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Fixed points of a Θ-contraction on metric spaces with a graph

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Abstract

The aim of this paper is to introduce a new type of contraction called Θ-G-contraction on a metric space endowed with a graph and establish some new fixed point theorems. Some examples are presented to support the results proved herein. Our results unify, generalize and extend various results related with G-contraction for a directed graph G.

Keywords: Metric space endowed with a graph, Θ-G-contractions, fixed point.
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1. Introduction and preliminaries

Banach's contraction principle [4] is one of the pivotal results of analysis. It establishes that, given a mapping F on a complete metric space (X, d) into itself and a constant $\alpha \in [0, 1)$ such that

$$d(Fx, Fy) \leq \alpha d(x, y), \tag{1.1}$$

holds for all $x, y \in X$. Then F has a unique fixed point in X.

Due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle (see [1–3, 5–11, 13, 14] and references therein). In 2008,

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Jachymski [12] proved some fixed point results in metric spaces endowed with a graph and generalized simultaneously Banach contraction principle from metric and partially ordered metric spaces. Consistent with Jachymski, let (X, d) be a metric space and Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, i.e., $\Delta \subseteq E(G)$. Also assume that the graph G has no parallel edges and thus, one can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [12]) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for each $i = 1, \ldots, N$.

Jachymski [12] gave the following definition of G-contraction:

Definition 1.1. [12] An operator $F : X \to X$ is called a Banach G-contraction or simply G-contraction if

(a) F preserves edges of G; for each $x, y \in X$ with $(x, y) \in E(G)$, we have $(F(x), F(y)) \in E(G)$;

(b) F decreases weights of edges of G ; there exists $\alpha \in [0, 1)$ such that for all $x, y \in X$ with $(x, y) \in E(G)$, we have

$$d(F(x), F(y)) \leq \alpha d(x, y). \tag{1.2}$$

Notice that a graph G is connected if there is a directed path between any two vertices and it is weakly connected if $\widetilde{G}$ is connected, where $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus, we have

$$V\left(G^{-1}\right) = V\left(G\right) \text{ and } E\left(G^{-1}\right) = \{(x, y) \in X \times X : (y, x) \in E\left(G\right)\}.$$

It is more convenient to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric, under this convention; we have that

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

By a subgraph of G we mean a graph H satisfying $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ such that $V(H)$ contains the vertices of all edges of $E(H)$. If $E(G)$ is symmetric, then for $x \in V(G)$, then the subgraph G_x consisting of all edges and vertices that are contained in some path in G beginning at x is called the component of G containing x. In this case, $V(G_x) = [x]_G$, where $[x]_G$ denotes the equivalence class of the relation R defined on $V(G)$ by the rule:

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

Clearly G_x is connected for all $x \in G$. We denote by $\Psi = \{G : G \text{ is a directed graph with } V(G) = X \text{ and } \Delta \subseteq E(G)\}$.

Consistent with [11, 15], the following definitions will be needed in the sequel.

Definition 1.2. [11] A mapping $F : X \to X$ is said to be orbitally continuous if for all $x, y \in X$ and any sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers $F^{k_n}x \to y \Longrightarrow F(F^{k_n}x) \to Fy$ as $n \to \infty$.

Definition 1.3. [11] A mapping $F : X \to X$ is said to be G-continuous if given $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, we have $Fx_n \to Fx$.

Definition 1.4. [11] A mapping $F : X \to X$ is said to be orbitally G-continuous if for all $x, y \in X$ and any sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers $F^{k_n}x \to y$ and $(F^{k_n}x, F^{k_{n+1}}x) \in E(G) \Longrightarrow F(F^{k_n}x) \to Fy$ as $n \to \infty$.

Definition 1.5. [15] Let (X, d) be a metric space and $F : X \to X$ be a self mapping. Then F is said to be a Picard operator if F has a unique fixed point x_* and $\lim_{n\to\infty} F^n x = x_*$ as $n \to \infty$.

Definition 1.6. [11] Let (X, d) be a metric space and $F : X \to X$ be a self mapping. Then F is said to be a weakly Picard operator if for any $x \in X$, $\lim_{n\to\infty} F^n x$ exists and is a fixed point of F.

Very recently, Jleli and Samet [13] introduced a new type of contraction called Θ-contraction and obtained new fixed point theorems for such contraction in the setting of generalized metric spaces.

Definition 1.7. Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying:

(Θ_1) Θ is nondecreasing;

(Θ_2) for each sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1$ if and only if $\lim_{n\to\infty}(\alpha_n) = 0$;

(Θ_3) there exists $0 < k < 1$ and $l \in (0, \infty]$ such that $\lim_{a\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^k} = l$.

