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# Some High-Order Convergence Modifications of the Householder Method for Nonlinear Equations

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## Abstract

One major setback of iterative methods that require the evaluation of high derivatives in their iterative procedures is high computational cost. The Householder's method is one of such methods that require second derivative evaluation in its implementation procedures. To circumvent this setback, the second derivative is annihilated by estimation via the use of the interpolating polynomial and the divided difference techniques. Consequently, three new modifications of the Householder's method that are of two and three steps were put forward in this article. To further improve the efficiency of the modified methods, a weight function is introduced to the iterative cycle to enhance the methods convergence order. From the convergence analysis conducted on the methods, revealed that they are of fifth, ninth and tenth order convergence respectively. To test the applicability of the methods, they were applied to locate the solutions of some nonlinear equations and modeled practical problems into nonlinear equations in scalar form. From the computational experience, it was observed that the methods performed better than the compared methods that are also modifications of the Householder methods in literature.

*Keywords:* Nonlinear equation, Householder method, Interpolation polynomial, Convergence order.
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## 1. Introduction

In many field of applied mathematics, real life problems are encountered and are continuously modeled into nonlinear equations (NE) with generic form $f(t) = 0$. In order to understudy these problems, solutions of the nonlinear models are usually required. But in many cases, analytic formulations to solve these problems fails, hence alternative measures are used to obtain the solutions. The alternative measures in most cases

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involves the use of continuous and repetitive computation process that keeps improving the initial guess approximation until a desired approximation or exact solution is achieved. This process is called iterative process. One of the foremost classical iterative process is the Newton method [1] given as:

$$t_{k+1} = t_k - \eta_k, \quad k = 0, 1, 2, \cdots \tag{1.1}$$

where $\eta_k = \frac{f(t_k)}{f'(t_k)}$. When the NM is implemented to solve NE, beginning with an initial guess, it will continuously correct the error of approximations of the exact solution by order two, until the desired accuracy level of the exact solution or exact solution of the NE is attained. This is referred to as method's convergence order (CO). Since each stage of the NM implementation will require two different function evaluation and its CO is two, then it is said to be optimal as conjecture by Traub in [1].

**Conjecture 1.1.** (Traub Conjecture [1]). An iterative method (IM) that is without memory is said to be optimal if it require $q$ number of different function evaluations in an iteration cycle to attain maximum CO of $2^{q-1}$.

The efficiency of the NM is 1.4142 and is obtained by using the efficiency index due to Ostrowski in [2].

**Definition 1.2.** (Efficiency Index [2]). The EI of an IM is a measure of its efficiency determined by the use of the metric $EI \approx \rho^{1/q}$.

The Householder method [3] is one of the early modification of the NM that is of CO three and $EI \approx 1.4422$. Its major pitfall is that it require second derivative of function evaluation during implementation. Consequent upon the above identified pitfall, the Householder method have enjoyed several modifications centered around eliminating the presence of second derivative as in [4, 5, 6, 7] or its enhancement to attain high convergence as put forward in [8, 9, 10].

Although, the variants of the Householder methods introduced in the works [5, 6, 8, 9, 10] are good, they all require the evaluation of first derivative $f'(\cdot)$ at two different iteration points. This will increase the chances of the methods (as in the case with NM), break down when $f'(\cdot) \approx 0$ during implementation. Further, the modifications in [4, 5] were made to attain CO nine at the expense of the introduction of third derivative evaluation of function. But in practice, not all functions have third derivative, therefore, modifications of this sort will suffer setbacks which includes breakdown and high computation cost.

Motivated by the highlighted pitfalls above, three families of modified Householder method that are of CO five, nine and tenth and with high EI were developed in this manuscript. The techniques used in their developments involved the composition, interpolation polynomial approximation and weight function.

The remaining parts of this manuscript is ordered as following: Section 2 presents the procedures for the methods development, and in Section 3, the developed methods convergence analysis was considered and established. In Section 4, the applicability of the developed methods in solving some NE and modeled problems in NE were illustrated. Finally, Section 5 is made up of conclusion remarks.

## 2. Methods Formulation

Consider the famous Householder method (HM) [3] given as:

$$t_{k+1} = t_k - \eta_k - \frac{\eta_k^2}{2}\frac{f''(t_k)}{f'(t_k)}, \quad k = 0, 1, 2. \cdots \tag{2.1}$$

The HM locates the solution of NE with CO three when implemented. However, it requires second order derivative which is considered as computationally expensive. To deal with this setback, we set $t_k = y_k$ and then approximate the second derivative $f''(t_k)$ as:

$$f''(t_k) \approx \frac{f'(y_k) - f'(t_k)}{y_k - t_k}. \tag{2.2}$$

Consequently, the following iterative protocol is proposed.

