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Complex Valued Bipolar Metric Spaces and Fixed Point Theorems

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Abstract

This article introduces the idea of complex valued bipolar metric space and derives some of its properties. Moreover, for complex valued bipolar metric spaces, various fixed point theorems of contravariant maps satisfying rational inequalities are proved. Additionally, the Kannan fixed point theorem and the Banach contraction principle are both generalised.

Keywords: Complex number, Bipolar metric space, Partial order, Cartesian product.
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1. Introduction

In [1], A. Azam et al. proposed the idea of complex valued metric spaces, deduced a few properties, and demonstrated fixed point outcomes for mappings fulfilling a rational inequality. See [2, 10, 11, 15] for a list of articles on fixed point theory in complex valued metric spaces.

Assume that $\mathbb{C}$ is the set of all complex numbers and $c_1, c_2 \in \mathbb{C}$. A partial order $\precsim$ on $\mathbb{C}$ should be defined as follows.
$c_1 \precsim c_2$ if and only if(or iff) $Re(c_1) \leq Re(c_2), Im(c_1) \leq Im(c_2)$. It follows that $c_1 \precsim c_2$ if one of the following axioms is fulfilled:

(i) $Re(c_1) = Re(c_2), Im(c_1) < Im(c_2)$,

(ii) $Re(c_1) < Re(c_2), Im(c_1) = Im(c_2)$,

(iii) $Re(c_1) < Re(c_2), Im(c_1) < Im(c_2)$,

(iv) $Re(c_1) = Re(c_2), Im(c_1) = Im(c_2)$.

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In particular we will write $c_1 \precnsim c_2$ if $c_1 \neq c_2$ and one of (i),(ii), and (iii) is fulfilled, and we will write $c_1 \prec c_2$ if only (iii) is fulfilled.
Note that

$$0 \precsim c_1 \precnsim c_2 \Rightarrow |c_1| < |c_2|$$
$$c_1 \precsim c_2, c_2 \prec c_3 \Rightarrow c_1 \prec c_3.$$

Definition 1.1. [1] Let G be a non empty set. A complex valued metric is a mapping $d : G \times G \to \mathbb{C}$ fulfilling the following conditions.

(I) $d(\aleph, \daleth) \succsim 0, \forall\ \aleph, \daleth \in G$,

(II) $d(\aleph, \daleth) = 0$ iff $\aleph = \daleth$ in G,

(III) $d(\aleph, \daleth) = d(\daleth, \aleph), \forall\ \aleph, \daleth \in G$,

(IV) $d(\aleph, \daleth) \precsim d(\aleph, \wp) + d(\wp, \daleth), \forall\ \aleph, \wp, \daleth \in G$.

The pair (G, d) is called a complex valued metric space.

A. Mutlu and U. Gurdal [8] established the idea of bipolar metric space, providing a novel definition of distance measurement between the elements of two distinct sets. A generalization of metric space is bipolar metric space. There are numerous articles that discuss fixed point theory in bipolar metric spaces; for instance, see [4, 5, 7, 9, 12, 13, 14] and its references.

Definition 1.2. [8] Let G and H be two non empty sets. A bipolar metric is a mapping $D : G \times H \to [0, \infty)$ fulfilling the following conditions.

(i) $D(\aleph, \daleth) = 0 \Rightarrow \aleph = \daleth$, whenever $(\aleph, \daleth) \in G \times H$,

(ii) $\aleph = \daleth \Rightarrow D(\aleph, \daleth) = 0$, whenever $(\aleph, \daleth) \in G \times H$,

(iii) $D(\aleph, \daleth) = D(\daleth, \aleph), \forall\ \aleph, \daleth \in G \cap H$,

(iv) $D(\aleph_1, \daleth_2) \leq D(\aleph_1, \daleth_1) + D(\aleph_2, \daleth_1) + D(\aleph_2, \daleth_2), \forall\ \aleph_1, \aleph_2 \in G$, and $\daleth_1, \daleth_2 \in H$.

The triple (G, H, D) is called a bipolar metric space.

In this paper, we present a new definition of complex valued bipolar metric space that generalises the notion of complex valued metric space by extending the domain of complex valued metric to a Cartesian product of two non-empty sets. Some complex valued bipolar metric space properties are derived. Moreover, in complex valued bipolar metric space, we demonstrate several fixed point solutions for contravariant mappings meeting various categories of rational inequalities. Additionally, we generalise the Kannan fixed point result and the Banach contraction principle [3, 6].

