{-\infty}^{\infty} k(x-\alpha,t)\varphi(\alpha)d\alpha + \int_0^t \int_{-\infty}^{\infty} k(x-\alpha,t-\lambda)\Big[f(\alpha,\lambda,w(\alpha,\lambda),w_x(\alpha,\lambda)) + g(\alpha,\lambda,v(\alpha,\lambda),v_x(\alpha,\lambda)) + h(\alpha,\lambda,u(\alpha,\lambda),u_x(\alpha,\lambda))\Big]d\alpha\, d\lambda \tag{3.4}$$

for all $x \in \mathbb{R},\ 0 < t \leq T$. Where

$$k(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left\{\frac{-x^2}{4t}\right\}$$

for all $x \in \mathbb{R}$ and $t > 0$. The initial value problem (3.3) possesses a unique solution if and only if the equation (3.4) possesses a unique solution (u, v, w) such that u, v, w and u_x, v_x, w_x are continuous and bounded for all $x \in \mathbb{R},\ 0 < t \leq T$. In our subsequent discussion we need following integral due to [7].

(i) $\int_0^t \int_{-\infty}^{\infty} k(x-\alpha,t-\lambda)d\alpha d\lambda \leq T$;

(ii) $\int_0^t \int_{-\infty}^{\infty} |\frac{\delta k}{\delta x}(x-\alpha,t-\lambda)|d\alpha d\lambda \leq 2\pi^{\frac{-1}{2}} T^{\frac{1}{2}}$.

Let the mapping $F : \Omega \times \Omega \times \Omega \to \Omega$ is defined by

$$F(u,v,w)(x,t) = \int_{-\infty}^{\infty} k(x-\alpha,t)\varphi(\alpha)d\alpha + \int_0^t \int_{-\infty}^{\infty} k(x-\alpha,t-\lambda)\Big[f(\alpha,\lambda,u(\alpha,\lambda),u_x(\alpha,\lambda)) + g(\alpha,\lambda,v(\alpha,\lambda),v_x(\alpha,\lambda)) + h(\alpha,\lambda,w(\alpha,\lambda),w_x(\alpha,\lambda))\Big]d\alpha\, d\lambda \tag{3.5}$$

for all $x \in \mathbb{R}$ and $t \in I$. It is easy to note that, if $(u, v, w) \in \Omega \times \Omega \times \Omega$ is a fixed point of F then (u, v, w) is a solution of the problem (3.3).
We show that all the conditions of the Corollary 2.11 and 2.12 are satisfied. From the condition (ii) of Theorem 3.3, one can easily prove that F has the mixed monotone property.
From (3.5), for $u_1, v_1, w_1, u_2, v_2, w_2 \in \Omega$ with $u_1 \geq u_2,\ v_1 \leq v_2$ and $w_1 \geq w_2$, we have

$$\begin{aligned}
&F(u_1,v_1,w_1)(x,t) - F(u_2,v_2,w_2)(x,t)\\
&= \int_0^t \int_{-\infty}^{\infty} k(x-\alpha,t-\lambda)\Big[f(\alpha,\lambda,u_1(\alpha,\lambda),(u_1)_x(\alpha,\lambda)) - f(\alpha,\lambda,u_2(\alpha,\lambda),(u_2)_x(\alpha,\lambda))\\
&\quad + g(\alpha,\lambda,v_1(\alpha,\lambda),(v_1)_x(\alpha,\lambda)) - g(\alpha,\lambda,v_2(\alpha,\lambda),(v_2)_x(\alpha,\lambda))\\
&\quad + h(\alpha,\lambda,w_1(\alpha,\lambda),(w_1)_x(\alpha,\lambda)) - h(\alpha,\lambda,w_2(\alpha,\lambda),(w_2)_x(\alpha,\lambda))\Big]d\alpha\, d\lambda\\
&\leq \int_0^t \int_{-\infty}^{\infty} k(x-\alpha,t-\lambda)\Big[c_f\, \psi\Big(\frac{u_1(\alpha,\lambda) - u_2(\alpha,\lambda) + (u_1)_x(\alpha,\lambda) - (u_2)_x(\alpha,\lambda)}{3}\Big)\\
&\quad + c_g\, \psi\Big(\frac{v_2(\alpha,\lambda) - v_1(\alpha,\lambda) + (v_2)_x(\alpha,\lambda) - (v_1)_x(\alpha,\lambda)}{3}\Big)\\
&\quad + c_h\, \psi\Big(\frac{w_1(\alpha,\lambda) - w_2(\alpha,\lambda) + (w_1)_x(\alpha,\lambda) - (w_2)_x(\alpha,\lambda)}{3}\Big)\Big]d\alpha\, d\lambda.
\end{aligned} \tag{3.6}$$