**Algorithm 2.1.** *For an initial guess $t_0$, obtain the solution $t_{k+1}$ of $f(t)=0$ using the protocol*

$$y_k = t_k - \frac{f(t_k)}{f\left[t_k, \eta\right]}, \quad \eta = t_k + \alpha \left(f(t_k)\right)^m, \quad m \geq 2 \tag{2.3}$$

$$t_{k+1} = y_k - \frac{f(y_k)}{f'\left(y_k\right)} + \frac{1}{2}\left(\frac{f(y_k)}{f'\left(y_k\right)}\right)^2 \left(\frac{f'(y_k) - f\left[t_k, \eta\right]}{\left(f'(y_k)\right)}\right) \frac{f\left[t_k, \eta\right]}{f(t_k)}, \tag{2.4}$$

where the operator $[\cdot, \cdot]$ is divided difference and $\alpha \in \Re - \{0\}$. Algorithm 2.1 is a double-step modified HM (MHM1) and it is of fifth-order convergence. Taking account of the definition of efficiency index (EI) in Definition 1.2, its $EI \approx 1.4953$, which is better than the HM efficiency index of $EI \approx 1.4422$ and another modification of HM due to Noor et al., [6] with $EI \approx 1.4141$. To further modify the HM to achieve better efficiency index, the MHM1 is utilized as a predictor iterative function, to a corrector iterative function, and then suggest a triple -step iterative protocol as following.

**Algorithm 2.2.** *For an initial guess $t_0$, obtain the solution $t_{k+1}$ of $f(t)=0$ using the protocol*

$$y_k = t_k - \frac{f(t_k)}{f\left[t_k, \eta\right]}, \quad \eta = t_k + \alpha \left(f(t_k)\right)^m, \quad m \geq 2 \tag{2.5}$$

$$z_k = y_k - \frac{f(y_k)}{f'\left(y_k\right)} + \frac{1}{2}\left(\frac{f(y_k)}{f'\left(y_k\right)}\right)^2 \left(\frac{f'(y_k) - f\left[t_k, \eta\right]}{\left(f'(y_k)\right)}\right) \frac{f\left[t_k, \eta\right]}{f(t_k)}, \tag{2.6}$$

$$t_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)}. \tag{2.7}$$

The Algorithm 2.2 possess CO ten and its procedure attracts six functions evaluation which implies that Algorithm 2.2 have an efficiency index $EI = 1.4679$. This is lower than the EI of Algorithm 2.1. To enhance its EI, we either reduce its number of function evaluation or increase its CO without using additional function evaluation. Suppose the first alternative is chosen, then we need to annihilate either of the functions $f(z)$ or $f'(z)$.

Now, consider the degree two Interpolating polynomial given as:

$$P(x) = \sum_{j=0}^{2} b_j \left(x - t_k\right)^j, \tag{2.8}$$

such that,

$$P'(x) = b_1 + 2b_2\left(x - t_k\right). \tag{2.9}$$

Now, suppose known values are substituted in (2.5) and (2.6) such that,

$$P(z_k) = f(z_k) = \sum_{j=0}^{2} b_j \left(z_k - t_k\right)^j, \tag{2.10}$$

$$P'(z_k) = f'(z_k) = b_1 + 2b_2\left(z_k - t_k\right), \tag{2.11}$$

where $b_0 = f(y_k)$, $b_1 = f'\left(y_k\right)$ and $b_2 = f''\left(y_k\right)$. Then, from (2.7) we can have

$$f'(z_k) \approx f'(y_k) + \left(z_k - y_k\right) f''\left(y_k\right). \tag{2.12}$$

Again, from (2.8), $f''\left(y_k\right)$ can be obtained as:

$$f''(y_k) \approx \frac{2}{\left(z_k - y_k\right)}\left(f\left[y_k, z_k\right] - f'(y_k)\right). \tag{2.13}$$

By the substitution of (2.9) into (2.8) the following approximation for $f'(z_k)$ is obtained as:

$$f'(z_k) \approx 2\left(f\left[y_k, z_k\right]\right) - f'(y_k). \tag{2.14}$$

If the relation in (2.10) is now used in Algorithm 2.2, a new triple-step iterative protocol is suggested below.

**Algorithm 2.3.** *For an initial guess $t_0$, obtain the solution $t_{k+1}$ of $f(t) = 0$ using the iterative protocol*

$$y_k = t_k - \frac{f(t_k)}{f\left[t_k, \eta\right]}, \quad \eta = t_k + \alpha\left(f(t_k)\right)^m, \quad m \geq 2, \tag{2.15}$$

$$z_k = y_k - \frac{f(y_k)}{f'\left(y_k\right)} + \frac{1}{2}\left(\frac{f(y_k)}{f'\left(y_k\right)}\right)^2\left(\frac{f'(y_k) - f\left[t_k, \eta\right]}{\left(f'(y_k)\right)}\right)\frac{f\left[t_k, \eta\right]}{f(t_k)}, \tag{2.16}$$

$$t_{k+1} = z_k - \frac{f(z_k)}{2\left(f\left[y_k, z_k\right]\right) - f'(y_k)}. \tag{2.17}$$

Algorithm 2.3 is a another triple-step modified HM (MHM2) and achieve CO nine by requiring five function evaluation in one complete iteration protocol. Consequently, Algorithm 2.3 has $EI \approx 1.5518$ which is better than that of Algorithm 2.2 and Algorithm 2.1.
To compensate for the shortfall in the CO of the MHM2 (since it is a modification of Algorithm 2.2 which is of CO ten) and to further improve its EI, a real valued weight function $G(v)$ such that its Taylor series expansion about 0 is

$$G(v) = G(0) + \sum_{i=1}^{10}\frac{1}{i!}G^{(i)}(0)(v)^i, \tag{2.18}$$

where $v = \frac{f(z_k)}{f(t_k)}$ and $G^{(i)}(0)$ is $ith$-derivative of $G(v)$, evaluated at $t = 0$ is introduced in its third step as following:

**Algorithm 2.4.** *For an initial guess $t_0$, obtain the solution $t_{k+1}$ of $f(t) = 0$ using the iterative protocol*

$$y_k = t_k - \frac{f(t_k)}{f\left[t_k, \eta\right]}, \quad \eta = t_k + \alpha\left(f(t_k)\right)^m, \quad m \geq 2, \tag{2.19}$$

$$z_k = y_k - \frac{f(y_k)}{f'\left(y_k\right)} + \frac{1}{2}\left(\frac{f(y_k)}{f'\left(y_k\right)}\right)^2\left(\frac{f'(y_k) - f\left[t_k, \eta\right]}{\left(f'(y_k)\right)}\right)\frac{f\left[t_k, \eta\right]}{f(t_k)}, \tag{2.20}$$

$$t_{k+1} = z_k - \left(\frac{f(z_k)}{2\left(f\left[y_k, z_k\right]\right) - f'(y_k)}\right) \times G\left(v\right). \tag{2.21}$$

By subjecting $G\left(v\right)$ to certain conditions, we shall prove that Algorithm 2.4 can attain minimum of CO ten. The implication is that, its EI would have improved from approximately 1.5518 to 1.5849.

## 3. Convergence Analysis of the Modifications

The analysis of convergence of the methods put forward in Section 2 is considered and established in this section. To do this successfully, we need to obtain an equation of the form $e_{k+1} = \theta e_k^{\rho} + O(e_k^{\rho+1})$ (where $e_k = t_k - t^*$ is iteration errror at $kth$ iteration cycle) from each of the $MHM1$, $MHM2$ and $MHM3$ via the use of the Taylor series expansion of $f(\cdot)$ and $f'(\cdot)$ as contained in the various methods, see [7, 11, 12, 13]. The expression for $e_{k+1}$ is referred to as iterative method's error equation, $\rho$ is CO and $\theta$ is error constant.

**Theorem 3.1.** *Consider the function $f : D \subset \Re \to \Re$ of real values that is sufficiently differentiable and that $f'(\cdot) \neq 0$ in $D$. If $t^* \in D$ is the simple solution of $f$, then given an initial guess $t_0$ close to $t^*$ , the approximations obtained when the Algorithm 2.1 and Algorithm 2.3 are implemented, will form sequence $\{t_k\}_{k\geq 0}, (t_k \in D)$ that converges to $t^*$ with CO five and nine respectively. Further, when $G(0) = 1$ and $G'(0) = \frac{2}{3}$, the CO of Algorithm 2.4 is ten.*

*Proof.* . Let $e_{k+1} = |e_k - t^*|$ be the error in the kth iteration. By replacing $t$=$t_k$ in the Taylor expansion of $f(t)$ and $f'(t)$ about $t^*$, the following expressions were obtained.

$$f(t_k) = f'(t^*)\left[e_k + \sum_{n=2}^{9} c_n e_k^n + O\left(e_k^{10}\right)\right], k = 0, 1, 2, \cdots \tag{3.1}$$

where

$$c_n = \frac{1}{n!}\frac{f^{(n)}(t^*)}{f'(t^*)}, \quad n = 2, 3, 4, \cdots$$

Using (3.1) the following expressions can be obtained.

$$f[t_k, \eta_k] = \frac{f(\eta_k) - f(t_k)}{\eta_{k}-_{k}} \tag{3.2}$$

$$= 1 + 2c_2e_k + 3c_3e_k^2 + (ac_2 + 4c_4)e_k^3 + (3\alpha(c_2^2 + c_3) + 5c_5)e_k^4 + 3(\alpha(c_2^3 + 4c_2c_3 + 2c_4) + 2c_6)e_k^5 \tag{3.3}$$

$$(\alpha^2c_3 + \alpha(c_2^4 + 15c_2^2 + 9c_3^2 + 21c_2c_4 + 10c_5) + 7c_7)e_k^6 + O(e_k^6). \tag{3.4}$$

By combining (3.1) and (3.2), we get

$$y_k = c_2e_k^2 + (2c_3 - 2c_2^2)e_k^3 + (3c_4 - 7c_2c_3 + 4c_2^3 + \alpha c_2)e_k^4 + (4c_5 - 10c_2c_4 - 6c_3^2 + 20c_2^2c_3 + 3\alpha c_3 - 8c_3^4)e_k^5 \tag{3.5}$$

$$(5c_6 - 13c_2c_5 + 2\alpha(3c_4 - c_2c_3 + c_2^3) + 16c_2^5 - 52c_2^3c_3 + 33c_2c_3^2 + 28c_2^2c_4 - 17c_3c_4)e_k^6 + O(e_k^7). \tag{3.6}$$

The Taylor expansion of $f(y_k)$ is

$$f(y_k) = f'(t^*)[c_2e_k^2 + (2c_3 - 2c_2^2)e_k^3 + (3c_4 - 7c_2c_3 + 5c_2^3 + \alpha c_2)e_k^4 \tag{3.7}$$

$$+ (4c_5 - 10c_2c_4 + (2c_2^2(2c_3 - 2c_2^2) - 6c_3^2 + 20c_2^2 + 3\alpha c_3 - 8c_2^4))e_k^5 \tag{3.8}$$

$$+ (16c_5^2 - 51c_2^3c_3 + 33c_2c_3^2 + 28c_2^2c_4 - 17c_3c_4 + 2\alpha(c_2^3 - c_2c_3 + 3c_4) \tag{3.9}$$

$$+ c_2((-2c_2^2 + 2c_3)^2 + 2c_2(\alpha c_2 + 14c_3^2 - 7c_2c_3 + 3c_4)))e_k^6 + O(e_k^7)]. \tag{3.10}$$