2. Complex Valued Bipolar Metric Spaces

Definition 2.1. Let G and H be two non empty sets. A complex valued bipolar metric is a mapping $d : G \times H \to \mathbb{C}$ fulfilling the following conditions.

(I) $d(\aleph, \daleth) \succsim 0$, whenever $(\aleph, \daleth) \in G \times H$,

(II) $d(\aleph, \daleth) = 0 \Rightarrow \aleph = \daleth$, whenever $(\aleph, \daleth) \in G \times H$,

(III) $\aleph = \daleth \Rightarrow d(\aleph, \daleth) = 0$, whenever $(\aleph, \daleth) \in G \times H$,

(IV) $d(\aleph, \daleth) = d(\daleth, \aleph), \forall\ \aleph, \daleth \in G \cap H$,

(V) $d(\aleph_1, \daleth_2) \precsim d(\aleph_1, \daleth_1) + d(\aleph_2, \daleth_1) + d(\aleph_2, \daleth_2)$, $\forall\ \aleph_1, \aleph_2 \in G$, and $\daleth_1, \daleth_2 \in H$.

The triple (G, H, d) is called a complex valued bipolar metric space(or, CVBMS).

Remark 2.2. Suppose (G,H,d) is a CVBMS. Then the space (G, H, d) is said to be disjoint if $G \cap H = \emptyset$. If $G \cap H \neq \emptyset$, the space (G, H, d) is referred to as a joint. The sets H and G are referred to, respectively, as the right pole and left pole of (G, H, d).

Example 2.3. Let G be the collection of functions such that $g : \mathbb{C} \to \{c : 1 \leq Re(c) \leq 3, Im(c) = 0\}$, $H = \mathbb{C}$. Define $d : G \times H \to \mathbb{C}$ as $d(g, c) = g(c)$, whenever $(g, c) \in G \times H$. Then (G, H, d) is a disjoint CVBMS.

Remark 2.4. Let (G, d) be a complex valued metric space, then (G, G, d) is a CVBMS. Conversely, if (G, H, d) is a CVBMS such that $G = H$, then (G, d) is a complex valued metric space.

Definition 2.5. Assume (G, H, d) is a CVBMS. Where the points of the sets H, G, and $G \cap H$ are referred to as the right, left, and central points. A right(or left, or central) sequence is one that only consists of right(or left, or central) points in (G, H, d).

Definition 2.6. Suppose (G, H, d) is a CVBMS. A left sequence $(\aleph_n)_{n=1}^{\infty}$ converges to a right point $\daleth$(or $(\aleph_n)_{n=1}^{\infty} \to \daleth$) iff for each $c \in \mathbb{C}$ with $c \succ 0$, there is an integer $n_0 \in \mathbb{N}$ such that $d(\aleph_n, \daleth) \prec c$, $\forall\ n \geq n_0$. Also a right sequence $(\daleth_n)_{n=1}^{\infty}$ converges to a left point $\aleph$ (or $(\daleth_n)_{n=1}^{\infty} \to \aleph$) iff for each $c \in \mathbb{C}$ with $c \succ 0$, there is an integer $n_0 \in \mathbb{N}$ such that $d(\aleph, \daleth_n) \prec c$, $\forall\ n \geq n_0$. When a CVBMS (G, H, d) is given with $(\wp_n)_{n=1}^{\infty} \to \hbar$ but no precise information on the sequence, this indicates that $(\wp_n)_{n=1}^{\infty}$ is either a right sequence and $\hbar$ is a left point, or $(\wp_n)_{n=1}^{\infty}$ is a left sequence and $\hbar$ is a right point.

Lemma 2.7. *Let (G, H, d) be a CVBMS. Then a left sequence $(\aleph_n)_{n=1}^{\infty}$ converges to a right point $\daleth$ iff $|d(\aleph_n, \daleth)| \to 0$, and also a right sequence $(\daleth_n)_{n=1}^{\infty}$ converges to a left point $\aleph$ iff $|d(\aleph, \daleth_n)| \to 0$.*

Proof. Let $(\aleph_n)_{n=1}^{\infty}$ be a left sequence, and $(\aleph_n)_{n=1}^{\infty} \to \daleth \in H$. Let $c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}$ for a certain real number $\epsilon > 0$,. For each $c \in \mathbb{C}$ with $c \succ 0$, there is an integer $n_0 \in \mathbb{N}$ such that $\forall\ n \geq n_0$, $d(\aleph_n, \daleth) \prec c$.