A mapping $F : X \to X$ is said to be Θ-contraction if there exist the function Θ satisfying (Θ_1)-(Θ_3) and a constant $\alpha \in (0, 1)$ such that for all $x, y \in X$,

$$d(Fx, Fy) \neq 0 \Longrightarrow \Theta(d(Fx, Fy)) \leq \Theta(d(x, y))]^\alpha. \tag{1.3}$$

Theorem 1.8 ([13])**.** *Let (X, d) be a complete metric space and $F : X \to X$ be a Θ-contraction, then F has a unique fixed point.*

To be consistent with Samet et al. [13], we denote by the Ω set of all functions $\Theta : (0, \infty) \to (1, \infty)$ satisfying the above conditions. In this paper, we define Θ-G-contraction and generalize the concepts of Θ-contraction and G-contraction. We prove some fixed point theorems which extend some results of Jachymski [11], Samet et al. [13] and thereby many more results by different authors. Throughout the article $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ will denote the set of natural numbers, real numbers and positive real numbers, respectively.

2. Main Result

Motivated by the work of Samet et al. [13], we give the following definition of Θ-G-contraction.

Definition 2.1. A self mapping $F : X \to X$ is said to be a Θ-G-contraction if there exist $\Theta \in \Omega$ and $G \in \Psi$, such that

(i) for all $x, y \in X$,
$$(x, y) \in E(G) \Longrightarrow (Fx, Fy) \in E(G). \tag{2.1}$$

(ii) there exists some $\alpha \in (0, 1)$ such that
$$\Theta(d(Fx, Fy)) \leq [\Theta(d(x, y))]^\alpha, \tag{2.2}$$
for all $x, y \in X$ with $(x, y) \in E(G)$ and $Fx \neq Fy$.

Remark 2.2. It follows from condition (2.1) that $(F(V(G)), (F \times F)(E(G)))$ is a subgraph of G where $(F \times F)(x, y) = (Fx, Fy)$ for all $x, y \in X$.

Example 2.3. Any constant self mapping $F : X \to X$ is a Θ-G-contraction for every $\Theta \in \Omega$ and $G \in \Psi$ because $E(G)$ contains all loops.

Example 2.4. Let $\Theta \in \Omega$ be an arbitrary. Then every Θ-contraction is a Θ-G_0-contraction for the complete graph G_0 given by $V(G_0) = X$ and $E(G_0) = X \times X$.

Example 2.5. Let $G \in \Psi$ be an arbitrary. Then every G-contraction is a Θ-G-contraction for Θ given by $\Theta(t) = e^{\sqrt{t}}$ for all $t > 0$.

Example 2.6. Let $\preceq$ be a partial order in X. Define the graph G_1 by $E(G_1) = \{(x, y) \in X \times X : x \preceq y\}$. Then $G \in \Psi$ and for any $\Theta \in \Omega$, a self mapping $F : X \to X$ is a Θ-G_1-contraction if it satisfies:

(i) F is nondecreasing w.r.t. $\preceq$;

(ii) there exists some $\alpha \in (0,1)$ such that

$$\Theta(d(Fx, Fy)) \leq [\Theta(d(x,y))]^{\alpha},$$

for all $x, y \in X$ with $x \preceq y$ and $Fx \neq Fy$.

Remark 2.7. Conditions (2.1) and (2.2) are independent. And this fact is shown by the examples [17,18].

Remark 2.8. Let G_d be the graph given by $V(G_d) = X$ and $E(G_d) = \Delta$. Then conditions (2.1) and (2.2) are satisfied for every mapping $F : X \to X$. Thus every $F : X \to X$ is a G_d-contraction. Consequently, there is no self mapping on X which is not a G-contraction for all $G \in \Psi$. But, for a fixed $G \in \Psi$ it is possible to find $\Theta \in \Omega$ and a mapping $F : X \to X$ such that F is a Θ-G-contraction but not a G-contraction.

Example 2.9. Consider the sequence

$$\begin{aligned} \tau_1 &= 1 \times 2, \\ \tau_2 &= 1 \times 2 + 3 \times 4, \\ &\vdots \\ \tau_n &= 1 \times 2 + 3 \times 4 + \ldots + (2n-1)(2n) = \frac{n(n+1)(4n-1)}{3}. \end{aligned}$$

Let $X = \{\tau_n : n \in \mathbb{N}\}$ and $d\left(\tau^*, \tau'\right) = \left|\tau^* - \tau'\right|$. Then (X, d) is a complete metric space. Define the mapping $F : X \to X$ by,

$$F(\tau_1) = \tau_1, \quad F(\tau_n) = \tau_{n-1}, \quad \text{for all } n \geq 2.$$

