Available online at https://rsgmodelling.co.uk/
Commun. Nonlinear Anal. 2 (2023), 1-6
Research Article

Communications in Nonlinear Analysis
Publisher
Research & Scinece Group Ltd.

New Common Fixed Point Theorem in Dislocated Quasi b-Metric Spaces

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Abstract

In this article, a new common fixed point theorem for a pair of continuous self mapping will be illustrated in the frame work of dislocated quasi b-metric space. The established theorem extend and generalize some well-known results in the literature. Example is given in the support of the constructed result.

Keywords: Complete dislocated quasi-b-metric space, Cauchy sequence, self-mapping, Fixed point.
2010 MSC: 34A12, 47H09, 47H30.

1. Introduction

In 1906, Frechet introduced the notion of metric space, which is one of the cornerstones of not only mathematics but also several quantitative sciences. Due to its importance and application potential, this notion has been extended, improved and generalized in many different ways.

In the field of metric fixed point theory the first significant result was proved by Banach in complete metric space which may be stated as following:
"Every contraction mapping on a complete metric space X has a unique fixed point".

The notion of b-metric space was introduced by Czerwik [6] in connection with some problems concerning with the convergence of non-measurable functions with respect to their measure. Fixed point theorems regarding b-metric spaces was obtained in [7]. Recently, Rahman and Sarwar [10] further generalized the concept of b-metric space and initiated the notion of dislocated quasi b-metric space. Rahman and Sarwar [10] open a new way to the researchers to work in the field of metric fixed point theory. Fixed point theorems regarding dislocated quasi b-metric spaces was obtained in [9].

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Received 2023-04-13

In the present work, we have proved a common fixed point theorem for new type of rational type of contraction conditions in the setting of dislocated quasi b-metric space which improve, extend and generalize similar type of fixed point results in dislocated quasi b-metric space. Example will be studied in order to validate our theorem.

2. Preliminaries

We need the following definitions which may be found in [10].

Definition 2.1. Let X be a non-empty set and $k \geq 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called dislocated quasi b-metric if $\forall\ x, y, z \in X$

(d_1) $d(x, y) = d(y, x) = 0$ implies that $x = y$;

(d_2) $d(x, y) \leq k[d(x, z) + d(z, y)]$.

The pair (X, d) is called dislocated quasi b-metric space or shortly (dq b-metric) space.

Remark 2.2. In the definition of dislocated quasi b-metric space if $k = 1$ then it becomes (usual) dislocated quasi-metric space. Therefore every dislocated quasi metric space is dislocated quasi b-metric space and every b-metric space is dislocated quasi b-metric space with same coefficient k and zero self distance. However, the converse is not true as clear from the following example.

Example 2.3. Let $X = \mathbb{R}$ and suppose

$$d(x, y) = |2x - y|^2 + |2x + y|^2.$$

Then (X, d) is a dislocated quasi b-metric space with the coefficient $k = 2$. But it is not dislocated quasi-metric space nor b-metric space.

Remark 2.4. Like dislocated quasi-metric space in dislocated quasi b-metric space the distance between similar points need not to be zero necessarily as clear from the above example.

Definition 2.5. A sequence $\{x_n\}$ is called dq-$b-$convergent in (X, d) if for $n \in N$ we have $\lim_{n\to\infty} d(x_n, x) = 0$. Then x is called the dq-b-limit of the sequence $\{x_n\}$.

Definition 2.6. A sequence $\{x_n\}$ in dq b-metric space (X, d) is called Cauchy sequence if for $\epsilon > 0$ there exists $n_0 \in N$, such that for $m, n \geq n_0$ we have $d(x_m, x_n) < \epsilon$ (OR) $\lim_{m,n\to\infty} d(x_m, x_n) = 0$.

Definition 2.7. A dq b-metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point of X.

Definition 2.8. Let (X, d_1) and (Y, d_2) be two dq b-metric spaces. A mapping $T : X \to Y$ is said to be continuous if for each $\{x_n\}$ which is dq-b convergent to x_0 in X, the sequence $\{Tx_n\}$ is dq b convergent to Tx_0 in Y.

The following well-known results can be seen in [10].

Lemma 2.9. *Limit of a convergent sequence in dislocated quasi b-metric space is unique.*

Lemma 2.10. *Let (X, d) be a dislocated quasi b-metric space and $\{x_n\}$ be a sequence in dq-b-metric space such that*

$$d(x_n, x_{n+1}) \leq \alpha \cdot d(x_{n-1}, x_n)$$

for $n = 1, 2, 3, \cdots$ and $0 \leq \alpha \cdot k < 1, \alpha \in [0, 1)$ and k is defined in dq b-metric space. Then $\{x_n\}$ is a Cauchy sequence in X.

Theorem 2.11. *Let (X, d) be a complete dislocated quasi b-metric space. Let $T : X \to X$ be a continuous contraction with $\alpha \in [0, 1)$ and $0 \leq k\alpha < 1$ where $k \geq 1$. Then T has a unique fixed point in X.*

Isufati in [2], proved the following result in the setting of dialocated quasi metric space which generalize the result of Dass and Gupta [5] in the metric spaces.