Consequent upon (3.4), the expansion of $f'(y_k)$ is obtained as

$$f'(y_k) = f'(t^*)[1 + c_2e_k^2 + (2c_3 - 2c_2^2)e_k^3 + (2c_2(3c_4 - 7c_2c_3 + 7c_2^3 + \alpha c_2) + 3c_2^2c_3)e_k^4 \tag{3.11}$$

$$+ 2c_2(6c_2c_3(2c_3 - 2c_2^2) - 2c_2(4c_5 - 10c_2c_4 - 6c_3^2 + 20c_2^2c_3 + 3\alpha c_3 - 8c_2^4))e_k^5 \tag{3.12}$$

$$+ (4c_2^3c_4 + 3c_3((2c_3 - 2c_2^2)^2 + 2c_2(2\alpha c_2 + 4c_2^3 - 7c_2c_3 + 3c_4)) \tag{3.13}$$

$$+ 2c_2(16c_2^5 - 52c_2)^3 + 33c_3c_3^2 + 28c_2^2c_4 - 17c_3c_4 + 2\alpha(c_2^3 - c_2c_3 + 3c_4) - 13c_2c_5 + 5c_6)e_k^6 + O(e_k^7)]. \tag{3.14}$$

Combining (3.4) and (3.5) the expansion of the quotient of $f(t_k)$ and $f'(t_k)$ is obtained as

$$\eta_k = \frac{f(t_k)}{f'(t_k)} = c_2e_k^2 + (2c_3 - 2c_2^2)e_k^3 + (3c_4 - 7c_2c_3 + 4c_2^3 + \alpha c_2)e_k^4 \tag{3.15}$$

$$+ (3c_4 - 16c_2^2c_3 - 4c_2^4 + 3\alpha c_3)e_k^5 \tag{3.16}$$

$$+ (5c_6 + c_2(29c_3^2 - 2\alpha c_3 - 13c_5) + 6c_2^5 - 32c_2^3 + 6\alpha c_4 + 22c_2^2c_4 - 17c_3c_4)e_k^6 + O(e_k^7). \tag{3.17}$$

From (3.2) and (3.5), we have that

$$\frac{f'(y_k)-f[t_k,\eta]}{f'(y_k)} = -c_2e_k+(2c_2^2-3c_3)e_k^2+(4c_2c_3-4c_4-\alpha c_2)e_k^3 \tag{3.18}$$
$$+(-4c_3^2+3c_2^2c_3-\alpha(c_2^2+3c_3)+6c_2c_4-5c_5)e_k^4 \tag{3.19}$$
$$+(8c_2^5-22c_2^3c_3+12c_2c_3^2+3\alpha(c_2^3-2c_2c_3-2c_4)+8c_2c_5-6c_6)e_k^5 \tag{3.20}$$
$$+(-8c_2^6-\alpha c_3+44c_2^4c_3-49c_2^2c_3^2+12c_3^2+12c_3^3-20c_2^3c_4 \tag{3.21}$$
$$+\alpha(3c_2^4-15c_2^2c_3+9c_3^2+9c_2c_4+10c_5)+2c_2(9c_3c_4+5c_6)-7c_7)c_k^6+O(e_k^7). \tag{3.22}$$

Using the expansions in (3.1), (3.2), (3.6) and (3.7), the following expression is put forward.

$$y_k-\frac{f(y_k)}{f'(y_k)}+\frac{1}{2}\left(\frac{f(y_k)}{f'(y_k)}\right)^2\left(\frac{f'(y_k)-f[t_k,\eta]}{(f'(y_k))}\right)\frac{f[t_k,\eta]}{f(t_k)} = t^*-\frac{3}{2}c_2^2c_3e_k^5 \tag{3.23}$$
$$+\frac{1}{2}(-\alpha c_2^2+4c_2^4+13c_2^2-12c_3^2-c_2c_4)e_k^6 \tag{3.24}$$
$$+(-12c_2^6-3c_2^4c_3-6c_3^3+6c_2^3c_4-17c_2c_4-\frac{c_2^2}{2}(13\alpha c_3-72c_3^2+5c_5))e_k^7 \tag{3.25}$$
$$+\frac{1}{2}(-2\alpha^2c_2^3+78c_2^7-120c_2^5c_3+30c_2^4c_4-52c_3^2c_4+\cdots+c_2^2(169c_3c_4-6c_6))e_k^8 \tag{3.26}$$
$$+\frac{1}{2}(-176c_2^8+553c_2^6c_3-5\alpha^2(c_2^4+4c_2^2c_3)-328c_2^5c_4+\cdots \tag{3.27}$$
$$+c_2^2(-607c_3^3+92c_4^2+218c_3c_5-7c_7))e_k^9+O(e_k^{10}). \tag{3.28}$$