Let us consider the mapping $\Theta : (0, \infty) \to (1, \infty)$ defined by

$$\Theta(t) = e^{\sqrt{te^t}}.$$

Let G be a graph given by $V(G) = X$ and $E(G) = \{(\tau_n, \tau_n) : n \in \mathbb{N}\} \cup \{(\tau_1, \tau_n) : n \in \mathbb{N}\}$. It is easy to see that F preserves edges. We show that F does not satisfy condition (1.2). Clearly $(x, y) \in E(G)$ with $Fx \neq Fy$ if and only if $x = \tau_1$ and $y = \tau_n$ for some $n > 2$. Thus, for $n > 2$, we have

$$\lim_{n\to\infty} \frac{d(F(\tau_n), F(\tau_1))}{d(\tau_n, \tau_1)} = \lim_{n\to\infty} \frac{\tau_{n-1} - 1}{\tau_n - 1} = \lim_{n\to\infty} \frac{4n^3 - 9n^2 + 5n - 6}{4n^3 + 3n^2 - n - 6} = 1.$$

Therefore F does not satisfy condition (1.2). But it satisfies condition (2.2) that is

$$e^{\sqrt{d(F(\tau_1),F(\tau_n))e^{d(F(\tau_1),F(\tau_n))}}} \leq e^{k\sqrt{d(\tau_1,\tau_n)e^{d(\tau_1,\tau_n)}}},$$

for some $\alpha \in (0,1)$. The above condition is equivalent to

$$d(F(\tau_1), F(\tau_n))e^{d(F(\tau_1),F(\tau_n))} \leq \alpha^2 d(\tau_1, \tau_n)e^{d(\tau_1,\tau_n)}.$$

So, we have to check that

$$\frac{d(F(\tau_1), F(\tau_n))e^{d(F(\tau_1),F(\tau_n))-d(\tau_1,\tau_n)}}{d(\tau_1, \tau_n)} \leq \alpha^2.$$

for some $\alpha \in (0,1)$. Consider

$$\begin{aligned}&\frac{d(F(\tau_1), F(\tau_n))e^{d(F(\tau_1),F(\tau_n))-d(\tau_1,\tau_n)}}{d(\tau_1,\tau_n)}\\ &=\frac{d(\tau_1,\tau_{n-1})e^{d(\tau_1,\tau_{n-1})-d(\tau_1,\tau_n)}}{d(\tau_1,\tau_n)}\\ &=\frac{4n^3-9n^2+5n-6}{4n^3+3n^2-n-6}e^{-6n(n-1)}\\ &\le e^{-1},\end{aligned}$$

with $\alpha = e^{-\frac{1}{2}}$. Hence F is a Θ-G-contraction which is not a G-contraction.

Example 2.10. Let $X = [0, 1]$ with usual metric. Define the mapping $F : X \to X$ by,

$$F(\tau) = \frac{1}{3} \text{ if } 0 \le \tau < 1 \text{ and } F(\tau) = \frac{1}{6} \quad \text{for} \quad \tau = 1.$$

Clearly F is not a Θ-contraction for any $\Theta \in \Omega$ because it is not a continuous mapping. Let G be a graph given by $V(G) = X$ and $E(G) = \{(\frac{1}{n}, \frac{1}{n+1}) : n \in \mathbb{N}\} \cup \{(\frac{1}{3}, \frac{1}{6}) : n \in \mathbb{N}\}$. Clearly $(\tau^*, \tau') \in E(G)$ with $F\tau^* \neq F\tau'$ if and only if $\tau^* = 1$ and $\tau' = \frac{1}{2}$. Now, we have

$$\Theta\left(d\left(F1, F\frac{1}{2}\right)\right) = \Theta\left(d\left(\frac{1}{6},\frac{1}{3}\right)\right) = \Theta\left(\frac{1}{6}\right) < \left[\Theta\left(\frac{1}{2}\right)\right]^{\alpha} = \left[\Theta\left(d\left(1,\frac{1}{2}\right)\right)\right]^{\alpha},$$

for some $\frac{\ln\Theta(\frac{1}{6})}{\ln\Theta(\frac{1}{2})} < \alpha \in (0, 1)$. Thus condition (2.2) holds for all $\tau^*, \tau' \in X$ with $(\tau^*, \tau') \in E(G)$ and $F\tau^* \neq F\tau'$. It is simple to observe that F preserves edges of G. Thus F is a Θ-G-contraction but not a Θ-contraction for every $\Theta \in \Omega$.