Theorem 2.12. *Let (X, d) be a complete metric space and $T : X \to X$ be a continues self mapping satisfying the following condition*

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) + \beta \cdot \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}$$

$\forall\ x, y \in X$, $\alpha, \beta > 0$, and $\alpha + \beta < 1$. Then T has a unique fixed point in X.

In [3], kohli et al. proved the following result in dislocated quasi metric space.

Theorem 2.13. *Let (X, d) be a complete dq-metric space and $T : X \to X$ be a continues self mapping satisfying the following condition*

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) + \beta \cdot d(y, Ty) + \gamma \cdot \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}$$

$\forall\ x, y \in X$, $\alpha, \beta, \gamma > 0$, and $\alpha + \beta + \gamma < 1$. Then T has a unique fixed point in X.

Recently Rahman and Sarwar [8], proved the following result in the context of dislocated metric space.

Theorem 2.14. *Let (X, d) be a complete dq-metric space and $T : X \to X$ be a continuous self mapping satisfying the condition*

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) + \beta \cdot \frac{d(x, Ty)d(y, Ty)}{d(x, y)d(y, Ty)} + \gamma \cdot \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + \mu \cdot \frac{d(x, Tx)d(x, Ty)}{1 + d(x, y)}$$

$\forall\ x, y \in X$ where $\alpha, \beta, \gamma, \mu > 0$ with $\alpha + \beta + \gamma + 2\mu < 1$. Then T has a unique fixed point in X.

Remark 2.15. it is obvious that the the following statement hold for real numbers a, b and c.
If $a < b$ and $c > 0$ then $ac < bc$.

3. Main Result

Theorem 3.1. *Let (X, d) be a complete dq b-metric space with coefficient $k \geq 1$ and S, T be a continuous self-mapping $S, T : X \to X$ satisfying the condition*

$$d(Sx, Ty) \leq \alpha \cdot d(x, y) + \beta \cdot \frac{d(y, Ty)[1 + d(x, Sx)]}{1 + d(x, y)} + \gamma \cdot \frac{d(x, Ty)d(y, Ty)}{k[d(x, y) + d(y, Ty)]}$$

$$+\mu \cdot \frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + \rho \cdot \frac{d(x, Sx)d(x, Ty)}{1 + d(x, y)}$$

$\forall\ x, y \in X$, where $\alpha, \beta, \gamma, \mu, \rho \geq 0$, with $k\alpha + \beta + \gamma + \mu + k(1 + k)\rho < 1$. Then S and T have a unique common fixed point in X.

Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ for $n = 0, 1, 2, \cdots$ using the rule

$$x_0, x_1 = Sx_0, x_3 = Sx_2, \cdots, x_{2n+1} = Sx_{2n} \ \text{ and } \ x_2 = Tx_1, x_4 = Tx_3, \cdots, x_{2n} = Tx_{2n-1}.$$

Now we have to show that $\{x_n\}$ is a Cauchy sequence in X for this consider

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1}).$$

Now using the given condition in the theorem and definition of the sequence defined above, we have

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha \cdot d(x_{2n}, x_{2n+1}) + \beta \cdot \frac{d(x_{2n+1}, Tx_{2n+1})[1 + d(x_{2n}, Sx_{2n})]}{1 + d(x_{2n}, x_{2n+1})}$$

$$+\gamma \cdot \frac{d(x_{2n}, Tx_{2n+1})d(x_{2n+1}, Tx_{2n+1})}{k[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, Tx_{2n+1})]} + \mu \cdot \frac{d(x_{2n}, Sx_{2n})d(x_{2n+1}, Tx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} + \rho \cdot \frac{d(x_{2n}, Sx_{2n})d(x_{2n}, Tx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})}.$$

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha \cdot d(x_{2n}, x_{2n+1}) + \beta \cdot \frac{d(x_{2n+1}, x_{2n+2})[1 + d(x_{2n}, x_{2n+1})]}{1 + d(x_{2n}, x_{2n+1})}$$

$$+\gamma \cdot \frac{d(x_{2n}, x_{2n+2})d(x_{2n+1}, x_{2n+2})}{k[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]} + \mu \cdot \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})} + \rho \cdot \frac{d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})}.$$

Now using triangular inequality and the Remark 2.15, we have the following

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha \cdot d(x_{2n}, x_{2n+1}) + \beta \cdot d(x_{2n+1}, x_{2n+2})\gamma \cdot d(x_{2n+1}, x_{2n+2})$$
$$+\mu \cdot d(x_{2n+1}, x_{2n+2}) + \rho \cdot k[d(x_{2n}, x_{2n+1} + d(x_{2n+1}, x_{2n+2}].$$

Simplification yields

$$d(x_{2n+1}, x_{2n+2}) \leq \left(\frac{\alpha + k\rho}{1 - \beta + \gamma + \mu + k\rho}\right) d(x_{2n}, x_{2n+1}).$$

