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Sliding window rough measurable function on Riesz Triple Almost $\left(\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}\right)$ Lacunary $\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}$ sequence spaces defined by a Orlicz function

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Abstract

In this paper we introduce a new concept for generalized sliding window rough measurable function on almost $\left(\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}\right)$ convergence in $\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}$-Riesz spaces strong P-convergent to zero with respect to an Orlicz function and examine some properties of the resulting sequence spaces. We also introduce and study sliding window rough statistical convergence of generalized sliding window rough measurable function on almost $\left(\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}\right)$ convergence in $\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}$-Riesz space and also some inclusion theorems are discussed.

Keywords: Analytic sequence, Orlicz function, double sequences, χ-sequence, Riesz space, Riesz convergence, Pringsheim convergence.
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1. Introduction

The idea of rough convergence was introduced by Phu [11], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. [10] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}\,(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types

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of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [12, 13], Esi et al. [2–4], Datta et al. [5], Subramanian et al. [14], Debnath et al. [6] and many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3. A triple sequence $x = (x_{mnk})$ is called triple gai sequence if

$$((m+n+k)! |x_{mnk}|)^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

The space of all triple gai sequences are usually denoted by χ^3.

In this paper we denote (γ, η) as a sliding window pair provided:

(i) γ and η are both nondecreasing nonnegative real valued measurable functions defined on $[0, \infty)$,

(ii) $\gamma(\alpha) < \eta(\alpha)$ for every positive real number α, and $\eta(\alpha) \to \infty$ as $\alpha \to \infty$,

(iii) $\lim\inf_{abc} (\eta(\alpha) - \gamma(\alpha)) > 0$.

(iv) $(0, \infty] = \bigcup \{(\gamma(s) - \eta(s)] : s \leq \alpha\}$ for all $\alpha > 0$.

Suppose $I_{abc} = (\gamma(\alpha), \eta(\alpha)]$ and $\eta(\alpha) - \gamma(\alpha) = \mu(I_{abc})$, where $\mu(A)$ denotes the Lebesgue measure of the set A.

2. Definitions and Preliminaries

A triple sequence $x = (x_{mnk})$ has limit 0 (denoted by $P-limx = 0$) (i.e) $((m+n+k)! |x_{mnk}|)^{1/m+n+k} \to 0$ as $m, n, k \to \infty$. We shall write more briefly as $P - convergent\ \ to\ \ 0$.

Definition 2.1. An Orlicz function ([see[7]) is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called modulus function.

Lindenstrauss and Tzafriri ([8]) used the idea of Orlicz function to construct Orlicz sequence space. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup \{|v| u - (f_{mnk})(u) : u \geq 0\}, m, n, k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, (see[9]) the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{x \in w^3 : I_f(|x_{mnk}|)^{1/m+n+k} \to 0\, as\, m, n, k \to \infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}\left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk}\right),$$

is an exteneded real number.

Let f be a Orlicz function; q be positive real number then we define the following definitions:

Let (γ, η) as a sliding window pair and $g : [0, \infty) \to \mathbb{R}^3$ a measurable function. Then;

Definition 2.2. The function g is $N(\gamma,\eta,f,q)$ summable to $\bar{0}$ and write

$$N(\gamma,\eta,f,q)-limg=\bar{0}\,(or\ g\rightarrow\bar{0}\ N(\gamma,\eta,f,q))$$

if and only if, $lim_{abc\rightarrow\infty}\frac{1}{\mu(I_{abc})}\int_{I_{abc}}f\left(|g(t),\bar{o}|^{q}\right)dt=0$.

Definition 2.3. Let $(q_{rst}),(\overline{q_{rst}}),(\overline{\overline{q_{rst}}})$ be sequences of positive numbers and

$$Q_r=\begin{bmatrix} q_{11} & q_{12} & ... & q_{1s} & 0... \\ q_{21} & q_{22} & ... & q_{2s} & 0... \\ . & & & & \\ . & & & & \\ . & & & & \\ q_{r1} & q_{r2} & ... & q_{rs} & 0... \\ 0 & 0 & ...0 & 0 & 0... \end{bmatrix}=q_{11}+q_{12}+\ldots+q_{rs}\neq 0,$$

$$\overline{Q}_s=\begin{bmatrix} \overline{q}_{11} & \overline{q}_{12} & ... & \overline{q}_{1s} & 0... \\ \overline{q}_{21} & \overline{q}_{22} & ... & \overline{q}_{2s} & 0... \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{q}_{r1} & \overline{q}_{r2} & ... & \overline{q}_{rs} & 0... \\ 0 & 0 & ...0 & 0 & 0... \end{bmatrix}=\overline{q}_{11}+\overline{q}_{12}+\ldots+\overline{q}_{rs}\neq 0,$$

$$\overline{\overline{Q}}_t=\begin{bmatrix} \overline{\overline{q}}_{11} & \overline{\overline{q}}_{12} & ... & \overline{\overline{q}}_{1s} & 0... \\ \overline{\overline{q}}_{21} & \overline{\overline{q}}_{22} & ... & \overline{\overline{q}}_{2s} & 0... \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{\overline{q}}_{r1} & \overline{\overline{q}}_{r2} & ... & \overline{\overline{q}}_{rs} & 0... \\ 0 & 0 & ...0 & 0 & 0... \end{bmatrix}=\overline{q}_{11}+\overline{q}_{12}+\ldots+\overline{q}_{rs}\neq 0.$$

Then the transformation is given by
$T_{rst}=\frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t}q_m\overline{q}_n\overline{\overline{q}}_k\left((m+n+k)!\,|x_{mnk}|\right)^{1/m+n+k}$ is called the Riesz mean of triple sequence $x=(x_{mnk})$. If $P-lim_{rst}T_{rst}(x)=0, 0\in\mathbb{R}$, then the sequence $x=(x_{mnk})$ is said to be Riesz convergent to 0. If $x=(x_{mnk})$ is Riesz convergent to 0, then we write $P_R-limx=0$.