Since the function ψ is non-decreasing, therefore we get

$$\begin{aligned}
&\psi\Big(\frac{u_1(\alpha,\lambda)-u_2(\alpha,\lambda)+(u_1)_x(\alpha,\lambda)-(u_2)_x(\alpha,\lambda)}{3}\Big)\\
&\leq \psi\Big(\frac{\sup_{\alpha\in\mathbb{R},\lambda\in I}|u_1(\alpha,\lambda)-u_2(\alpha,\lambda)|+\sup_{\alpha\in\mathbb{R},\lambda\in I}|(u_1)_x(\alpha,\lambda)-(u_2)_x(\alpha,\lambda)|}{3}\Big)\\
&\leq \psi\Big(\frac{d(u_1,u_2)}{3}\Big).
\end{aligned}$$

Similarly

$$\begin{aligned}
&\psi\Big(\frac{v_2(\alpha,\lambda)-v_1(\alpha,\lambda)+(v_2)_x(\alpha,\lambda)-(v_1)_x(\alpha,\lambda)}{3}\Big)\\
&\leq \psi\Big(\frac{\sup_{\alpha\in\mathbb{R},\lambda\in I}|v_1(\alpha,\lambda)-v_2(\alpha,\lambda)|+\sup_{\alpha\in\mathbb{R},\lambda\in I}|(v_1)_x(\alpha,\lambda)-(v_2)_x(\alpha,\lambda)|}{3}\Big)\\
&\leq \psi\Big(\frac{d(v_1,v_2)}{3}\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\psi\Big(\frac{w_1(\alpha,\lambda)-w_2(\alpha,\lambda)+(w_1)_x(\alpha,\lambda)-(w_2)_x(\alpha,\lambda)}{3}\Big)\\
&\leq \psi\Big(\frac{\sup_{\alpha\in\mathbb{R},\lambda\in I}|w_1(\alpha,\lambda)-w_2(\alpha,\lambda)|+\sup_{\alpha\in\mathbb{R},\lambda\in I}|(w_1)_x(\alpha,\lambda)-(w_2)_x(\alpha,\lambda)|}{3}\Big)\\
&\leq \psi\Big(\frac{d(w_1,w_2)}{3}\Big).
\end{aligned}$$

Thus from inequality (3.6), we arrive at

$$\begin{aligned}
&\sup_{x\in\mathbb{R},t\in I}|F(u_1,v_1,w_1)(x,t)-F(u_2,v_2,w_2)(x,t)|\\
&\leq \int_0^t\int_{-\infty}^{\infty}k(x-\alpha,t-\lambda)\Big[c_f\,\psi\Big(\frac{d(u_1,u_2)}{3}\Big)+c_g\,\psi\Big(\frac{d(v_1,v_2)}{3}\Big)+c_h\,\psi\Big(\frac{d(w_1,w_2)}{3}\Big)\Big]d\alpha\,d\lambda \qquad (3.7)\\
&\leq T\Big[c_f\,\psi\Big(\frac{d(u_1,u_2)}{3}\Big)+c_g\,\psi\Big(\frac{d(v_1,v_2)}{3}\Big)+c_h\,\psi\Big(\frac{d(w_1,w_2)}{3}\Big)\Big].
\end{aligned}$$