$$|d(\aleph_n, \daleth)| \prec |c| = \epsilon,\ \forall\ n \geq n_0.$$

Thus, it follows $|d(\aleph_n, \daleth)| \to 0$ as $n \to \infty$. Conversely, let $|d(\aleph_n, \daleth)| \to 0$ as $n \to \infty$. Then for $c \in \mathbb{C}$ with $c \succ 0$, there is a real number $\delta > 0$ such that for $z \in \mathbb{C}$

$$|z| < \delta \Rightarrow z \prec c$$

For this δ, there is an integer $n_0 \in \mathbb{N}$ such that

$$|d(\aleph_n, \daleth)| < \delta,\ \forall\ n \geq n_0.$$

It follows that $d(\aleph_n, \daleth) \prec c, \forall n \geq n_0$. Therefore $\aleph_n \to \daleth \in H$.
Obviously, a right sequence $(\daleth_n)_{n=1}^{\infty}$ converges to a left point $\aleph$ iff $|d(\aleph, \daleth_n)| \to 0$ and the proof is now complete. □

Lemma 2.8. *Suppose (G, H, d) is a CVBMS. When a central point serves as a sequence's limit, the central point acts as the sequence's unique limit.*

Proof. Suppose $(\aleph_n)_{n=1}^{\infty}$ is a left sequence, $(\aleph_n)_{n=1}^{\infty} \to \aleph \in G \cap H$, and $(\aleph_n)_{n=1}^{\infty} \to \daleth \in H$. Let $c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}$ for a certain real number $\epsilon > 0$. For each $c \in \mathbb{C}$ with $c \succ 0$, there is an integer $n_0 \in \mathbb{N}$ such that, $\forall\ n \geq n_0$, we have $d(\aleph_n, \aleph) \prec \frac{c}{2}$, and $d(\aleph_n, \daleth) \prec \frac{c}{2}$, and then

$$d(\aleph, \daleth) \precsim d(\aleph, \aleph) + d(\aleph_n, \aleph) + d(\aleph_n, \daleth) \prec 0 + \frac{c}{2} + \frac{c}{2}.$$

$$|d(\aleph, \daleth)| \precsim |d(\aleph, \aleph) + d(\aleph_n, \aleph) + d(\aleph_n, \daleth)| \prec |0 + \frac{c}{2} + \frac{c}{2}| = |c| = \epsilon.$$

We conclude that $d(\aleph, \daleth) = 0$, because $\epsilon > 0$ is arbitrary. Therefore $\aleph = \daleth$. □

Lemma 2.9. *Suppose (G, H, d) is a CVBMS. If a left sequence $(\aleph_n)_{n=1}^{\infty}$ converges to $\daleth$ and a right sequence $(\daleth_n)_{n=1}^{\infty}$ converges to $\aleph$, then $d(\aleph_n, \daleth_n) \to d(\aleph, \daleth)$ as $n \to \infty$.*

Proof. Let $(\aleph_n)_{n=1}^{\infty} \to \daleth \in H$, and $(\daleth_n)_{n=1}^{\infty} \to \aleph \in G$. Let $c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}$, for a certain real number $\epsilon > 0$. For each $c \in \mathbb{C}$ with $c \succ 0$, there is an integer $n_0 \in \mathbb{N}$ such that, $\forall\ n \geq n_0$, we have $d(\aleph_n, \daleth) \prec \frac{c}{2}$, and $d(\aleph, \daleth_n) \prec \frac{c}{2}$, then

$$d(\aleph, \daleth) \precsim d(\aleph, \daleth_n) + d(\aleph_n, \daleth_n) + d(\aleph_n, \daleth)$$

implies

$$d(\aleph, \daleth) - d(\aleph_n, \daleth_n) \precsim d(\aleph, \daleth_n) + d(\aleph_n, \daleth) \prec \frac{c}{2} + \frac{c}{2},$$

$$|d(\aleph_n, \daleth_n) - d(\aleph, \daleth)| \precsim |d(\aleph, \daleth_n) + d(\aleph_n, \daleth)| \prec |c| = \epsilon, \forall n \geq n_0,$$

and therefore $d(\aleph_n, \daleth_n) \to d(\aleph, \daleth)$ as $n \to \infty$. □

Definition 2.10. Let (G_α, H_α) and (G_β, H_β) be two complex valued bipolar metric spaces, and $g : G_\alpha \cup H_\alpha \to G_\beta \cup H_\beta$.