Definition 2.4. Let $\lambda=(\lambda_{m_i}),\mu=(\mu_{n_\ell})$ and $\gamma=(\gamma_{k_j})$ be three non-decreasing sequences of positive real numbers such that each tending to ∞ and

$$\lambda_{m_i+1}\leq\lambda_{m_i}+1,\lambda_1=1,\ \ \mu_{n_\ell+1}\leq\mu_{n_\ell}+1,\mu_1=1\ \ \gamma_{k_j+1}\leq\gamma_{k_j}+1,\gamma_1=1.$$

Let $I_{m_i}=[m_i-\lambda_{m_i}+1,m_i],I_{n_\ell}=[n_\ell-\mu_{n_\ell}+1,n_\ell]$ and $I_{k_j}=\left[k_j-\gamma_{k_j}+1,k_j\right]$. For any set $K\subseteq\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, the number

$$\delta_{\lambda,\mu,\gamma}(K)=lim_{m,n,k\rightarrow\infty}\frac{1}{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}\left|\left\{(i,j):i\in I_{m_i},j\in I_{n_\ell},k\in I_{k_j},(i,\ell,j,)\in K\right\}\right|,$$

is called the $(\lambda,\mu,\gamma)-$ density of the set K provided the limit exists.

Definition 2.5. The function g is $N(\gamma,\eta,f,p)$ summable to 0. A triple sequence $x=(x_{mnk})$ of numbers is said to be $(\lambda,\mu,\gamma)-$ sliding window rough statistical convergent to a number ξ of measurable function provided that for each $\epsilon>0$,

$$lim_{m,n,k\rightarrow\infty}\frac{1}{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}\frac{1}{Q_i\overline{Q}_\ell\overline{\overline{Q}}_j}\mu\left(\left|\left\{(t)\in I_{m_in_\ell k_j}:f\left(q_m\overline{q}_n\overline{\overline{q}}_k\,|x_{mnk}(t)-\xi|^{p}\right)\geq r+\epsilon\right\}\right|\right)=0,$$

(i.e) the set

$$K(\epsilon) = \frac{1}{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}} \frac{1}{Q_i\overline{Q}_\ell\overline{\overline{Q}}_j} \mu\left(\left|\left\{(t) \in I_{m_i n_\ell k_j} : f\left(q_m\overline{q}_n\overline{\overline{q}}_k\left|x_{mnk}(t) - \xi\right|^p\right) \geq r + \epsilon\right\}\right|\right)$$

has (λ, μ, γ) − density zero. In this case the number ξ is called the (λ, μ, γ) − sliding window rough statistical measurable function of limit of the sequence and we write $st^r_{(\lambda,\mu,\gamma)} lim_{m,n,k\to\infty} = \xi$.

Definition 2.6. The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$m_0 = 0, h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty.$$

$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty.$$

$$k_0 = 0, \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i\overline{h_\ell h_j}$, and $\theta_{i,\ell,j}$ is determine by

$$I_{i,\ell,j} = \{(m,n,k) : m_{i-1} < m < m_i,\ n_{\ell-1} < n \leq n_\ell,\ k_{j-1} < k \leq k_j\}, q_k = \frac{m_k}{m_{k-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}.$$

Using the notations of lacunary sequence and Riesz mean for triple sequences. $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ be a triple lacunary sequence and $q_m\overline{q}_n\overline{\overline{q}}_k$ be sequences of positive real numbers such that

$$Q_{m_i} = \sum_{m\in(0,m_i]} p_{m_i}, Q_{n_\ell} = \sum_{n\in(0,n_\ell]} p_{n_\ell}, Q_{n_j} = \sum_{k\in(0,k_j]} p_{k_j}$$

and

$$H_i = \sum_{m\in(0,m_i]} p_{m_i}, \overline{H} = \sum_{n\in(0,n_\ell]} p_{n_\ell}, \overline{\overline{H}} = \sum_{k\in(0,k_j]} p_{k_j}.$$

Clearly, $H_i = Q_{m_i} - Q_{m_{i-1}}, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}}, \overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}}$. If the Riesz transformation of triple sequences is RH-regular, and $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty, \overline{H} = \sum_{n\in(0,n_\ell]} p_{n_\ell} \to \infty$ as $\ell \to \infty, \overline{\overline{H}} = \sum_{k\in(0,k_j]} p_{k_j} \to \infty$ as $j \to \infty$, then $\theta'_{i,\ell,j} = \{(m_i, n_\ell, k_j)\} = \left\{\left(Q_{m_i}Q_{n_j}Q_{k_k}\right)\right\}$ is a triple lacunary sequence. If the assumptions $Q_r \to \infty$ as $r \to \infty$, $\overline{Q}_s \to \infty$ as $s \to \infty$ and $\overline{\overline{Q}}_t \to \infty$ as $t \to \infty$ may be not enough to obtain the conditions $H_i \to \infty$ as $i \to \infty, \overline{H}_\ell \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j \to \infty$ as $j \to \infty$ respectively. For any lacunary sequences $(m_i), (n_\ell)$ and (k_j) are integers.
Throughout the paper, we assume that

$$Q_r = q_{11} + q_{12} + \ldots + q_{rs} \to \infty\,(r \to \infty), \overline{Q}_s = \overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \to \infty\,(s \to \infty),$$

$$\overline{\overline{Q}}_t = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \ldots + \overline{\overline{q}}_{rs} \to \infty\,(t \to \infty),$$

such that

$$H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty\ (i \to \infty), \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}} \to \infty\ (\ell \to \infty),$$

$$\overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}} \to \infty\ (j \to \infty).$$