Using (3.8), we obtain the expansion for $f(z_k)$ as:

$$f(z_k) = f'(t^*)[-\frac{3}{2}c_2^2c_3e_k^5+\frac{1}{2}(-\alpha c_2^2+4c_2^4+13c_2^2-12c_3^2-c_2c_4)e_k^6 \tag{3.29}$$
$$+(-12c_2^6-3c_2^4c_3-6c_3^3+6c_2^3c_4-17c_2c_4-\frac{c_2^2}{2}(13\alpha c_3-72c_3^2+5c_5))e_k^7 \tag{3.30}$$
$$+\frac{1}{2}(-12\alpha^2c_3^3+78c_2^7-120c_2^5c_3+30c_2^4-52c_3^2c_4+\cdots+c_2^2(169c_3c_4-6c_6))e_k^8 \tag{3.31}$$
$$+\frac{1}{2}(-176c_2^6+97c_2^4c_3-48c_3^3+6c_2^3c_4+\cdots+c_2^2(-607c_3^3+92c_4^2+218c_3c_5-7c_7))e_k^9+O(e_k^9)], \tag{3.32}$$

and

$$f[t_k,\eta_k] = 1+2c_2e_k^2+(2c_2c_3-2c_2^3)e_k^3+c_2(\alpha c_2+4c_2^3-6c_2c_3+3c_4)e_k^4)e_k^4 \tag{3.33}$$
$$+\frac{c_2}{2}(3\alpha(-16c_4^2+6\alpha c_3+29c_2^2c_3-4c_2^3-20c_2c_4+8c_5)e_k^5 \tag{3.34}$$
$$+(18c_2^6-\frac{67}{2}c_2^4c_3+4c_3^3+\cdots+c_2^2(49c_3c_4-16c_6)-2c_2(14c_3^3+6c_4^2+7c_3c_5-3c_7))e_k^7 \tag{3.35}$$
$$+(103c_2^8-282c_2^6c_3-\cdots c_2(55c_2^3c_4+31c_4c_5+17c_3c_6-7c_8))e_k^8+O(e_k^9). \tag{3.36}$$

By combining the expansions in (3.5), (3.8), (3.9) and (3.10), we get

$$z_k-\frac{f(z_k)}{2\left(f[y_k,z_k]\right)-f'(y_k)} = t^*+\frac{3}{2}c_2^4c_3^2e_k^9 \tag{3.37}$$
$$+\frac{1}{4}c_2^3c_3\left(2\alpha c_2^2-8c_2^4-41c_2^2c_3+48c_3^2+8c_2c_4\right)e_k^{10}+O\left(e_k^{11}\right) \tag{3.38}$$

Now, using (3.1) and (3.9), the Taylor's expansion of $v$ is

$$v=\frac{f(z_k)}{f(t_k)}=-\frac{3}{2}c_2^2c_3e_k^4+\left(-\frac{\alpha}{2}c_2^3+2c_2\left(c_2^4+4c_2^2c_3-3c_3^2-c_2c_4\right)\right)e_k^5 \tag{3.39}$$

$$+\frac{1}{2}\left(\left(-28c_2^6-22c_2^4c_3-12c_3^3+\alpha\left(c_2^4-13c_2^2c_3\right)+\cdots+c_2^2\left(87c_3^2-5c_5\right)\right)\right)e_k^6 \tag{3.40}$$

$$+\cdots+\left(-\frac{\alpha^3c_3^2}{2}+291c_2^9-1110c_2^7c_3+\cdots+c_2^2\left(-1344c_3^2c_4+153c_4c_5+165c_3c_6-4c_8\right)\right)e_k^9 \tag{3.41}$$

$$+O\left(e_k^{10}\right). \tag{3.42}$$

From (2.12), (3.11) and (3.12), we have

$$z_k-\left(\frac{f(z_k)}{2\left(f\left[y_k,z_k\right]\right)-f'(y_k)}\right)G(v)=t^*+\frac{3}{2}c_2^2c_3\left(G(0)-1\right)e_k^5 \tag{3.43}$$

$$+\frac{c_2}{2}\left(\alpha c_2^2-4c_2^4-13c_2^2c_3+12c_3^2+4c_2c_4\right)\left(G(0)-1\right)e_k^6 \tag{3.44}$$

$$+\frac{1}{2}(24c_2^6+6c_2^4c_3+12c_3^3-12c_2^3+34c_2c_3c_4+c_2^2(13\alpha c_3-72c_3^2)+5c_5))\left(G(0)-1\right)e_k^7 \tag{3.45}$$

$$+\frac{1}{2}(2\alpha^2c_2^3-78c_2^7+\cdots+\alpha c_2(-15c_2^4-16c_2^2c_3+46c_3^2+20c_2c_4)+\cdots+c_2^2(6c_6-169c_3c_4))\left(G(0)-1\right)e_k^8 \tag{3.46}$$

$$+\frac{1}{4}\left(352c_2^8(G(0)-1)-1106c_2^6c_3(G(0)-1)+\cdots-3c_2^4\left(22c_5(G(0)-1)+c_3^2\left(-70+68G(0)+3G'(0)\right)\right)\right)e_k^9 \tag{3.47}$$

$$+O\left(e_k^{10}\right). \tag{3.48}$$

To make vanish the coefficients of $e_k^n,\ \ 5\le n\le 9$ in (3.13), we set $G(0)=1$ and $G'(0)=\frac{2}{3}$. Therefore, (3.13) becomes

$$z_k-\left(\frac{f(z_k)}{2\left(f\left[y_k,z_k\right]\right)-f'(y_k)}\right)G(v)=t^*+\frac{1}{4}c_2^3c_3\left(-2\alpha c_2+8c_2^3+17c_2c_3-8c_4\right)e_k^{10}+O\left(e_k^{11}\right). \tag{3.49}$$