Let $h = \frac{\alpha+k\rho}{1-\beta+\gamma+\mu+k\rho}$, so the above equation become

$$d(x_{2n+1}, x_{2n+2}) \leq h \cdot d(x_{2n}, x_{2n+1}).$$

Similarly we can get

$$d(x_{2n}, x_{2n+1}) \leq h \cdot d(x_{2n-1}, x_{2n}).$$

Continuing the same procedure we can get

$$d(x_n, x_{n+1}) \leq h \cdot d(x_{n-1}, x_n).$$

So by using Lemma 2.10, we get that x_n is a Cauchy sequence in complete dislocated quasi b-metric space X. Consequently there exist $u \in X$, such that, $lim_{n \to \infty} x_n = u$. Also the subsequences $\{x_{2n+1}\}$ and $\{x_{2n+2}\}$ will also converges to $u \in X$

Using the continuity of S and T, we have

$$T \cdot \lim_{n \to \infty} x_{2n+1} = Tu \quad \Rightarrow \quad Tu = u.$$

Similarly we can show that $Su = u$. Therefore u is the common fixed point of S and T.
Uniqueness: Let u, v are two distinct common fixed points of S and T. Using given condition in the theorem and triangular inequality, one can easily prove that

$$d(u, u) = d(v, v) = 0. \tag{3.1}$$

Now consider

$$d(u, v) = d(Su, Tv) \leq \alpha \cdot d(u, v) + \beta \cdot \frac{d(v, Tv)[1 + d(u, Su)]}{1 + d(u, v)} + \gamma \cdot \frac{d(u, Tv)d(v, Tv)}{k[d(u, v) + d(v, Tv)]}$$

$$+\mu \cdot \frac{d(u, Su)d(v, Tv)}{1 + d(u, v)} + \rho \cdot \frac{d(u, Su)d(u, Tv)}{1 + d(u, v)}.$$

Now using (3.1) and the fact that u, v are fixed points of S and T, we have

$$d(u,v) = d(Su,Tv) \leq \alpha \cdot d(u,v) + \beta \cdot \frac{d(v,v)[1+d(u,u)]}{1+d(u,v)} + \gamma \cdot \frac{d(u,v)d(v,v)}{k[d(u,v)+d(v,v)]}$$

$$+\mu \cdot \frac{d(u,u)d(v,v)}{1+d(u,v)} + \rho \cdot \frac{d(u,u)d(u,v)}{1+d(u,v)}.$$

$$d(u,v) \leq \alpha d(u,v).$$

Which implies that $d(u,v) = 0$, as $k\alpha < 1$. Similarly we can show that $d(u,v) = 0$. Thus, $d(u,v) = d(v,u) = 0$. Which gives $u = v$. Hence u is the common unique fixed point of S and T. □

By putting $S = T$, in Theorem 3.1, we get the following corollary.

Corollary 3.2. *Let (X,d) be a complete dq b-metric space with coefficient $k \geq 1$ and T be a continuous self-mapping $T : X \to X$ satisfying the condition*

$$d(Tx,Ty) \leq \alpha \cdot d(x,y) + \beta \cdot \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \gamma \cdot \frac{d(x,Ty)d(y,Ty)}{k[d(x,y)+d(y,Ty)]}$$

$$+\mu \cdot \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} + \rho \cdot \frac{d(x,Tx)d(x,Ty)}{1+d(x,y)}$$

$\forall\ x, y \in X$, where $\alpha, \beta, \gamma, \mu, \rho \geq 0$, with $k\alpha + \beta + \gamma + \mu + k(1+k)\rho < 1$. Then T has a unique fixed point in X.

Remark 3.3. If $\beta = 0$, in Corollary 3.2, we get Theorem 2.14.

Remark 3.4. If $\gamma = \mu = \rho = 0$, in Corollary 3.2, we get Theorem 2.12.

Remark 3.5. By putting different constants equal to zero, our establish result will generalize much more results of the literature including Banach contraction theorem in the context of dislocated quasi b-metric space.

Now, we provide a particular example in support of of Theorem 3.1.

Example 3.6. Let $X = [0,1]$, with complete dq b-metric space defined by $d(x,y) = |2x-y|^2 + |2x+y|^2$, for all $x, y \in X$. The continuous self mappings $S, T : X \to X$, are defined by

$$Sx = 0 \quad \text{and} \quad Tx = \frac{x}{3}, \quad \text{for all} \quad x \in X.$$

In particular, if we take $\alpha = \frac{1}{6}, \beta = \frac{1}{4}, \gamma = \frac{1}{8}, \mu = \frac{1}{10}, \rho = \frac{1}{12}$. All the conditions of the Theorem 3.1 are satisfied and $x = 0 \in [0,1]$ is the unique common fixed point of S and T.

4. Conclusion

This work will contribute more generalized, unified and extended result for the future literature in the setting of newly initiated notion of dislocated quasi b-metric space.

Conflict of Interest

The authors have no conflict of interest regarding the publication of this article.

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