Let $Q_{m_i,n_\ell,k_j} = Q_{m_i}\overline{Q}_{n_\ell}\overline{\overline{Q}}_{k_j}, H_{i\ell j} = H_i\overline{H}_\ell\overline{\overline{H}}_j$,

$$I'_{i\ell j} = \left\{(m,n,k) : Q_{m_{i-1}} < m < Q_{m_i}, \overline{Q}_{n_{\ell-1}} < n < Q_{n_\ell},\ \overline{Q}_{k_{j-1}} < k < \overline{Q}_{k_j}\right\},$$

$$V_i = \frac{Q_{m_i}}{Q_{m_{i-1}}}, \overline{V}_\ell = \frac{Q_{n_\ell}}{Q_{n_{\ell-1}}}, \ \overline{\overline{V}}_j = \frac{Q_{k_j}}{Q_{k_{j-1}}}, \ V_{i\ell j} = V_i \overline{V}_\ell \overline{\overline{V}}_j.$$

If we take $q_m = 1, \overline{q}_n = 1 \; and \; \overline{\overline{q}}_k = 1$ for all m, n and k then $H_{i\ell j}, Q_{i\ell j}, V_{i\ell j}$ and $I'_{i\ell j}$ reduce to $h_{i\ell j}, q_{i\ell j}, v_{i\ell j}$ and $I_{i\ell j}$.
Let f be a Musielak Orlicz function; p be any factorable triple sequence of strictly positive real number then we define the following definitions:
Let (γ, η) as a sliding window pair and $g : [0, \infty) \to \mathbb{R}^3$ a measurable function, we define the following sequence spaces:

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right] = \mu\Big(P - lim_{i,\ell,j\to\infty} \frac{1}{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}} \frac{1}{H_{i,\ell j}} \sum_{i\in I_{i\ell j}} \sum_{\ell \in I_{i\ell j}} \sum_{j\in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k$$
$$\left[f\left((m+n+k)! \left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{p_{mnk}}\right]\Big) = 0,$$

uniformly in i, ℓ and j.

$$\left[\Lambda^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right] = \mu\Big(P - sup_{i,\ell,j} \frac{1}{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}} \frac{1}{H_{i,\ell j}} \sum_{i\in I_{i\ell j}} \sum_{\ell \in I_{i\ell j}} \sum_{j\in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k$$
$$\left[f \left|x_{m+i,n+\ell,k+j}(t)\right|^{p_{mnk}}\right]\Big) < \infty\Big),$$

uniformly in i, ℓ and j.
Consider $Q_r = q_{11} + \cdots q_{rs}$, $\overline{Q}_s = \overline{q}_{11} \cdots \overline{q}_{rs}$ and $\overline{\overline{Q}}_t = \overline{\overline{q}}_{11} \cdots \overline{\overline{q}}_{rs}$. If we choose $q_m = 1, \overline{q}_n = 1$ and $\overline{\overline{q}}_k = 1$ for all m, n and k, then we obtain the following sequence spaces.

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, q, f, p\right] = \mu\Big(P - lim_{i,\ell,j\to\infty} \frac{1}{\lambda_i\mu_\ell\gamma_j} \frac{1}{Q_i\overline{Q}_\ell\overline{\overline{Q}}_j} \sum_{m=1}^{i} \sum_{n=1}^{\ell} \sum_{k=1}^{j} q_m \overline{q}_n \overline{\overline{q}}_k$$
$$\left[f\left((m+n+k)! \left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{p_{mnk}}\right]\Big) = 0,$$

uniformly in i, ℓ and j.

$$\left[\Lambda^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, q, f, p\right] = \mu\Big(P - sup_{i,\ell,j} \frac{1}{\lambda_i\mu_\ell\gamma_j} \frac{1}{Q_i\overline{Q}_\ell\overline{\overline{Q}}_j} \sum_{m=1}^{i} \sum_{n=1}^{\ell} \sum_{k=1}^{j} q_m \overline{q}_n \overline{\overline{q}}_k$$
$$\left[f\left((m+n+k)! \left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{p_{mnk}}\right]\Big) < \infty,$$

uniformly in i, ℓ and j.

3. Main Result

Theorem 3.1. *If f be any Orlicz function and a bounded factorable positive sliding window rough measurable function of triple sequence, then* $\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right]$ *is linear space.*

Proof. The proof is easy. Theorefore, we omit the proof. □

Theorem 3.2. *For any Orlicz function f and a bounded factorable positive sliding window rough measurable function of triple sequence we have*

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right] \subset \left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, p\right].$$

Proof. Let $x \in \left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, p\right]$ so that for each i, ℓ and j

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right] = \mu\Big(P - lim_{i,\ell,j\to\infty} \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{H_{i,\ell j}} \sum_{i\in I_{i\ell j}}\sum_{\ell\in I_{i\ell j}}\sum_{j\in I_{i\ell j}} q_m\overline{q}_n\overline{\overline{q}}_k \left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{p_{mnk}}\right]\Big) = 0,$$

uniformly in i, ℓ and j. Since f is continuous at zero, for $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for every t with $0 \le t \le \delta$. We obtain the following:

$$\frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{h_{i\ell j}}(h_{i\ell j}\epsilon) + \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{h_{i\ell j}} \sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}, |x_{m+i,n+\ell,k+j}-0|>\delta} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}\right]$$
$$\frac{1}{h_{i\ell j}}(h_{i\ell j}\epsilon) + \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{h_{i\ell j}} K\delta^{-1} f(2)\, h_{i\ell j}\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, p\right].$$