Moreover, from the above inequality, we acquire

$$\begin{aligned}
&\sup_{x\in\mathbb{R},t\in I}\Big|\frac{\delta F(u_1,v_1,w_1)}{\delta x}(x,t)-\frac{\delta F(u_2,v_2,w_2)}{\delta x}(x,t)\Big|\\
&\leq \int_0^t\int_{-\infty}^{\infty}\Big|\frac{\delta k}{\delta x}(x-\alpha,t-\lambda)\Big|\Big[c_f\,\psi\Big(\frac{d(u_1,u_2)}{3}\Big)+c_g\,\psi\Big(\frac{d(v_1,v_2)}{3}\Big)+c_h\,\psi\Big(\frac{d(w_1,w_2)}{3}\Big)\Big]d\alpha\,d\lambda \qquad (3.8)\\
&\leq 2\pi^{\frac{-1}{2}}T^{\frac{1}{2}}\Big[c_f\,\psi\Big(\frac{d(u_1,u_2)}{3}\Big)+c_g\,\psi\Big(\frac{d(v_1,v_2)}{3}\Big)+c_h\,\psi\Big(\frac{d(w_1,w_2)}{3}\Big)\Big].
\end{aligned}$$

By adding (3.7) and (3.8) and utilizing the condition (iv) of Theorem 3.3, we have

$$\begin{aligned}
d(F(u_1,v_1,w_1),F(u_2,v_2,w_2)) &\leq (T+2\pi^{\frac{-1}{2}}T^{\frac{1}{2}})\Big[c_f\,\psi\Big(\frac{d(u_1,u_2)}{3}\Big)+c_g\,\psi\Big(\frac{d(v_1,v_2)}{3}\Big)+c_h\,\psi\Big(\frac{d(w_1,w_2)}{3}\Big)\Big]\\
&\leq \frac{1}{3}\Big[\psi\Big(\frac{d(u_1,u_2)}{3}\Big)+\psi\Big(\frac{d(v_1,v_2)}{3}\Big)+\psi\Big(\frac{d(w_1,w_2)}{3}\Big)\Big].
\end{aligned} \qquad (3.9)$$

As ψ is non-decreasing, which yields

$$\psi\Big(\frac{d(u_1,u_2)}{3}\Big)+\psi\Big(\frac{d(v_1,v_2)}{3}\Big)+\psi\Big(\frac{d(w_1,w_2)}{3}\Big)\leq 3\psi\Big(\frac{d(u_1,u_2)+d(v_1,v_2)+d(w_1,w_2)}{3}\Big).$$

Hence, from (3.9), we obtain

$$\begin{aligned}
d(F(u_1,v_1,w_1),F(u_2,v_2,w_2)) &\le \psi\Big(\frac{d(u_1,u_2)+d(v_1,v_2)+d(w_1,w_2)}{3}\Big)\\
&\le \psi\Big(M(u_1,v_1,w_1,u_2,v_2,w_2)\Big)\\
&= \frac{\psi\Big(M(u_1,v_1,w_1,u_2,v_2,w_2)\Big)}{M(u_1,v_1,w_1,u_2,v_2,w_2)}.M(u_1,v_1,w_1,u_2,v_2,w_2)\\
&= \beta\Big(M(u_1,v_1,w_1,u_2,v_2,w_2)\Big).M(u_1,v_1,w_1,u_2,v_2,w_2)
\end{aligned}$$

Finally, Let $(u,v,w)\in\Omega\times\Omega\times\Omega$ be a tripled lower-upper-lower solution of (3.3) then we have

$$u(x,t)\le F(u(x,t),v(x,t),w(x,t)),$$

$$v(x,t)\ge F(v(x,t),u(x,t),v(x,t)),$$
$$w(x,t)\le F(w(x,t),v(x,t),u(x,t)),$$

for all $x\in\mathbb{R}$ and $t\in(0,T]$. Therefore from Corollary 2.11 and 2.12, F has unique tripled fixed point. □