Proposition 2.11. *If a mapping $F : X \to X$ is such that the condition (2.1) (resp. condition (2.2)) holds, then condition (2.1) (resp. condition (2.2)) is also satisfied for G^{-1} and $\tilde{G}$. Thus if F is a Θ-G-contraction then F is both Θ-G^{-1}-contraction and Θ-$\tilde{G}$-contraction.*

Proof. This is an obvious consequence of symmetry of d and condition (1.1). □

Lemma 2.12. *Let $F : X \to X$ be a Θ-G-contraction. For $x \in X$ and $y \in [x]_{\tilde{G}}$, we have $d(F^n x, F^n y) \to 0$ as $n \to \infty$.*

Proof. Let $x \in X$ and $y \in [x]_{\tilde{G}}$. Then there exists a path $(x_j)_{j=0}^N$ in $\tilde{G}$ from x to y. That is, $x_0 = x$, $x_N = y$ and $(x_{j-1}, x_j) \in E(\tilde{G})$ for all $j = 1, 2, \ldots, N$. Proposition 2.11 shows that F is a Θ-$\tilde{G}$-contraction. So, inductively $(F^n x_{j-1}, F^n x_j) \in E(\tilde{G})$ for all $n \in \mathbb{N}$ and $j = 1, 2, \ldots, N$ and there exists some $\alpha \in (0, 1)$ such that

$$\Theta(d(F^n x_{j-1}, F^n x_j)) \le [\Theta(d(x_{j-1}, x_j))]^{\alpha^n}, \tag{2.3}$$

for all $n \in \mathbb{N}$ and $j = 1, 2, \ldots, N$ with $F^n x_{j-1} \neq F^n x_j$. If for some $j = 1, 2, \ldots, N$ and $k \in \mathbb{N}$, $F^k x_{j-1} = F^k x_j$, then $F^n x_{j-1} = F^n x_j$ for all $n \ge k$. Hence $d(F^n x_{j-1}, F^n x_j) \to 0$ as $n \to \infty$ for all $j = 1, 2, \ldots, N$. Consider the case, when $F^n x_{j-1} \neq F^n x_j$ for all $n \in \mathbb{N}$. Then the condition (2.3) is satisfied for all $n \in \mathbb{N}$. Letting $n \to \infty$ in condition (2.3), we have $\lim_{n\to\infty} \Theta(d(F^n x_{j-1}, F^n x_j)) = 1$. By (Θ_2), we get $\lim_{n\to\infty} d(F^n x_{j-1}, F^n x_j) = 0$. Hence, for all $j = 1, 2, \ldots, N$, we get $\lim_{n\to\infty} d(F^n x_{j-1}, F^n x_j) = 0$. By triangular inequality, we have

$$d(F^n x, F^n y) \le \sum_{j=1}^{N} d(F^n x_{j-1}, F^n x_j) \to 0 \text{ as } n \to \infty.$$

□

Theorem 2.13. *Let (X,d) be a complete metric space. Then following statements are equivalent:*

(i) G is weakly connected;

(ii) for any Θ-G-contraction, a mapping $F : X \to X$ and $x, y \in X$, the sequences $\{F^n x\}$ and $\{F^n y\}$ are Cauchy and equivalent;

(iii) for any Θ-G-contraction, a mapping $F : X \to X$, $Card(FixF) \leq 1$.

Proof. (i) $\implies$ (ii) Let G be weakly connected, $F : X \to X$ be a Θ-G-contraction and $x, y \in X$. Then $X = [x]_{\tilde{G}}$. Take $y = Tx \in [x]_{\tilde{G}}$ in Lemma 2.12. We can find a path $(x_j)_{j=0}^N$ in $\tilde{G}$ such that $x_0 = x$, $x_N = Tx$ and $(x_{j-1}, x_j) \in E(\tilde{G})$ for all $j = 1, 2, \ldots, N$. If for some $k \in \mathbb{N}$, $F^{k+1}x = F^k x$, then $\{F^n x\}$ becomes eventually constant and hence Cauchy. So, without loss of generality we assume that $F^{n+1}x = F^n x$ that is, $d(F^n x, F^{n+1}x) > 0$ for all $n \in \mathbb{N}$. By triangular inequality, we have

$$d(F^n x, F^{n+1}x) \leq \sum_{j=1}^{N} d(F^n x_{j-1}, F^n x_j) \leq \sum_{j=1}^{\infty} d(F^n x_{j-1}, F^n x_j). \tag{2.4}$$

We show that $\sum_{j=1}^{\infty} d(F^n x_{j-1}, F^n x_j)$ is convergent for all j. Fix $j \in \{1, 2, ..., N\}$. If $d(F^{n_0} x_{j-1}, F^{n_0} x_j)$ for some n_0, then $d(F^n x_{j-1}, F^n x_j) = 0$ for all $n \geq n_0$. Hence $\sum_{j=1}^{\infty} d(F^n x_{j-1}, F^n x_j)$ becomes a finite sum and thus convergent. So, we assume that $d(F^n x_{j-1}, F^n x_j) > 0$ for all $n \in \mathbb{N}$. Then as in Lemma 2.12, $\Theta(d(F^n x_{j-1}, F^n x_j)) \leq [\Theta(d(x_{j-1}, x_j))]^{\alpha^n}$ for all $n \in \mathbb{N}$. By (Θ_2), we have

$$\lim_{n\to\infty} \Theta(d(F^n x_{j-1}, F^n x_j)) = 1 \Longleftrightarrow \lim_{n\to\infty} d(F^n x_{j-1}, F^n x_j) = 0. \tag{2.5}$$