Hence i, ℓ and j goes to infinity, we are granted $x \in \left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right]$. □

Theorem 3.3. *Let $\theta_{i,\ell,j} = \{m_i, n_\ell, k_j\}$ be a sliding window rough measurable function of triple lacunary sequence and $q_i, \overline{q}_\ell\overline{\overline{q}}_j$ with $\liminf_i V_i > 1$, $\liminf_\ell \overline{V_\ell} > 1$ and $\liminf_j V_j > 1$, then for any Orlicz function f,*

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, f, q, p\right] \subseteq \left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, p\right].$$

Proof. Suppose $\liminf_i V_i > 1, \liminf_\ell \overline{V_\ell} > 1$ and $\liminf_j \overline{\overline{V}}_j > 1$, then there exists $\delta > 0$ such that $V_i > 1+\delta$, $\overline{V_\ell} > 1+\delta$ and $\overline{\overline{V}}_j > 1+\delta$. This implies $\frac{H_i}{Q_{m_i}} \ge \frac{\delta}{1+\delta}$, $\frac{\overline{H_\ell}}{\overline{Q}_{n_\ell}} \ge \frac{\delta}{1+\delta}$ and $\frac{\overline{\overline{H}}_j}{\overline{\overline{Q}}_{k_j}} \ge \frac{\delta}{1+\delta}$. Then for $x(t) \in \left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, f, q, p\right]$, we can write for each i, ℓ and j,

$$A_{i,\ell,j} = \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{H_{i\ell j}} \sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$
$$= \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{H_{i\ell j}} \sum_{m=1}^{m_i}\sum_{n=1}^{n_\ell}\sum_{k=1}^{k_j} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$
$$- \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{H_{i\ell j}} \sum_{m=1}^{m_{i-1}}\sum_{n=1}^{n_{\ell-1}}\sum_{k=1}^{k_{i-1}} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$
$$- \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{H_{i\ell j}} \sum_{m=m_{i-1}+1}^{m_i}\sum_{n=1}^{n_{\ell-1}}\sum_{k=1}^{k_{j-1}} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$
$$- \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{H_{i\ell j}} \sum_{k=k_j+1}^{k_j}\sum_{n=n_{\ell-1}+1}^{n_\ell}\sum_{m=1}^{m_{k-1}} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$
$$= \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{Q_{m_i}\overline{Q}_{n_\ell}\overline{Q}_{k_j}}{Hh_{i\ell j}}$$
$$\left(\frac{1}{Q_{m_i}\overline{Q}_{n_\ell}\overline{\overline{Q}}_{k_j}} \sum_{m=1}^{m_i}\sum_{n=1}^{n_\ell}\sum_{k=1}^{k_j} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$$

$$- \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{Q_{m_{k-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}}{H_{i\ell j}}$$
$$\left(\frac{1}{Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{kj-1}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} f \left[((m+n+k)! \left| x_{m+i,n+\ell,k+j}(t) \right|)^{1/m+n+k} \right]^{p_{mnk}} \right)$$
$$- \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{Q_{k_{j-1}}}{H_{i\ell j}} \left(\frac{1}{Q_{k_{j-1}}} \sum_{m=m_{i-1}+1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} f \left[((m+n+k)! \left| x_{m+i,n+\ell,k+j}(t) \right|)^{1/m+n+k} \right]^{p_{mnk}} \right)$$
$$- \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{\overline{Q}_{n_{\ell-1}}}{H_{i\ell j}} \left(\frac{1}{\overline{Q}_{n_{\ell-1}}} \sum_{m=m_{k-1}+1}^{m_k} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} f \left[((m+n+k)! \left| x_{m+i,n+\ell,k+j}(t) \right|)^{1/m+n+k} \right]^{p_{mnk}} \right)$$
$$- \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{\overline{\overline{Q}}_{m_{k-1}}}{H_{i\ell j}} \left(\frac{1}{\overline{\overline{Q}}_{m_{k-1}}} \sum_{k=1}^{k_j} \sum_{n=n_{\ell-1}+1}^{n_\ell} \sum_{m=1}^{m_{k-1}} f \left[((m+n+k)! \left| x_{m+i,n+\ell,k+j}(t) \right|)^{1/m+n+k} \right]^{p_{mnk}} \right).$$

Since $x \in \left[\chi^3_{R_{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}}}, f, q, p \right]$, the last three terms tend to zero uniformly in m, n, k in the sense, thus, for each i, ℓ and j

$$A_{i,\ell,j} = \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}}{H_{i\ell j}}$$
$$\left(\frac{1}{Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)! \left| x_{m+i,n+\ell,k+j}(t) \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right)$$
$$- \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}}{H_{i\ell j}}$$
$$\left(\frac{1}{Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)! \left| x_{m+i,n+\ell,k+j}(t) \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right) + O(1).$$

Since $\frac{1}{\lambda_i \mu_\ell \gamma_j} H_{i\ell j} = \frac{1}{\lambda_i \mu_\ell \gamma_j} Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j} - \frac{1}{\lambda_i \mu_\ell \gamma_j} Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}$ we are granted for each i, ℓ and j the following

$$\frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}}{H_{i\ell j}} \leq \frac{1+\delta}{\delta}, \ \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}}{H_{i\ell j}} \leq \frac{1}{\delta}.$$