The equations in (3.8), (3.11) and (3.14) enabled us to obtain error equations in relation to Algorithm 2.1, Algorithm 2.3 and Algorithm 2.4 as:

$$e_{k+1}=-\frac{3}{2}c_2^2c_3e_k^5+O\left(e_k^6\right), \tag{3.50}$$

$$e_{k+1}=\frac{3}{2}c_2^4c_3^2e_k^9+O\left(e_k^{10}\right), \tag{3.51}$$

and

$$e_{k+1}=\frac{1}{4}c_2^3c_3\left(-2\alpha c_2+8c_2^3+17c_2c_3-8c_4\right)e_k^{10}+O\left(e_k^{11}\right), \tag{3.52}$$

respectively. The implications of the equations (3.15), (3.16) and (3.17), is that the iterative protocols in (2.3), (2.11) and (2.13) will locate the solution of NE with CO five, nine and ten respectively. This concludes the proof.

□

*Remark* 3.2. Consequent upon the proof of Theorem 2.1, a concrete form of Algorithm 2.4 can be obtained for $G(v) = 1 + \frac{2}{3}v$ as:

*Algorithm* 3.3. *For an initial guess $t_0$, obtain the solution $t_{k+1}$ of $f(t) = 0$ using the iterative protocol*

$$y_k = t_k - \frac{f(t_k)}{f[t_k, \eta]}, \quad \eta = t_k + \alpha (f(t_k))^m, \quad m \geq 2, \tag{3.53}$$

$$z_k = y_k - \frac{f(y_k)}{f'(y_k)} + \frac{1}{2}\left(\frac{f(y_k)}{f'(y_k)}\right)^2 \left(\frac{f'(y_k) - f[t_k, \eta]}{(f'(y_k))}\right) \frac{f[t_k, \eta]}{f(t_k)}, \tag{3.54}$$

$$t_{k+1} = z_k - \left(\frac{f(z_k)}{2(f[y_k, z_k]) - f'(y_k)}\right)\left(1 + \frac{2}{3}\frac{f(z_k)}{f(t_k)}\right). \tag{3.55}$$

Algorithm 3.2 is referred as triple-step tenth-order modified HM (MHM3) with $EI \approx 1.5849$.

*Remark* 3.4. We know that iterative methods with only $f'(\cdot)$ in its denominator (as in the case of the NM), breakdown when $f'(\cdot) \approx 0$. Therefore, the more evaluation of $f'(\cdot)$ at different points in an iterative protocol, will increase its chances of breakdown. The developed methods (Algorithm 2.1, Algorithm 2.2 and Algorithm 3.3) require evaluation of function derivative $f'(\cdot)$ at only one point $y_k$ in an iterative cycle, unlike other modifications of HM in Nazeer et al. [5, 14], Tanveer et al., [4], Noor and Gupta [6] and Hafiz and Al-Goria [10] that require evaluation of $f'(\cdot)$ at the two points $t_k$ and $y_k$.

## 4. Numerical Implementation

The applicability of methods developed in this work (MHM1, MHM2 and MHM3) are illustrated in this section by comparing their performance with some good existing methods in literature. The compared methods includes the HM, CO nine method of Noor et al., [15](NKEM) put forward as:

$$y_k = t_k - \eta_k; \tag{4.1}$$

$$z_k = y_k - \frac{2f(y_k)f'(y_k)}{2(f'(y_k))^2 - \frac{2f(y_k)}{y_k - t_k}\left\{2f'(y_k) + f'(t_k) - 3\frac{f(y_k) - f(t_t)}{y_k - t_k}\right\}}; \tag{4.2}$$

$$t_{k+1} = z_k - \left(\frac{f(z_k)}{f'(t_k)}\right)\frac{f'(t_k) + f'(y_k)}{3f'(y_k) - f'(t_k)}, \tag{4.3}$$

the method of Nazeer et al., [5] (NNEM) given as

$$y_k = t_k - \eta_k - \frac{\eta_k^2}{2}\frac{f''(t_k)}{f'(t_k)}; \tag{4.4}$$

$$t_{k+1} = y_k - \frac{f(y_k)}{f'(y_k)} - \frac{1}{2}\left(\frac{f(y_k)}{f'(y_k)}\right)^2 \frac{f''(y_k)}{f'(y_k)} - \frac{1}{6}\left(\frac{f(y_k)}{f'(y_k)}\right)^3 \left(\frac{f'''(y_k)}{f'(y_k)}\right), \tag{4.5}$$

the method of Tanveer et al., [4] (TMHM) put forward as:

$$y_k = t_k - \frac{2f(t_k)f'(t_k)}{2f'^2(t_k) - f(t_k)f''(t_k)}; \tag{4.6}$$

$$t_{k+1} = y_k - \frac{f(y_k)}{f'(y_k)} - \frac{1}{2}\left(\frac{f(y_k)}{f'(y_k)}\right)^2 \frac{f''(y_k)}{f'(y_k)} - \frac{1}{6}\left(\frac{f(y_k)}{f'(y_k)}\right)^3 \left(\frac{f'''(y_k)}{f'(y_k)}\right), \tag{4.7}$$

and the tenth-order method of Hafiz and Al-Goria [10] (HAM) developed as:

$$y_k = t_k - \eta_k, \tag{4.8}$$

$$z_k = y_k - \frac{f(y_k)}{f'(y_k)} - \frac{1}{2}\frac{[f(y_k)][2f'(y_k) + f'(t_k) - 3f[t_k, y_k]]}{(f'(y_k))}, \tag{4.9}$$

$$t_{k+1} = z_k - \frac{(z_k - y_k)f(z_k)}{(z_k - y_k)f[z_k, y_k] + (f[z_k, y_k - f'(y_k)])}. \tag{4.10}$$