3.2. Application to system of integral equations

Consider the following system of integral equations:

$$\begin{aligned}
u(t) &= p(t)+\int_0^T \lambda(t,s)[f(s,u(s))+g(s,v(s))+h(s,w(s))]ds,\\
v(t) &= p(t)+\int_0^T \lambda(t,s)[f(s,v(s))+g(s,u(s))+h(s,v(s))]ds,\\
w(t) &= p(t)+\int_0^T \lambda(t,s)[f(s,w(s))+g(s,v(s))+h(s,u(s))]ds.
\end{aligned}\tag{3.10}$$

We consider the space $X=C([0,T],\mathbb{R})$ of continuous functions defined on $[0,T]$. Obviously, the space with the metric given by

$$d(u,v)=\sup_{t\in[0,T]}|u(t)-v(t)|,\quad u,v\in C([0,T],\mathbb{R}),$$

is a complete metric space. Consider on $X=C([0,T],\mathbb{R})$ the natural partial order relation, that is,

$$u,v\in C([0,T],\mathbb{R}),\quad u\le v \iff u(t)\le v(t),\quad t\in[0,T].$$

Theorem 3.4. *Consider the problem (3.10) and assume that the following conditions are satisfied:*

(i) $f,\ g,\ h:[0,T]\times\mathbb{R}\to\mathbb{R}$ *are continuous;*

(ii) $p:[0,T]\to\mathbb{R}$ *is continuous;*

(iii) $\lambda:[0,T]\times\mathbb{R}\to[0,\infty)$ *is continuous;*

(iv) There exists $c>0$ *and* $\psi\in\Psi$ *such that for all* $u,v\in\mathbb{R}$, $v\ge u$,

$$\begin{aligned}
0\le f(s,v)-f(s,u) &\le \frac{c}{3}\psi(\frac{v-u}{3}),\\
0\le g(s,u)-g(s,v) &\le \frac{c}{3}\psi(\frac{v-u}{3}),\\
0\le h(s,v)-h(s,u) &\le \frac{c}{3}\psi(\frac{v-u}{3});
\end{aligned}$$

(v) Assume that

$$c \sup_{t\in[0,T]} \int_0^T \lambda(t,s)ds \leq 1;$$

(vi) A pair $(\alpha,\eta,\gamma)\in X^3$ with $(X=C([0,T],\mathbb{R}))$ is called a lower-upper-lower solution of (3.10), if

$$\begin{aligned}
\alpha(t) &\leq p(t) + \int_0^T \lambda(t,s)[f(s,u(s))+g(s,v(s))+h(s,w(s))]ds,\\
\eta(t) &\geq p(t) + \int_0^T \lambda(t,s)[f(s,v(s))+g(s,u(s))+h(s,v(s))]ds,\\
\gamma(t) &\leq p(t) + \int_0^T \lambda(t,s)[f(s,w(s))+g(s,v(s))+h(s,u(s))]ds.
\end{aligned}$$

Then the system of integral equation (3.10) has a unique solution in X^3 with $(X=C([0,T],\mathbb{R}))$.

Proof. Consider the mapping $F: X\times X\times X\to X$ defined by

$$F(u,v,w)(t) = p(t) + \int_0^T \lambda(t,s)[f(s,u(s))+g(s,v(s))+h(s,w(s))]ds,$$

for all $u,v,w\in X$ and $t\in[0,T]$. We prove that all the conditions of Corollary 2.11 and 2.12 are satisfied. By the condition (iv) of the Theorem 3.4, it is not difficult to show that F has mixed monotone property. Now, for $u_1,v_1,w_1,u_2,v_2,w_2\in X$ with $u_1\geq u_2,\ v_1\leq v_2$ and $w_1\geq w_2$, we obtain

$$\begin{aligned}
&F(u_1,v_1,w_1)(t) - F(u_2,v_2,w_2)(t)\\
&= \int_0^T \lambda(t,s)\Big[f(s,u_1(s)) - f(s,u_2(s))\Big]ds + \int_0^T \lambda(t,s)\Big[g(s,v_1(s)) - g(s,v_2(s))\Big]ds\\
&\quad + \int_0^T \lambda(t,s)\Big[h(s,w_1(s)) - h(s,w_2(s))\Big]ds\\
&\leq \frac{c}{3}\Big[\psi\Big(\frac{u_1(s)-u_2(s)}{3}\Big) + \psi\Big(\frac{v_2(s)-v_1(s)}{3}\Big) + \psi\Big(\frac{w_1(s)-w_2(s)}{3}\Big)\Big]\int_0^T \lambda(t,s)ds.
\end{aligned}$$