(i) If $g(G_\alpha) \subseteq G_\beta$ and $g(H_\alpha) \subseteq H_\beta$, then g is called a covariant map from (G_α, H_α) to (G_β, H_β), and we write $g : (G_\alpha, H_\alpha) \rightrightarrows (G_\beta, H_\beta)$.

(ii) If $g(G_\alpha) \subseteq H_\beta$ and $g(H_\alpha) \subseteq G_\beta$, then g is called a contravariant map from (G_α, H_α) to (G_β, H_β), and we write $g : (G_\alpha, H_\alpha) \rightleftarrows (G_\beta, H_\beta)$.

Remark 2.11. Suppose d_α, and d_β are two complex valued bipolar metrics on (G_α, H_α) and (G_β, H_β) respectively. We can also use the symbols $g : (G_\alpha, H_\alpha, d_1) \rightrightarrows (G_\beta, H_\beta, d_2)$ and $g : (G_\alpha, H_\alpha, d_1) \rightleftarrows (G_\beta, H_\beta, d_2)$ in the place of $g : (G_\alpha, H_\alpha) \rightrightarrows (G_\beta, H_\beta)$ and $g : (G_\alpha, H_\alpha) \rightleftarrows (G_\beta, H_\beta)$.

Definition 2.12. Let (G, H, d) be a CVBMS.

(i) A sequence $(\aleph_n, \daleth_n)$ on the set $G \times H$ is called a bisequence on (G, H, d).

(ii) If both $(\aleph_n)_{n=1}^{\infty}$ and $(\daleth_n)_{n=1}^{\infty}$ converges, then the bisequence $(\aleph_n, \daleth_n)$ is called convergent. If both $(\aleph_n)_{n=1}^{\infty}$ and $(\daleth_n)_{n=1}^{\infty}$ converges to a same point $\aleph \in G \cap H$, then the bisequence is called biconvergent.

(iii) If for each $c \in \mathbb{C}$ with $c \succ 0$, there is an $n_0 \in \mathbb{N}$ such that $d(\aleph_n, \daleth_{n+m}) \prec c$, $\forall\ n \geq n_0$, then a bisequence $(\aleph_n, \daleth_n)$ is called a Cauchy bisequence on (G, H, d).

Lemma 2.13. *Let (G, H, d) be a CVBMS. Then $(\aleph_n, \daleth_n)$ is a Cauchy bisequence iff $|d(\aleph_n, \daleth_{n+m})| \to 0$ as $n \to \infty$.*

Proof. Let $(\aleph_n, \daleth_n)$ is a Cauchy bisequence. Let $c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}$ for a certain real number $\epsilon > 0$. For each $c \in \mathbb{C}$ with $c \succ 0$, there is an integer $n_0 \in \mathbb{N}$ such that, $\forall\ n \geq n_0$, $d(\aleph_n, \daleth_{n+m}) \prec c$.

$$|d(\aleph_n, \daleth_{n+m})| \prec |c| = \epsilon,\ \forall\ n \geq n_0.$$

Thus, it follows $|d(\aleph_n, \daleth_{n+m})| \to 0$ as $n \to \infty$. Conversely, let $|d(\aleph_n, \daleth_{n+m})| \to 0$ as $n \to \infty$. Then given $c \in \mathbb{C}$ with $c \succ 0$, there is a real number $\delta > 0$ such that for $z \in \mathbb{C}$

$$|z| < \delta \Rightarrow z \prec c$$

For this δ, there is an integer $n_0 \in \mathbb{N}$ such that

$$|d(\aleph_n, \daleth_{n+m})| < \delta,\ \forall\ n \geq n_0.$$

It follows that $d(\aleph_n, \daleth_{n+m}) \prec c, \forall n \geq n_0$. Therefore $(\aleph_n, \daleth_n)$ is a Cauchy bisequence. □

Proposition 2.14. *Every biconvergent bisequence is a Cauchy bisequence in CVBMS (G, H, d).*