For computation purpose, computer programs codes were designed in MAPLE 2017 version environment, with 2000 digits of mantissa using Intel Celeron(R) CPU 1.6 GHz with 2 GB of RAM processor. The criteria used for program stoppage is $|f(t_{k+1})| \leq 10^{-1000}$. For the purpose of comparison, the metrics: Number of function evaluation (NFE), Absolute value of function iteration point value $|f(t_{k+1})|$ and Computational order of convergence ($\rho_{coc}$) due to Jay [16] given as,

$$\rho_{coc} = \frac{\log |f(x_{k+1})|}{\log} |f(x_k)|, \tag{4.11}$$

were used. Some NE taken from [13, 17] were used in testing the applicability of the $MHM1$ , $MHM2$ and $MHM3$. The NE $(f_i(t)), \quad i = 1, 2, 3, \ldots$ are as given next:

$$f_1(t) = \sin(2\cos t) - 1 - t^2 + e^{(\sin\ t^3)} = 0, \quad t^* = 1.3061752018\ldots, \tag{4.12}$$

$$f_2(t) = e^{-t^2+t+2} - \cos(t+1) + t^3 + 1 = 0, \quad t^* = 1.0. \tag{4.13}$$

$$f_3(t) = -2 + (t-1)^3 = 0, \quad t^* = 2.2599210498\ldots \tag{4.14}$$

$$f_4(t) = 1 - t^2 + \sin^2\ t = 0, \quad t^* = 1.40449164821\ldots \tag{4.15}$$

Table 1-2 presents numerical results obtained from various methods when implemented on the NE $f_i(t) = 0$.

Table 1: Methods computational results comparison

| $f_i(x)$ | $Methods$ | $x_0$ | $IT$ | $NFE$ | $\lvert f(x_{k+1})\rvert$ | $\rho_{coc}$ |
|---|---|---|---|---|---|---|
| $f_1(x)$ | HM | 1.2 | 8 | 24 | $1.8_{-0909}$ | 3.0 |
| | MHM1 | | 5 | 20 | $2.4_{-1018}$ | 5.0 |
| | NKEM | | 3 | 15 | $9.4_{-0340}$ | 9.0 |
| | NNEM | | 4 | 28 | $4.3_{-0875}$ | 9.0 |
| | TMHM | | 3 | 21 | $2.1_{-0281}$ | 9.0 |
| | MHM2 | | 3 | 15 | $2.8_{-0320}$ | 9.0 |
| | HAM | | 3 | 15 | $2.3_{-0250}$ | 9.0 |
| | MHM3 | | 3 | 15 | $1.8_{-0493}$ | 10.0 |
| $f_2(x)$ | HM | 0.1 | 9 | 27 | $7.3_{-0822}$ | 3.0 |
| | MHM1 | | 4 | 16 | $2.9_{-0442}$ | 5.0 |
| | NKEM | | 3 | 15 | $3.2_{-0356}$ | 9.0 |
| | NNEM | | 4 | 28 | $8.3_{-0540}$ | 9.0 |
| | TMHM | | 4 | 28 | $2.7_{-1388}$ | 9.0 |
| | MHM2 | | 3 | 15 | $1.9_{-0513}$ | 9.0 |
| | HAM | | 3 | 15 | $7.8_{-0386}$ | 9.0 |
| | MHM3 | | 3 | 15 | $7.8_{-0627}$ | 10.0 |

### *4.1. Some Real Life Problems Applications*

In this part of the article, the developed methods $MHM1$, $MHM2$ and $MHM3$ were applied to obtain solutions to some modeled real life problems into NE. Their computational performance were compared with that of $HM$, $NKEM$, $NNEM$, $TMHM$ and $HAM$.

**Problem 1:** (Chemical equilibrium [18]) In obtaining the fraction of converted nitrogen-hydrogen to ammonia in a fractional conversion process under the pressure of 250 atm and temperature of $500^0C$, require solving the solution of the modeled problem given as:

$$f(t) = \frac{8t^2(4-t)^2}{(6-3t)^2(2-t)} - 0.186 = 0. \tag{4.16}$$

The solutions of the model in (4.5) are : $t_1^* = -0.384094\ldots$; $t_2^* = 0.27776\ldots$; $t_3^* = 3.94854 \pm 0.316124i$ and $t_4^* = 3.94854 \pm 316124i$ . But in this process, the value of the factional conversion must be between 0 and 1. Hence, the second solution $t_2^* = 0.27776\ldots$ meets this condition because in practice it is meaningful. The developed methods and the compared methods computation results are presented in Table 3.