The terms

$$\left(\frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{1}{Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)! \left| x_{m+r,n+s,k+u}(t) \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right)$$

and

$$\left(\frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{1}{Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)! \left| x_{m+i,n+\ell,k+j}(t) \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right)$$

are both gai sequences for all r, s and u. Thus $A_{i\ell j}$ is a gai sequence for each i, ℓ and j. Hence $x \in \left[\chi^3_{R_{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}}}, \theta_{i\ell j}, q, p \right]$. □

Theorem 3.4. *Let $\theta_{i,\ell,j} = \{m_i, n_\ell, k_j\}$ be a sliding window rough measurable function of triple lacunary sequence and $q_m\overline{q}_n\overline{\overline{q}}_k$ with $\limsup_i V_i < \infty$, $\limsup_\ell \overline{V}_\ell < \infty$ and $\limsup_j \overline{\overline{V}}_j < \infty$, then for any Orlicz function f,*

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right] \subseteq \left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, q, f, p\right].$$

Proof. Since $\limsup_i V_i < \infty$, $\limsup_\ell \overline{V}_\ell < \infty$ and $\limsup_j \overline{\overline{V}}_j < \infty$, there exists $H > 0$ such that $V_i < H$, $\overline{V_\ell} < H$ and $\overline{\overline{V}}_j < H$ for all i, ℓ and j. Let $x \in \left[\chi^3_{R_{\lambda_i\mu_\ell\gamma_j}}, \theta_{i\ell j}, q, f, p\right]$ and $\epsilon > 0$. Then, there exist $i_0 > 0, \ell_0 > 0$ and $j_0 > 0$, such that for every $a \geq i_0$, $b \geq \ell_0$ and $c \geq j_0$ and for all i, ℓ and j,

$$A'_{abc} = \frac{1}{\lambda_i\mu_\ell\gamma_j H_{abc}} \sum_{m\in I_{a,b,c}} \sum_{n\in I_{a,b,c}} \sum_{k\in I_{a,b,c}} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$

$\to 0$ *as* $m, n, k \to \infty$.
Let $G' = max\left\{A'_{a,b,c} : 1 \leq a \leq i_0,\ 1 \leq b \leq \ell_0 \text{ and } 1 \leq c \leq j_0\right\}$ and p, r and t be such that $m_{i-1} < p \leq m_i$, $n_{\ell-1} < r \leq n_\ell$ and $k_{j-1} < t \leq k_j$. Thus, we obtain the following:

$$\frac{1}{\lambda_i\mu_\ell\gamma_j Q_p\overline{Q}_r\overline{\overline{Q}}_t} \sum_{m=1}^{p}\sum_{n=1}^{r}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$

$$\leq \frac{1}{\lambda_i\mu_\ell\gamma_j Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}} \sum_{m=1}^{m_i}\sum_{n=1}^{n_\ell}\sum_{k=1}^{k_j} \left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$

$$\leq \frac{1}{\lambda_i\mu_\ell\gamma_j Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}} \sum_{a=1}^{i}\sum_{b=1}^{\ell}\sum_{c=1}^{j}$$

$$\left(\sum_{m\in I_{a,b,c}} \sum_{n\in I_{a,b,c}} \sum_{k\in I_{a,b,c}} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$$

$$= \frac{1}{\lambda_i\mu_\ell\gamma_j Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{Q}_{k_{j-1}}} \sum_{a=1}^{i_0}\sum_{b=1}^{\ell_0}\sum_{c=1}^{j_0} H_{a,b,c}A'_{a,b,c}$$

$$+ \frac{1}{\lambda_i\mu_\ell\gamma_j Q_{m_{k-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}} \sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq\ell)\bigcup(j_0<c\leq j)} H_{a,b,c}A'_{a,b,c}$$

$$\leq \frac{G' Q_{m_{i_0}}\overline{Q}_{n_{i_0}}\overline{\overline{Q}}_{k_{i_0}}}{\lambda_i\mu_\ell\gamma_j Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}} + \frac{1}{\lambda_i\mu_\ell\gamma_j Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}} \sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq\ell)\bigcup(j_0<c\leq J)} H_{a,b,c}A'_{a,b,c}$$

$$\leq \frac{G' Q_{m_{i_0}}\overline{Q}_{n_{\ell_0}}\overline{\overline{Q}}_{k_{j_0}}}{\lambda_i\mu_\ell\gamma_j m_{i-1}n_{\ell-1}k_{j-1}} + \frac{1}{\lambda_i\mu_\ell\gamma_j Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{j_{j-1}}} \sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq\ell)\bigcup(j_0<c\leq j)} H_{a,b,c}A'_{a,b,c}$$

$$\leq \frac{G' Q_{m_{i_0}}\overline{Q}_{n_{\ell_0}}\overline{\overline{Q}}_{k_{j_0}}}{\lambda_i\mu_\ell\gamma_j Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}}$$

$$+ \left(sup_{a\geq i_0\bigcup b\geq\ell_0\bigcup c\geq j_0} A'_{a,b,c}\right) \frac{1}{\lambda_i\mu_\ell\gamma_j Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}} \sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq\ell)\bigcup(j_0<c\leq j)} H_{a,b,c}$$

$$\leq \frac{G' Q_{m_{i_0}}\overline{Q}_{n_{\ell_0}}\overline{\overline{Q}}_{k_{j_0}}}{\lambda_i\mu_\ell\gamma_j Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}} + \frac{\epsilon}{\lambda_i\mu_\ell\gamma_j Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}} \sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq\ell)\bigcup(j_0<c\leq j)} H_{a,b,c}$$

$$= \frac{G' Q_{m_{i_0}} \overline{Q}_{n_{\ell_0}} \overline{\overline{Q}}_{k_{j_0}}}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} + V_i \overline{V}_\ell \overline{\overline{V}}_j \epsilon$$

$$\leq \frac{G' Q_{m_{i_0}} \overline{Q}_{n_{\ell_0}} \overline{\overline{Q}}_{k_{j_0}}}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} + \epsilon H^3.$$