By (Θ_3), there exists $0 < k_j < 1$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\Theta(d(F^n x_{j-1}, F^n x_j)) - 1}{(d(F^n x_{j-1}, F^n x_j))^{k_j}} = l.$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $m_j \in \mathbb{N}$ such that

$$\left| \frac{\Theta(d(F^n x_{j-1}, F^n x_j)) - 1}{(d(F^n x_{j-1}, F^n x_j))^{k_j}} - l \right| \leq B,$$

for all $n > m_j$. This implies that

$$\frac{\Theta(d(F^n x_{j-1}, F^n x_j)) - 1}{(d(F^n x_{j-1}, F^n x_j))^{k_j}} \geq l - B = \frac{l}{2} = B,$$

for all $n > m_j$. Then

$$(d(F^n x_{j-1}, F^n x_j))^{k_j} \leq An[\Theta(d(F^n x_{j-1}, F^n x_j)) - 1].$$

Then, there exists $m_j \in \mathbb{N}$ such that

$$d(F^n x_{j-1}, F^n x_j))^{k_j} \leq \frac{1}{n^{1/k_j}}, \tag{2.6}$$

for all $n > m_j$. This implies that $\sum_{j=1}^{\infty} d(F^n x_{j-1}, F^n x_j)$ is convergent. By inequality (2.4), it is clear that $\sum_{j=1}^{\infty} d(F^n x, F^n x)$ is also convergent. Now for $m > n > m_j$, we have

$$\begin{aligned} d(F^n x, F^m x) &\leq d(F^n x, F^{n+1}x) + d(F^{n+1}x, F^{n+2}x) + \cdots + d(F^{m-1}x, F^m x) \\ &= \sum_{j=n}^{m-1} d(F^j x, F^{j+1}x) < \sum_{j=n}^{\infty} d(F^j x, F^{j+1}x) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $\{F^n x\}$ is a Cauchy sequence. By Lemma 2.12, we get $d(F^n x, F^n y) \to 0$ as $n \to \infty$. Thus $\{F^n y\}$ is also Cauchy sequence.

(ii) $\Longrightarrow$ (iii) Let a self mapping $F : X \to X$ be a Θ-G-contraction and $x, y \in FixF$. By the condition (ii) that $\{F^n x\}$ and $\{F^n y\}$ are equivalent. This gives $x = y$. Thus $Card(FixF) \leq 1$.

(iii) $\Longrightarrow$ (i) Let $Card(FixF) \leq 1$ and suppose on the contrary that G be not weakly connected. Then $\tilde{G}$ is also disconnected. Let $x_0 \in X$. Then both $[x_0]_{\tilde{G}}$ and $X \backslash [x_0]_{\tilde{G}}$ are nonempty. Choose $y_0 \in X \backslash [x_0]_{\tilde{G}}$. Define

$$F(x) = \begin{cases} x_0 \text{ if } x \in [x_0]_{\tilde{G}}, \\ y_0 \text{ if } x \in X \backslash [x_0]_{\tilde{G}}. \end{cases}$$

Then $FixF = \{x_0, y_0\}$. We now show that F is a Θ-G-contraction. Let $(x, y) \in E(G)$ be arbitrary. Then $[x]_{\tilde{G}} = [y]_{\tilde{G}}$. So $x, y \in [x_0]_{\tilde{G}}$ or $x, y \in X \backslash [x_0]_{\tilde{G}}$. In both cases, we have $Fx = Fy$. This shows that $(Fx, Fy) \in E(G)$ because $\Delta \subseteq E(G)$. So condition (2.1) holds and since there is no $(x, y) \in E(G)$ with $Fx \neq Fy$. Therefore, the inequality (2.2) is vacuously satisfied. Thus F is a Θ-G-contraction having two fixed points. That is a contradiction because the assumption $Card(FixF) \leq 1$ is hold. Hence G must be weakly connected. □

Corollary 2.14. *Let (X, d) be a complete metric space. Then following statements are equivalent:*

(i) G is weakly connected;

(ii) for any Θ-G-contraction, a mapping $F : X \to X$, there exists $z \in X$ such that $F^n x = z$ for all $x \in X$.