Proof. Suppose a bisequence $(\aleph_n, \daleth_n)$ is biconvergent to a point $\aleph \in G \cap H$. Let $c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}$, for a certain real number $\epsilon > 0$. For each $c \in \mathbb{C}$ with $c \succ 0$, there is an integer $n_0 \in \mathbb{N}$ such that $\forall\ n \geq n_0$, $d(\aleph_n, \aleph) \prec \frac{c}{2}$, and $\forall\ n \geq n_0$, $d(\aleph, \daleth_{n+m}) \prec \frac{c}{2}$. Then we have

$$d(\aleph_n, \daleth_{n+m}) \precsim d(\aleph_n, \aleph) + d(\aleph, \aleph) + d(\aleph, \daleth_{n+m}) \prec \frac{c}{2} + 0 + \frac{c}{2}, \forall n \geq n_0.$$

$$|d(\aleph_n, \daleth_{n+m})| \precsim |d(\aleph_n, \aleph) + d(\aleph, \aleph) + d(\aleph, \daleth_{n+m})| \prec |\frac{c}{2} + 0 + \frac{c}{2}| = |c| = \epsilon, \forall n \geq n_0.$$

So $(\aleph_n, \daleth_n)$ is a Cauchy bisequence. □

Proposition 2.15. *Every convergent Cauchy bisequence is biconvergent in a CVBMS (G, H, d).*

Proof. Let $(\aleph_n, \daleth_n)$ be a Cauchy bisequence such that $(\aleph_n)_{n=1}^{\infty}$ convergent to $\daleth$ in H and $(\daleth_n)_{n=1}^{\infty}$ convergent to $\aleph$ in G. Let $c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}$ for a certain real number $\epsilon > 0$. For each $c \in \mathbb{C}$ with $c \succ 0$, there is an integer $n_0 \in \mathbb{N}$ such that $d(\aleph_n, \daleth) \prec \frac{c}{3}$, $d(\aleph, \daleth_{n+m}) \prec \frac{c}{3}$, for all $n \geq n_0$, and $d(\aleph_n, \daleth_{n+m}) \prec \frac{c}{3}$, $\forall\ n \geq n_0$. Then

$$d(\aleph, \daleth) \precsim d(\aleph, \daleth_{n+m}) + d(\aleph_n, \daleth_{n+m}) + d(\aleph_n, \daleth) \prec \frac{c}{3} + \frac{c}{3} + \frac{c}{3}, \forall n \geq n_0.$$

$$|d(\aleph, \daleth)| \precsim |d(\aleph, \daleth_{n+m}) + d(\aleph_n, \daleth_{n+m}) + d(\aleph_n, \daleth)| \prec |\frac{c}{3} + \frac{c}{3} + \frac{c}{3}| = |c| = \epsilon, \forall n \geq n_0.$$

Therefore $d(\aleph, \daleth) = 0$ and so that $\aleph = \daleth$. Then $(\aleph_n, \daleth_n)$ is biconvergent. □

Definition 2.16. If every Cauchy bisequence is convergent, or equivalently, biconvergent, the CVBMS (G, H, d) is said to be complete.

3. Main Results

Theorem 3.1. *Let (G, H, d) be a complete CVBMS. If a contravariant map $g : (G, H, d) \rightleftarrows (G, H, d)$ satisfies $d(g(\daleth), g(\aleph)) \precsim \lambda d(\aleph, \daleth) + \frac{\mu d(\aleph, g(\aleph)) d(g(\daleth), \daleth)}{1 + d(\aleph, \daleth)}$, whenever $(\aleph, \daleth) \in G \times H$, for some $\lambda, \mu \in (0, 1)$ with $\lambda + \mu < 1$, then the function $g : G \cup H \to G \cup H$ has a UFP.*

Proof. Let $\aleph_0 \in G$, $\daleth_0 = g(\aleph_0) \in H$, and $\aleph_1 = g(\daleth_0)$. Suppose, $\daleth_n = g(\aleph_n)$ and $\aleph_{n+1} = g(\daleth_n)$, $\forall\ n \in \mathbb{N}$. Then $(\aleph_n, \daleth_n)$ is a bisequence on (G, H, d). For every $n \in \mathbb{N}$, from