Table 2: Methods computational results comparison

| $f_i(x)$ | $Methods$ | $x_0$ | $IT$ | $NFE$ | $\|f(x_{k+1})\|$ | $\rho_{coc}$ |
|---|---|---|---|---|---|---|
| $f_3(x)$ | HM | 3.0 | 7 | 21 | $2.4_{-0704}$ | 3.0 |
| | MHM1 | | 5 | 20 | $2.1_{-1229}$ | 5.0 |
| | NKEM | | 3 | 15 | $1.4_{-0282}$ | 9.0 |
| | NNEM | | 4 | 24 | $2.78_{-0242}$ | 9.0 |
| | TMHM | | 3 | 21 | $6.9_{-0299}$ | 9.0 |
| | MHM2 | | 3 | 15 | $3.7_{-0414}$ | 9.0 |
| | HAM | | 4 | 20 | $8.8_{-1579}$ | 9.0 |
| | MHM3 | | 3 | 15 | $5.0_{-0465}$ | 10.0 |
| $f_4(x)$ | HM | 2.0 | 7 | 21 | $3.0_{-0858}$ | 3.0 |
| | MHM1 | | 5 | 20 | $4.8_{-1400}$ | 5.0 |
| | NKEM | | 3 | 15 | $1.1_{-0358}$ | 9.0 |
| | NNEM | | 4 | 24 | $3.0_{-0288}$ | 9.0 |
| | TMHM | | 3 | 21 | $1.4_{-0258}$ | 9.0 |
| | MHM2 | | 3 | 15 | $1.7_{-0470}$ | 9.0 |
| | HAM | | 3 | 15 | $1.1_{-0337}$ | 9.0 |
| | MHM3 | | 3 | 15 | $7.7_{-0564}$ | 10.0 |

Table 3: Methods computational results comparison

| | $Methods$ | $x_0$ | $IT$ | $NFE$ | $\|f(x_{k+1})\|$ | $\rho_{coc}$ |
|---|---|---|---|---|---|---|
| | HM | 0.8 | 8 | 24 | $1.3_{-0703}$ | 3.0 |
| | MHM2 | | 5 | 20 | $1.0_{-0405}$ | 5.0 |
| | NKEM | | 4 | 20 | $5.5_{-0968}$ | 9.0 |
| | NNEM | | 4 | 28 | $1.4_{-0776}$ | 9.0 |
| | TMHM | | 4 | 28 | $10.0_{-1199}$ | 9.0 |
| | MHM2 | | 4 | 20 | $8.8_{-1152}$ | 9.0 |
| | HAM | | 4 | 20 | $7.7_{-0617}$ | 9.0 |
| | MHM3 | | 4 | 15 | $2.6_{-1453}$ | 10.0 |

**Problem 2:** (Chemical reactor conversion) Consider the model presented in [19] on the fractional species conversion in chemical reactor where $t$ is the require fractional conversion in the model

$$f(t) = 4.45977 + \frac{t}{1-t} - 5ln\left[\frac{0.4(1-t)}{0.4-0.5t}\right] = 0. \tag{4.17}$$

The model solution is $t^* = 0 : 7573962463\ldots$. For an initial solution guess of $t_0^* = 0 : 77$, the obtained results by the various methods are given in Table 4.

Table 4: Methods computational results comparison

| | $Methods$ | $x_0$ | $IT$ | $NFE$ | $\|f(x_{k+1})\|$ | $\rho_{coc}$ |
|---|---|---|---|---|---|---|
| | HM | 0.77 | 7 | 21 | $6.2_{-1545}$ | 3.0 |
| | MHM1 | | 5 | 20 | $3.2_{-1538}$ | 5.0 |
| | NKEM | | 3 | 15 | $1.1_{-0544}$ | 9.0 |
| | NNEM | | 3 | 15 | $4.3_{-0602}$ | 9.0 |
| | TMHM | | 3 | 21 | $5.2_{-0847}$ | 9.0 |
| | MHM2 | | 3 | 15 | $3.7_{-0433}$ | 9.0 |
| | HAM | | 3 | 15 | $1.1_{-0445}$ | 9.0 |
| | MHM3 | | 3 | 15 | $2.4_{-0589}$ | 10.0 |

### *4.2. Results discussion*

For brevity, computation results are presented in the format $a.b_{-c}$ to represent $a.b \times 10^{-c}$, where $a, b, c \in \Re$. From Table 1-4, observe that the numerical results produced by the methods developed herein, the computational CO (see $\rho_{coc}$ in the last column of Table 1-4) agrees with derived theoretical CO in Theorem 3.1. In addition, the developed method located the solutions of all the NE with a competitive error margins with methods compared. Again, the NFE of the new methods required to achieve convergence to the solution of NE, in most of the problem solved, are fewer than the compared modified HM. It was also observed that the developed modified HM (HAM) in [10], was theoretically proven to have tenth order convergence. Whereas, its computational CO did not agree with the theoretical CO claimed (see Eq. 20 in [10]), rather it is of CO nine (see all computation results of HAM in Table 1-4). On the other hand, the developed methods herein, has no such conflict of CO.

## 5. Conclusions

In this article, we have successfully modified the famous Householder's method that is of CO four with EI of 1.4422 to methods with CO five , nine and ten with EI of 1.4953, 1.5518 and 1.5849 respectively. The modified methods require no second order derivative evaluation and only require the the evaluation of function derivative at one point, which is another advantage of the methods. The computational performance of the methods reveals that they are better than some modified HM in literature compared.

## Conflict of Interest

The authors have no conflict of interest regarding the publication of this article.

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