Theorem 2.15. *Let (X, d) be a complete metric space and $F : X \to X$ be a Θ-G-contraction such that $Fx_0 \in [x_0]_{\tilde{G}}$ for some $x_0 \in X$. Let $\tilde{G}_{x_0}$ be component of $\tilde{G}$ containing x_0. Then $[x_0]_{\tilde{G}}$ is F-invariant and $F|_{[x_0]_{\tilde{G}}}$ is a Θ-$\tilde{G}_{x_0}$-contraction. Moreover, if $x, y \in [x_0]_{\tilde{G}}$ then the sequences $\{F^n x\}$ and $\{F^n y\}$ are Cauchy and equivalent.*

Proof. Let $x \in [x_0]_{\tilde{G}}$ be an arbitrary point. Then there exists a path $(x_j)_{j=0}^N$ in $\tilde{G}$ from x_0 to x. That is $x_N = x$ and $(x_{j-1}, x_j) \in E(\tilde{G})$ for all $j = 1, 2, \ldots, N$. Proposition 2.11 shows that F is a Θ-$\tilde{G}$-contraction. So, inductively $(Fx_{j-1}, Fx_j) \in E(\tilde{G})$ for all $j = 1, 2, \ldots, N$. Consequently $(Fx_j)_{j=0}^N$ is a path in $\tilde{G}$ from Fx_0 to Fx. Thus $Fx \in [Fx_0]_{\tilde{G}}$. But it is given that $Fx_0 \in [x_0]_{\tilde{G}}$. So $[Fx_0]_{\tilde{G}} = [x_0]_{\tilde{G}}$. Hence $Fx \in [x_0]_{\tilde{G}}$. Thus $[x_0]_{\tilde{G}}$ is F-invariant. Now, let $(x, y) \in E(\tilde{G}_{x_0})$ be an arbitrary. Then, there exists a path $(x_j)_{j=0}^N$ in $\tilde{G}$ from x_0 to y such that $x_N = x$. Repeating the argument from the first part of the proof we infer that $(Fx_j)_{j=0}^N$ is a path in $\tilde{G}$ from Fx_0 to Fy. It is given that $Fx_0 \in [x_0]_{\tilde{G}}$. Therefore, there exists a path $(y_j)_{j=0}^M$ in $\tilde{G}$ from x_0 to Fx_0. It follows that there is a path $(y_0, y_1, \ldots, y_M, Fx_1, Fx_2, \ldots, Fx_N)$ in $\tilde{G}$ from x_0 to Fy. In particular, $(Fx_{N-1}, Fx_N) \in E(\tilde{G}_{x_0})$ that is $(Fx, Fy) \in E(\tilde{G}_{x_0})$. Since $E(\tilde{G}_{x_0}) \subseteq E(\tilde{G})$ and F is a Θ-G-contraction. Therefore the condition (2.2) holds for the graph $\tilde{G}_{x_0}$ as well. Hence $F|_{[x_0]_{\tilde{G}}}$ is a Θ-$\tilde{G}_{x_0}$-contraction. Finally Theorem 2.13 and connectedness of $\tilde{G}_{x_0}$ implies that $\{F^n x\}$ and $\{F^n y\}$ are Cauchy and equivalent for all $x, y \in [x_0]_{\tilde{G}}$. □

Theorem 2.16. *Let (X, d) be a complete metric space and (X, d, G) satisfy the following property. For any sequence $(x_n)_{n \in \mathbb{N}}$ in X with $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$, there exists a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ satisfying $(x_{k_n}, x) \in E(G)$ for all $n \in \mathbb{N}$. Let $F : X \to X$ be a Θ-G-contraction and $X_F = \{x \in X : (x, Tx) \in E(G)\}$. Then*

(a) $Card(FixF) = Card\ \{[x]_{\tilde{G}} : x \in X_F\}$;

(b) $Fix\ F \neq \emptyset$ if and only if $X_F \neq \emptyset$;

(c) F has a unique fixed point if and only if there exists a point $x_0 \in X_F$ such that $X_F \subseteq [x_0]_{\tilde{G}}$;

(d) for any $x \in X_F$, $F|_{[x]_{\tilde{G}}}$ *is a Picard operator;*

(e) if $X_F \neq \emptyset$ *and* G *is weakly connected then* F *is a Picard operator;*

(f) if $X^{'} = \cup\{[x]_{\tilde{G}} : x \in X_F\}$, *then* $F|_{X'}$ *is a weakly Picard operator;*

(g) if $F \subseteq E(G)$, *then* F *is a weakly Picard operator.*

Proof. We start from the proof of (d). Let $x \in X_F = \{x \in X : (x, Fx) \in E(G)\}$ be an arbitrary point. Then $(x, Fx) \in E(G)$. This implies that $Fx \in [x]_{\tilde{G}}$. So by Theorem 2.15, for any $y \in X$ sequences $\{F^n x\}$ and $\{F^n y\}$ are Cauchy and equivalent. Since (X, d) be a complete metric space, so there exists $z \in X$ such that

$$\lim_{n\to\infty} F^n x = z = \lim_{n\to\infty} F^n y.$$

Since $(x, Fx) \in E(G)$. So (2.1) yields that

$$(F^n x, F^{n+1} x) \in E(G), \tag{2.7}$$

for all $n \in \mathbb{N}$. Condition (2.7) implies that there exists a subsequence $\{F^{k_n} x\}_{n\in}$ of $\{F^k x\}_{n\in\mathbb{N}}$ such that $(F^{k_n} x, z) \in E(G)$ for all $n \in \mathbb{N}$. Thus from (2.7), we have $(x, Fx, F^2 x, \dots, F^{k_1}, z)$ a path in G (and hence in $\tilde{G}$) from x to z. So $z \in [x]_{\tilde{G}}$. Now as $(F^{k_n} x, z) \in E(G)$ and $F : X \to X$ is a Θ-G-contraction, so there exists some $k \in (0, 1)$ such that

$$\Theta(d(F(F^{k_n} x), F(z))) \leq [\Theta(d(F^{k_n} x, z))]^{\alpha} < \Theta(d(F^{k_n} x, z)).$$