$$\begin{aligned} d(\aleph_n, \daleth_n) &= d(g(\daleth_{n-1}), g(\aleph_n)) \\ &\precsim \lambda d(\aleph_n, \daleth_{n-1}) + \frac{\mu d(\aleph_n, g(\aleph_n)) d(g(\daleth_{n-1}), \daleth_{n-1})}{1 + d(\aleph_n, \daleth_{n-1})} \\ &= \lambda d(\aleph_n, \daleth_{n-1}) + \frac{\mu d(\aleph_n, \daleth_n) d(\aleph_n, \daleth_{n-1})}{1 + d(\aleph_n, \daleth_{n-1})} \\ &\precsim \lambda d(\aleph_n, \daleth_{n-1}) + \mu d(\aleph_n, \daleth_n) \end{aligned}$$

we conclude that

$$d(\aleph_n, \daleth_n) \precsim \frac{\lambda}{1-\mu} d(\aleph_n, \daleth_{n-1}),$$

and

$$\begin{aligned} d(\aleph_n, \daleth_{n-1}) &= d(g(\daleth_{n-1}), g(\aleph_{n-1})) \\ &\precsim \lambda d(\aleph_{n-1}, \daleth_{n-1}) + \frac{\mu d(\aleph_{n-1}, g(\aleph_{n-1})) d(g(\daleth_{n-1}), \daleth_{n-1})}{1 + d(\aleph_{n-1}, \daleth_{n-1})} \\ &= \lambda d(\aleph_{n-1}, \daleth_{n-1}) + \frac{\mu d(\aleph_{n-1}, \daleth_{n-1}) d(\aleph_n, \daleth_{n-1})}{1 + d(\aleph_{n-1}, \daleth_{n-1})} \\ &\precsim \lambda d(\aleph_{n-1}, \daleth_{n-1}) + \mu d(\aleph_n, \daleth_{n-1}) \end{aligned}$$

so that

$$d(\aleph_n, \daleth_{n-1}) \precsim \frac{\lambda}{1-\mu} d(\aleph_{n-1}, \daleth_{n-1}),$$

Therefore, by putting $\alpha = \frac{\lambda}{1-\mu}$, we have

$$d(\aleph_n, \daleth_n) \precsim \alpha^{2n} d(\aleph_0, \daleth_0)$$

and

$$d(\aleph_n, \daleth_{n-1}) \precsim \alpha^{2n-1} d(\aleph_0, \daleth_0).$$

For every $m, n \in \mathbb{N}$,

$$\begin{aligned} d(\aleph_n, \daleth_m) &\precsim d(\aleph_n, \daleth_n) + d(\aleph_{n+1}, \daleth_n) + d(\aleph_{n+1}, \daleth_m) \\ &\precsim (\alpha^{2n} + \alpha^{2n+1}) d(\aleph_0, \daleth_0) + d(\aleph_{n+1}, \daleth_m) \\ &\precsim ... \\ &\precsim (\alpha^{2n} + \alpha^{2n+1} + ... + \alpha^{2m-1}) d(\aleph_0, \daleth_0) + d(\aleph_m, \daleth_m) \\ &\precsim (\alpha^{2n} + \alpha^{2n+1} + ... + \alpha^{2m}) d(\aleph_0, \daleth_0), \text{ if } m > n, \end{aligned}$$

$$|d(\aleph_n, \daleth_m)| \leq (\alpha^{2n} + \alpha^{2n+1} + ... + \alpha^{2m}) |d(\aleph_0, \daleth_0)|, \text{ if } m > n,$$

and similarly, if $m < n$, then

$$d(\aleph_n, \daleth_m) \precsim (\alpha^{2m+1} + \alpha^{2m+2} + ... + \alpha^{2n+1}) d(\aleph_0, \daleth_0),$$

$$|d(\aleph_n, \daleth_m)| \leq (\alpha^{2m+1} + \alpha^{2m+2} + ... + \alpha^{2n+1}) |d(\aleph_0, \daleth_0)|.$$

By $\alpha \in (0, 1)$, $|d(\aleph_n, \daleth_m)| \to 0$, as $n, m \to \infty$, we conclude that $(\aleph_n, \daleth_n)$ is a Cauchy bisequence. Also, $(\aleph_n, \daleth_n)$ converges, and biconverges to a point $\wp \in G \cap H$, because (G, H, d) is complete. Hence, $g(\aleph_n) = \daleth_n \to \wp \in G \cap H$ as $n \to \infty$ implies $d(g(\wp), g(\aleph_n)) \to d(g(\wp), \wp)$ as $n \to \infty$, by using Lemma 2.9. Also by taking the limit from