Since $Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}} \to \infty$ as $i, \ell, j \to \infty$ approaches infinity, it follows that

$$\frac{1}{\lambda_i \mu_\ell \gamma_j Q_p \overline{Q}_r \overline{\overline{Q}}_t} \sum_{m=1}^{p} \sum_{n=1}^{q} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)! \left| x_{m+i,n+\ell,k+j}(t) \right| \right)^{1/m+n+k} \right]^{p_{mnk}} = 0,$$

uniformly in i, ℓ and j. Hence $x \in \left[\chi^3_{R_{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}}}, q, f, p \right]$. □

Corollary 3.5. *Let $\theta_{i,\ell,j} = \{m_i, n_\ell, k_j\}$ be a sliding window rough measurable function of Orlicz function on triple lacunary sequence and $q_m \overline{q}_n \overline{\overline{q}}_k$ be sequences of positive numbers. If $1 < lim_{i\ell j} V_{i\ell j} \leq lim_{i\ell j} sup V_{i\ell j} < \infty$, then for any Orlicz function f,*

$$\left[\chi^3_{R_{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}}}, \theta_{i\ell j}, q, f, p \right] = \left[\chi^3_{R_{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}}}, q, f, p \right].$$

Definition 3.6. Let $\theta_{i,\ell,j} = \{m_i, n_\ell, k_j\}$ be a sliding window rough measurable function of Orlicz function on triple lacunary sequence. The triple number sequence $x(t)$ is said to be $s^r_{\left[\chi^3_{R_{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}}}, \theta_{i\ell j} \right]} - P$ convergent to 0 provided that for every $\epsilon > 0$,

$$\mu\Big(P - lim_{i\ell J} \frac{1}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}} sup_{i\ell j} \Big| \Big\{ (m,n,k) \in I'_{i\ell j} : q_m \overline{q}_n \overline{\overline{q}}_k \; f\left[\left((m+n+k)! \left| x_{mnk}(t) \right| \right)^{1/m+n+k}, \bar{0} \right] \Big\} \geq r + \epsilon \Big| \Big) = 0.$$

In this case we write $s^r_{\left[\chi^3_{R_{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}}}, \theta_{i\ell j} \right]} - P - lim x = 0$.

Theorem 3.7. *Let $\theta_{i,\ell,j} = \{m_i, n_\ell, k_j\}$ be a sliding window rough measurable function of Orlicz function on triple lacunary sequence. If $I'_{i,\ell,j} \subseteq I_{i,\ell,j}$, then the inclusion $\left[\chi^3_{R_{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}}}, \theta_{i\ell j}, q \right] \subset s^r_{\left[\chi^3_{R_{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}}}, \theta_{i\ell j} \right]}$ is strict and $\left[\chi^3_{R_{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}}}, \theta_{i\ell j}, q \right] - \mu \left(P - lim x \right) = s^r_{\left[\chi^3_{R_{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}}}, \theta_{i\ell j} \right]} - \mu \left(P - lim x \right) = 0$.*

Proof. Let

$$K_{Q_{i\ell j}}(\epsilon) = \left| \left\{ (m,n,k) \in I'_{i\ell j} : q_m \overline{q}_n \overline{\overline{q}}_k f\left[\left((m+n+k)! \left| x_{m+i,n+\ell,k+j}(t) \right| \right)^{1/m+n+k}, \bar{0} \right] \right\} \geq r + \epsilon \right| \tag{3.1}$$

Suppose that $x \in \left[\chi^3_{R_{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}}}, \theta_{i\ell j}, q \right]$. Then for each i, ℓ and j,

$$\mu \left(P - lim_{i\ell j} \frac{1}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}} \sum_{m \in I_{i\ell j}} \sum_{n \in I_{i\ell j}} \sum_{k \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k f \left[\left((m+n+k)! \left| x_{m+i,n+\ell,k+j}(t) \right| \right)^{1/m+n+k}, \bar{0} \right] \right) = 0.$$

Since

$$
\begin{aligned}
&\frac{1}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}} \sum_{m \in I_{i\ell j}} \sum_{n \in I_{i\ell j}} \sum_{k \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}, \bar{0}\right] \\
&\geq \frac{1}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}} \sum_{m \in I_{i\ell j}} \sum_{n \in I_{i\ell j}} \sum_{k \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}, \bar{0}\right] \\
&= \frac{\left|K_{Q_{i\ell j}}(\epsilon)\right|}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}},
\end{aligned}
$$

for all i, ℓ and j, we get $\mu\left(P - lim_{i,\ell,j} \frac{\left|K_{Q_{i\ell j}}(\epsilon)\right|}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}}\right) = 0$ for each i, ℓ and j.

This implies that $x \in s^r_{\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]}$.