By (Θ_1), we have

$$d(F(F^{k_n} x), F(z)) < \Theta(d(F^{k_n} x, z)).$$

Letting $n \to \infty$, we get

$$d(z, F(z)) = 0.$$

Thus $z = F(z)$. Hence $F|_{[x]_{\tilde{G}}}$ is a Picard operator.

(e) Further suppose that G is weakly connected and $x \in X_F$, then $X = [x]_{\tilde{G}}$. Thus F is a Picard operator.

(f) Suppose $X^{'} = \cup\{[x]_{\tilde{G}} : x \in X_F\}$, then $F|_{X'}$ is a Picard operator from (d). Hence $F|_{X'}$ is a weakly Picard operator.

(g) Suppose that $F \subseteq E(G)$. Then $X = X_F$ which gives $X^{'} = X$ and hence F is a weakly Picard operator by (d).

(a) Consider the mapping $\tau : FixF \to \Omega$ by

$$\tau(x) = [x]_{\tilde{G}}$$

for all $x \in FixF$, where $\Omega = \{[x]_{\tilde{G}} : x \in Fx\}$. To prove that $Card(FixF) = Card\ \{[x]_{\tilde{G}} : x \in X_F\}$, it suffices to show that τ is a bijection mapping. Let $x \in Fx$ be an arbitrary point. By (d), we have $F|_{[x]_{\tilde{G}}}$ is a Picard operator. Let $z = \lim_{n\to\infty} F^n x$. Then $z \in FixF \cap [x]_{\tilde{G}}$ and $\tau z = [z]_{\tilde{G}} = [x]_{\tilde{G}}$. So τ is surjective. Now, let $x_1, x_2 \in FixF$ be an arbitrary with $[x_1]_{\tilde{G}} = [x_2]_{\tilde{G}}$. Then $x_2 \in [x_1]_{\tilde{G}}$. By (d), we have

$$\lim_{n\to\infty} F^n x_2 \in FixF \cap [x_1]_{\tilde{G}} = \{x_1\}.$$

But $F^n x_2 = x_2$ for all $n \in \mathbb{N}$. Thus, we get $x_1 = x_2$. Thus τ is surjective. Hence τ is a bijection mapping.

(b) Suppose $FixF \neq \emptyset$. Then $Card(FixF) \neq 0$. From (a), we have $Card\ \{[x]_{\tilde{G}} : x \in X_F\} \neq 0$. This implies that $X_F \neq \emptyset$. Conversely, suppose that $X_F \neq \emptyset$. Then $Card\ \{[x]_{\tilde{G}} : x \in X_F\} \neq 0$. From (a), we have $Card(FixF) \neq 0$. This implies that $FixF \neq \emptyset$.

(c) Suppose F has a unique fixed point. As $Card(FixF) = Card\ \{[x]_{\tilde{G}} : x \in X_F\}$, so there exists a point $x_0 \in X_F$ such that $X_F \subseteq [x_0]_{\tilde{G}}$. And similarly the converse can be obtained from (a). □

Corollary 2.17. *Let (X, d) be a complete metric space and (X, d, G) satisfy the following property. For any sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$, there exists a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ satisfying $(x_{k_n}, x) \in E(G)$ for all $n \in \mathbb{N}$. Then the following statements are equivalent.*

(a) G is weakly connected;

(b) every Θ-G-contraction, a mapping $F : X \to X$ such that $(Fx_0, x_0) \in X$ for some $x_0 \in E(G)$, is a Picard operator;

(c) for any Θ-G-contraction, a mapping $F : X \to X$, $Card(FixF) \leq 1$.

Proof. (a) $\Longrightarrow$ (b): Suppose that G is weakly connected and $F : X \to X$ be a Θ-G-contraction such that $(Fx_0, x_0) \in E(G)$ for some $x_0 \in X$. That is $x_0 \in X_F$. So $X_F \neq \emptyset$. Thus by (e) of Theorem 2.16, F is a Picard operator.

(b) $\Longrightarrow$ (c): Let $F : X \to X$ be a Θ-G-contraction. If $X_F = \emptyset$, then $FixF = \emptyset$ as $FixF \subseteq X_F$. But, if $X_F \neq \emptyset$, then by (b), $FixF$ is singleton. In both cases, $Card(FixF) \leq 1$.