$$d(g(\wp), g(\aleph_n)) \precsim \lambda d(\aleph_n, \wp) + \frac{\mu d(\aleph_n, \daleth_n) d(g(\wp), \wp)}{1 + d(\aleph_n, \wp)}$$

$$|d(g(\wp), g(\aleph_n))| \leq \lambda |d(\aleph_n, \wp)| + \frac{\mu |d(\aleph_n, \daleth_n) d(g(\wp), \wp)|}{|1 + d(\aleph_n, \wp)|}$$

as $n \to \infty$, we get $d(g(\wp), \wp) = 0$. Hence $g(\wp) = \wp$. Therefore $\wp$ is a fixed point of g.
If $\hbar$ is another fixed point of g, then $g(\hbar) = \hbar$, $\hbar \in G \cap H$, and hence,

$$d(\wp, \hbar) = d(g(\wp), g(\hbar)) \precsim \lambda d(\wp, \hbar) + \frac{\mu d(\wp, g(\wp)) d(g(\hbar), \hbar)}{1 + d(\wp, \hbar)} \precsim \lambda d(\wp, \hbar)$$

Therefore $d(\wp, \hbar) = 0$ so that $\wp = \hbar$. So g has a UFP. □

A Corollary 5 of [1] is generalised by the previous Theorem. Also if $\mu = 0$ then the above Theorem generalizes a Banach contraction principle (see [6]).

Example 3.2. Let $G = \{0, \frac{1}{2}, 2\}$ and $H = \{0, \frac{1}{2}\}$. Let $d(\aleph, \daleth) = |\aleph - \daleth| + i|\aleph - \daleth|$, where $(\aleph, \daleth) \in G \times H$. Then (G, H, d) is a complete CVBMS. Define a contravariant map $g : (G, H, d) \rightleftarrows (G, H, d)$ by $g(0) = 0$, $g(\frac{1}{2}) = 0$, and $g(2) = \frac{1}{2}$. Then, g satisfies the inequality $d(g(\daleth), g(\aleph)) \precsim \lambda d(\aleph, \daleth) + \frac{\mu d(\aleph, g(\aleph)) d(g(\daleth), \daleth)}{1 + d(\aleph, \daleth)}$ for $\lambda = \frac{1}{3}$ and $\mu = \frac{1}{6}$. By Theorem 3.1, g has a UFP zero in $G \cap H$.

Theorem 3.3. *Let* (G, H, d) *be a complete CVBMS. If a contravariant map* $g : (G, H, d) \rightleftarrows (G, H, d)$ *satisfies* $d(g(\daleth), g(\aleph)) \precsim \lambda[d(\aleph, g(\aleph)) + d(g(\daleth), \daleth)] + \frac{\mu d(\aleph, g(\aleph)) d(g(\daleth), \daleth)}{1 + d(\aleph, \daleth)}$*, whenever* $(\aleph, \daleth) \in G \times H$*, for some* $\lambda, \mu \in (0, 1)$ *with* $2\lambda + \mu < 1$*, then the function* $g : G \cup H \to G \cup H$ *has a UFP.*

Proof. Let $\aleph_0 \in G$, $\daleth_0 = g(\aleph_0) \in H$, and $\aleph_1 = g(\daleth_0)$. Suppose, $\daleth_n = g(\aleph_n)$ and $\aleph_{n+1} = g(\daleth_n)$, $\forall\ n \in \mathbb{N}$. Then $(\aleph_n, \daleth_n)$ is a bisequence on (G, H, d). For every $n \in \mathbb{N}$, from

$$\begin{aligned} d(\aleph_n, \daleth_n) &= d(g(\daleth_{n-1}), g(\aleph_n)) \\ &\precsim \lambda[d(\aleph_n, g(\aleph_n)) + d(g(\daleth_{n-1}), \daleth_{n-1})] + \frac{\mu d(\aleph_n, g(\aleph_n)) d(g(\daleth_{n-1}), \daleth_{n-1})}{1 + d(\aleph_n, \daleth_{n-1})} \\ &= \lambda[d(\aleph_n, \daleth_n) + d(\aleph_n, \daleth_{n-1})] + \frac{\mu d(\aleph_n, \daleth_n) d(\aleph_n, \daleth_{n-1})}{1 + d(\aleph_n, \daleth_{n-1})} \\ &\precsim \lambda[d(\aleph_n, \daleth_n) + d(\aleph_n, \daleth_{n-1})] + \mu d(\aleph_n, \daleth_n) \end{aligned}$$