To show that this inclusion is strict, let $x = (x_{mnk})$ be defined as

$$
(x_{mnk}(t)) =
$$

$$
\begin{bmatrix}
1 & 2 & 3 & \ldots f\left(\frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(m+n+k)!}\right) & 0 & \ldots \\
1 & 2 & 3 & \ldots f\left(\frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(m+n+k)!}\right) & 0 & \ldots \\
\vdots & & & & & \\
f\left(\frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(m+n+k)!}\right) & 2 & 3 & \ldots f\left(\frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(m+n+k)!}\right) & 0 & \ldots \\
\vdots & & & & & \\
f\left(\frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!}\right) & f\left(\frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!}\right) & f\left(\frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!}\right) & 0 & \ldots & \\
\vdots & & & & & \\
0 & 0 & 0 & \ldots 0 & 0 & \ldots \\
\vdots & & & & &
\end{bmatrix};
$$

and $q_m = 1; \overline{q}_n = 1; \overline{\overline{q}}_k = 1$ for all m, n and k. Clearly, x is unbounded sequence. For $\epsilon > 0$ and for all i, ℓ and j we have

$$
\begin{aligned}
&\left|\left\{(m,n,k) \in I'_{i\ell j} : q_m \overline{q}_n \overline{\overline{q}}_k f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}, \bar{0}\right]\right\} \geq r + \epsilon\right| \\
&= \mu\left(P - lim_{i\ell j}\ f\left(\frac{\lambda_i\mu_\ell\gamma_j (m+n+k)!\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}(m+n+k)!}\right)^{1/m+n+k}\right) \\
&= 0.
\end{aligned}
$$

Therefore, $x \in s^r_{\left[\chi^3_{R_{\lambda_{m_i n_\ell k_j}}}, \theta_{i\ell j}\right]}$ with the $\mu(P - lim) = 0$. Also note that

$$
\begin{aligned}
&\mu\left(P - lim_{i\ell j} \frac{1}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}} \sum_{m \in I_{i\ell j}} \sum_{n \in I_{i\ell j}} \sum_{k \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}, \bar{0}\right]\right) \\
&= \mu\left(P - \frac{1}{2}\left(lim_{i\ell j}\ f\left(\frac{\lambda_i\mu_\ell\gamma_j (m+n+k)!\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}(m+n+k)!}\right)^{1/m+n+k} + 1\right)\right)
\end{aligned}
$$

$$= \frac{1}{2}.$$

Hence $x \notin \left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q\right]$. □

Theorem 3.8. *A sliding window rough triple sequence of Orlicz function of f, let $I'_{i\ell j} \subseteq I_{i\ell j}$. If the following conditions hold:*

(1) $0 < \mu < 1$ *and* $0 \leq f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}, \bar{0}\right] < 1.$

(2) $1 < \mu < \infty$ *and* $1 \leq f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}, \bar{0}\right] < \infty.$

Then

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q\right]_\mu \subset s^r_{\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]}$$

and

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q\right]_\mu - \mu\left(P - limx\right) = s^r_{\left[\chi^3_{R_{\lambda_i\mu_\ell\gamma_j}}, \theta_{i\ell j}\right]} - \mu\left(P - limx\right) = 0.$$

Proof. Let $x = (x_{mnk})$ be strongly $\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{n_j}}}, \theta_{i\ell j}, q\right]_\mu$-almost P-convergent to the limit 0. Since

$$q_m\bar{q}_n\bar{\bar{q}}_k f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}, \bar{0}\right]^\mu$$
$$\geq q_m\bar{q}_n\bar{\bar{q}}_k f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}, \bar{0}\right]$$

for (1) and (2), for all i, ℓ and j, we have

$$\frac{1}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}} \sum_{m\in I_{i\ell j}} \sum_{n\in I_{i\ell j}} \sum_{k\in I_{i\ell j}} q_m\bar{q}_n\bar{\bar{q}}_k f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}, \bar{0}\right]^\mu$$
$$\geq \frac{1}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}} \sum_{m\in I_{i\ell j}} \sum_{n\in I_{i\ell j}} \sum_{k\in I_{i\ell j}} q_m\bar{q}_n\bar{\bar{q}}_k f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}, \bar{0}\right]$$
$$\geq \frac{\epsilon\left|K_{Q_{i\ell j}}(\epsilon)\right|}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}},$$

where $K_{Q_{i\ell j}}(\epsilon)$ is as in (3.1). Taking limit $i, \ell, j \to \infty$ in both sides of the above inequality, we conclude that $s^r_{\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]} - \mu\left(P - limx\right) = 0.$ □

Definition 3.9. A sliding window rough measurable function of Orlicz function on triple sequence of $x = (x_{mnk})$ is said to be Riesz lacunary of χ almost P-convergent 0 if $\mu\left(P - lim_{i,\ell,j} w^{i\ell j}_{mnk}(x(t))\right) = 0$, uniformly in i, ℓ and j, where

$$w^{i\ell j}_{mnk}(x(t)) = w^{i\ell j}_{mnk} = \frac{1}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}} \sum_{m\in I_{i\ell j}} \sum_{n\in I_{i\ell j}} \sum_{k\in I_{i\ell j}} q_m\bar{q}_n\bar{\bar{q}}_k f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}(t)\right|\right)^{1/m+n+k}, \bar{0}\right].$$

Definition 3.10. A sliding window rough measurable function of Orlicz function on triple sequence (x_{mnk}) is said to be Riesz lacunary χ almost statistically summable to 0, if for every $\epsilon > 0$ the set

$K_\epsilon = \left\{(i,\ell,j) \in \mathbb{N}^3 : f\left(\left|w^{i\ell j}_{mnk}, \bar{0}\right|\right) \geq r+\epsilon\right\}$ has triple natural density zero, (i.e) $\delta_3(K_\epsilon) = 0$. In this we write

$$\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]_{st_2} - \mu\left(P - limx\right) = 0.$$

That is, for every $\epsilon > 0$,

$$\mu\left(P - lim_{rst}\frac{1}{rst}\left|\left\{i \leq r, \ell \leq s, j \leq t : f\left(\left|w^{i\ell j}_{mnk}, \bar{0}\right|\right) \geq r+\epsilon\right\}\right|\right) = 0,$$

uniformly in i, ℓ and j.