(c) $\Longrightarrow$ (a): It is a direct consequence of Theorem 2.13. □

Theorem 2.18. *Let (X, d) be a complete metric space and $F : X \to X$ be an orbitally G-continuous Θ-G-contraction. Let $X_F = \{x \in X : (x, Tx) \in E(G)\}$. Then the following statements hold:*

(a) $FixF \neq \emptyset$ if and only if $X_F \neq \emptyset$;

(b) for any $x \in X_F$ and $y \in [x]_{\tilde{G}}$, the sequence $\{F^n y\}_{n\in\mathbb{N}}$ converges to a fixed point of F and $\lim_{n\to\infty} F^n y$;

(c) if $X_F \neq \emptyset$ and G is weakly connected, then F is a Picard operator does not depend on y;

(d) if $F \subseteq E(G)$, then F is a weakly Picard operator.

Proof. (a) The proof is same as of (b) Theorem 2.16.

(b) Let $x \in X_F$ be an arbitrary and suppose $y \in [x]_{\tilde{G}}$. Then $(x, Fx) \in E(G)$. It follows by Theorem 2.16 that both sequences $\{F^n x\}_{n\in\mathbb{N}}$ and $\{F^n y\}_{n\in\mathbb{N}}$ converges to one point x^{Λ}. Moreover $(F^n x, F^{n+1}x) \in E(G)$ for all $n \in \mathbb{N}$. Since $F : X \to X$ be an orbitally G-continuous. So, we get

$$\lim_{n\to\infty} F^{n+1}x = \lim_{n\to\infty} F(F^n x) = Fx^{\Lambda}.$$

This implies that $x^{\Lambda} = Fx^{\Lambda}$. Thus (b) proved.

(c) Suppose $X_F \neq \emptyset$ and G is weakly connected. Let $x_0 \in X_F$, then $[x_0]_{\tilde{G}} = X$. Then (b) yield that F has fixed point which is unique. Thus F is a Picard operator.

(d) Suppose $F \subseteq E(G)$. This means that $X_F = X$. Hence F has a fixed point. Thus F is a weakly Picard operator. □

Now, we take $F : X \to X$ as orbitally continuous instead of orbitally G-continuous. This will make the previous theorem strengthen.

Theorem 2.19. *Let (X, d) be a complete metric space and $F : X \to X$ be an orbitally continuous Θ-G-contraction. Let $X_F = \{x \in X : (x, Tx) \in E(G)\}$. Then, the following statements hold:*

(a) $FixF \neq \emptyset$ if and only if there exists $x_0 \in X$ with $Fx_0 \in [x_0]_{\tilde{G}}$;

(b) if $x \in X$ and $Fx \in [x]_{\tilde{G}}$, then for $y \in [x]_{\tilde{G}}$, the sequence $\{F^n y\}_{n\in\mathbb{N}}$ converges to a fixed point of F and $\lim_{n\to\infty} F^n y$ does not depend on y;

(c) if G is weakly connected, then F is a Picard operator;

(d) if $Fx \in [x]_{\tilde{G}}$ for any $x \in X$, then F is a weakly Picard operator.

Proof. (a) It is obvious.

(b) Let $x \in X$ be such that $Tx \in [x]_{\tilde{G}}$ and let $y \in [x]_{\tilde{G}}$. It follows by Theorem 2.16 that both sequences $\{F^n x\}_{n\in\mathbb{N}}$ and $\{F^n y\}_{n\in\mathbb{N}}$ converges to one point x^{Λ}. Since $F : X \to X$ be an orbitally continuous so, we get

$$\lim_{n\to\infty} F^{n+1}x = \lim_{n\to\infty} F(F^n x) = Fx^{\Lambda}.$$

This implies that $x^{\Lambda} = Fx^{\Lambda}$. Thus (b) proved.

(c) Suppose G is weakly connected. Then $[x_0]_{\tilde{G}} = X$. Thus (b) yield that F has fixed point which is unique. Thus F is a Picard operator.

(d) Suppose $Fx \in [x]_{\tilde{G}}$ for any $x \in X$. Hence F has a fixed point. Thus F is a weakly Picard operator. □

Corollary 2.20. *Let (X, d) be a complete metric space. Then the following statements are equivalent.*

(a) G is weakly connected;

(b) every orbitally continuous Θ-G-contraction, a mapping $F : X \to X$ is a Picard operator;

(c) for any orbitally continuous Θ-G-contraction, a mapping $F : X \to X$, $Card(FixF) \leq 1$.

Hence if $\tilde{G}$ is disconnected. Then, there exists at least one orbitally continuous Θ-G-contraction, a mapping $F : X \to X$ which has at least two fixed points.

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