As ψ is non-decreasing function, we have

$$\begin{aligned}
\psi\Big(\frac{u_1(s)-u_2(s)}{3}\Big) &\leq \psi\Big(\frac{\sup_{s\in[0,T]}|u_1(s)-u_2(s)|}{3}\Big)\\
&= \psi\Big(\frac{d(u_1,u_2)}{3}\Big).
\end{aligned}$$

Similarly,

$$\psi\Big(\frac{v_2(s)-v_1(s)}{3}\Big) \leq \psi\Big(\frac{d(v_1,v_2)}{3}\Big)$$

and

$$\psi\Big(\frac{w_1(s)-w_2(s)}{3}\Big) \leq \psi\Big(\frac{d(w_1,w_2)}{3}\Big).$$

Hence, from the above inequality, we arrive at

$$\begin{aligned}
&\sup_{t\in[0,T]} |F(u_1,v_1,w_1)(t) - F(u_2,v_2,w_2)(t)|\\
&\qquad \leq \frac{1}{3}\Big[\psi\Big(\frac{d(u_1,u_2)}{3}\Big) + \psi\Big(\frac{d(v_1,v_2)}{3}\Big) + \psi\Big(\frac{d(w_1,w_2)}{3}\Big)\Big] \sup_{t\in[0,T]} c\int_0^T \lambda(t,s)ds.
\end{aligned}$$

Which yields

$$d(F(u_1,v_1,w_1)(t)-F(u_2,v_2,w_2)(t)) \leq \frac{1}{3}\Big[\psi\Big(\frac{d(u_1,u_2)}{3}\Big)+\psi\Big(\frac{d(v_1,v_2)}{3}\Big)+\psi\Big(\frac{d(w_1,w_2)}{3}\Big)\Big]. \tag{3.11}$$

Since, ψ is a non-decreasing function, we get

$$\psi\Big(\frac{d(u_1,u_2)}{3}\Big)+\psi\Big(\frac{d(v_1,v_2)}{3}\Big)+\psi\Big(\frac{d(w_1,w_2)}{3}\Big) \leq 3\psi\Big(\frac{d(u_1,u_2)+d(v_1,v_2)+d(w_1,w_2)}{3}\Big).$$

So by (3.11), we have

$$\begin{aligned} d(F(u_1,v_1,w_1)(t)-F(u_2,v_2,w_2)(t)) &\leq \psi\Big(\frac{d(u_1,u_2)+d(v_1,v_2)+d(w_1,w_2)}{3}\Big) \\ &\leq \psi\Big(M(u_1,v_1,w_1,u_2,v_2,w_2)\Big) \\ &= \frac{\psi\Big(M(u_1,v_1,w_1,u_2,v_2,w_2)\Big)}{M(u_1,v_1,w_1,u_2,v_2,w_2)}.M(u_1,v_1,w_1,u_2,v_2,w_2) \\ &= \beta\Big(M(u_1,v_1,w_1,u_2,v_2,w_2)\Big).M(u_1,v_1,w_1,u_2,v_2,w_2). \end{aligned}$$

This show that the contractive condition in Corollary 2.11 is satisfied.
Let $(\alpha,\eta,\gamma) \in X^3$ be a tripled lower-upper-lower solution of problem (3.10) appearing in condition (vi) of Theorem 3.4 then we have

$$\alpha \leq F(\alpha,\eta,\gamma), \quad \eta \geq F(\eta,\alpha,\eta) \text{ and } \gamma \leq \mathrm{F}(\gamma,\eta,\alpha).$$

Therefore from Corollary 2.11 and 2.12, F has a unique tripled point in X, that is, the system of integral equations has a unique solution.

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