Theorem 3.11. *A sliding window rough measurable function of Orlicz function on triple sequence of* $I'_{i\ell j} \subseteq I_{i\ell j}$ *and* $q_m\bar{q}_n\bar{\bar{q}}_k\, f\left[\left((m+n+k)!\,|x_{m+i,n+\ell,k+j}(t)|\right)^{1/m+n+k}, \bar{0}\right] \leq M$ *for all* $m,n,k \in \mathbb{N}^3$ *and for each* i,ℓ *and* j. *Let* $x = (x_{mnk})$ *be* $s^r_{\left[\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]} - \mu\left(P - limx\right) = 0.$

Let

$$K_{Q_{i\ell j}}(\epsilon) = \left|\left\{(m,n,k) \in I'_{i\ell j} : q_m\bar{q}_n\bar{\bar{q}}_k f\left[\left((m+n+k)!\,|x_{m+i,n+\ell,k+j}(t)|\right)^{1/m+n+k}, \bar{0}\right]\right\} \geq r+\epsilon\right|.$$

Then

$$\begin{aligned} f\left(\left|w^{i\ell j}_{mnk}, \bar{0}\right|\right) &= \left|\frac{1}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}}\sum_{m\in I_{i\ell j}}\sum_{n\in I_{i\ell j}}\sum_{k\in I_{i\ell j}} q_m\bar{q}_n\bar{\bar{q}}_k f\left[\left((m+n+k)!\,|x_{m+i,n+\ell,k+j}(t)|\right)^{1/m+n+k}, \bar{0}\right]\right| \\ &\leq \left|\frac{1}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}}\sum_{m\in I'_{i\ell j}}\sum_{n\in I'_{i\ell j}}\sum_{k\in I'_{i\ell j}} q_m\bar{q}_n\bar{\bar{q}}_k f\left[\left((m+n+k)!\,|x_{m+i,n+\ell,k+j}(t)|\right)^{1/m+n+k}, \bar{0}\right]\right| \\ &\leq \frac{M\left|K_{Q_{i\ell j}}(\epsilon)\right|}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}} + (r+\epsilon), \end{aligned}$$

for each i,ℓ *and* j, *which implies that* $\mu\left(P - lim_{i,\ell,j}w^{i\ell j}_{mnk}(x(t))\right) = 0$, *uniformly* i,ℓ *and* j. *Hence,* $st^r_2 - \mu\left(P - lim_{i\ell j}w^{i\ell j}_{mnk}\right) = 0$ *uniformly in* i,ℓ,j. *Therefore,* $\left[\chi^3_{R_{\lambda_i\mu_\ell\gamma_j}}, \theta_{i\ell j}\right]_{st^r_2} - \mu\left(P - limx\right) = 0.$

To see that the converse is not true, consider the sliding window rough measurable function of Orlicz function on triple lacunary sequence $\theta_{i\ell j}\left\{\left(2^{i-1}3^{\ell-1}4^{j-1}\right)\right\}, q_m = 1, \bar{q}_n = 1, \bar{\bar{q}}_k = 1$ *for all* m,n *and* k, *and the sliding window rough measurable of Orlicz function on triple sequence* $x = (x_{mnk})$ *defined by* $x_{mnk}(t) = f\left(\frac{(-1)^{m+n+k}}{(m+n+k)!}\right)$ *for all* m,n *and* k.

References

[1] S. Aytar, *Rough statistical Convergence*, Numer. Funct. Anal. Optim., **29** (2008), 291–303. 1

[2] A. Esi, *On some triple almost lacunary sequence spaces defined by Orlicz functions*, Research and Reviews:Discrete Mathematical Structures, **1** (2014), 16–25. 1

[3] A. Esi, M. Necdet Catalbas, *Almost convergence of triple sequences*, Global Journal of Mathematical Analysis, **2** (2014), 6–10.

[4] A. Esi, E. Savas, *On lacunary statistically convergent triple sequences in probabilistic normed space*, Appl. Math. Inf. Sci., **9** (2015), 2529–2534. 1

[5] A. J. Dutta, A. Esi, B. C. Tripathy, B. Chandra,*Statistically convergent triple sequence spaces defined by Orlicz function*, J. Math. Anal., **4** (2013), 16–22. 1

[6] S. Debnath, B. Sarma, B. C. Das, *Some generalized triple sequence spaces of real numbers,* J. Nonlinear Anal. Optim., **6** (2015), 71–79. 1

[7] P. K. Kamthan, M. Gupta, *Sequence spaces and series, Lecture notes, Pure and Applied Mathematics*, 65 Marcel Dekker, Inc., New York, (1981). 2.1

[8] J. Lindenstrauss, L. Tzafriri, *On Orlicz sequence spaces*, Israel J. Math., **10** (1971), 379–390. 2

[9] J. Musielak, *Orlicz spaces and modular spaces.Lectures Notes in Mathematics.,1034*, Springer-Verlag, Berlin, (1983). 2

[10] S. K. Pal, D. Chandra, S. Dutta, *Rough ideal Convergence*, Hacet. J. Math. Stat., **42** (2013), 633–640. 1

[11] H. X. Phu, *Rough convergence in normed linear spaces*, Numer. Funct. Anal. Optim., **22** (2001), 201–224. 1

[12] A. Sahiner, M. Gurdal, F. K. Duden, *Triple sequences and their statistical convergence*, Selcuk J. Appl. Math., **8** (2007), 49–55. 1

[13] A. Sahiner, B. C. Tripathy, *Some I related properties of triple sequences,* Selcuk J. Appl. Math., **9** (2008), 9–18. 1

[14] N. Subramanian, A. Esi, *The generalized tripled difference of* χ^3 *sequence spaces*, Global Journal of Mathematical Analysis, **3** (2015), 54–60. 1