we conclude that

$$d(\aleph_n, \daleth_n) \precsim \frac{\lambda}{1 - \lambda - \mu} d(\aleph_n, \daleth_{n-1}),$$

and

$$\begin{aligned} d(\aleph_n, \daleth_{n-1}) &= d(g(\daleth_{n-1}), g(\aleph_{n-1})) \\ &\precsim \lambda[d(\aleph_{n-1}, g(\aleph_{n-1})) + d(g(\daleth_{n-1}), \daleth_{n-1})] + \frac{\mu d(\aleph_{n-1}, g(\aleph_{n-1})) d(g(\daleth_{n-1}), \daleth_{n-1})}{1 + d(\aleph_{n-1}, \daleth_{n-1})} \\ &= \lambda[d(\aleph_{n-1}, \daleth_{n-1}) + d(\aleph_n, \daleth_{n-1})] + \frac{\mu d(\aleph_{n-1}, \daleth_{n-1}) d(\aleph_n, \daleth_{n-1})}{1 + d(\aleph_{n-1}, \daleth_{n-1})} \\ &\precsim \lambda[d(\aleph_{n-1}, \daleth_{n-1}) + d(\aleph_n, \daleth_{n-1})] + \mu d(\aleph_n, \daleth_{n-1}) \end{aligned}$$

so that

$$d(\aleph_n, \daleth_{n-1}) \precsim \frac{\lambda}{1 - \lambda - \mu} d(\aleph_{n-1}, \daleth_{n-1}),$$

Therefore, by putting $\alpha = \frac{\lambda}{1-\lambda-\mu}$, we have

$$d(\aleph_n, \daleth_n) \precsim \alpha^{2n} d(\aleph_0, \daleth_0)$$

and

$$d(\aleph_n, \daleth_{n-1}) \precsim \alpha^{2n-1} d(\aleph_0, \daleth_0).$$

For every $m, n \in \mathbb{N}$,

$$\begin{aligned} d(\aleph_n, \daleth_m) &\precsim d(\aleph_n, \daleth_n) + d(\aleph_{n+1}, \daleth_n) + d(\aleph_{n+1}, \daleth_m) \\ &\precsim (\alpha^{2n} + \alpha^{2n+1}) d(\aleph_0, \daleth_0) + d(\aleph_{n+1}, \daleth_m) \\ &\precsim \ldots \\ &\precsim (\alpha^{2n} + \alpha^{2n+1} + \ldots + \alpha^{2m-1}) d(\aleph_0, \daleth_0) + d(\aleph_m, \daleth_m) \\ &\precsim (\alpha^{2n} + \alpha^{2n+1} + \ldots + \alpha^{2m}) d(\aleph_0, \daleth_0), \text{ if } m > n, \end{aligned}$$

$$|d(\aleph_n, \daleth_m)| \leq (\alpha^{2n} + \alpha^{2n+1} + \ldots + \alpha^{2m}) |d(\aleph_0, \daleth_0)|, \text{ if } m > n,$$

and similarly, if $m < n$, then

$$d(\aleph_n, \daleth_m) \precsim (\alpha^{2m+1} + \alpha^{2m+2} + \ldots + \alpha^{2n+1}) d(\aleph_0, \daleth_0),$$

$$|d(\aleph_n, \daleth_m)| \leq (\alpha^{2m+1} + \alpha^{2m+2} + \ldots + \alpha^{2n+1}) |d(\aleph_0, \daleth_0)|.$$

By $\alpha \in (0,1)$, $|d(\aleph_n, \daleth_m)| \to 0$, as $n, m \to \infty$, we conclude that $(\aleph_n, \daleth_n)$ is a Cauchy bisequence. Also, $(\aleph_n, \daleth_n)$ converges, and biconverges to a point $\wp \in G \cap H$, because (G, H, d) is complete. Hence, $g(\aleph_n) = \daleth_n \to \wp \in G \cap H$ as $n \to \infty$ implies $d(g(\wp), g(\aleph_n)) \to d(g(\wp), \wp)$ as $n \to \infty$, by using Lemma 2.9. Also by taking the limit from

$$d(g(\wp), g(\aleph_n)) \precsim \lambda[d(\aleph_n, \daleth_n) + d(g(\wp), \wp)] + \frac{\mu d(\aleph_n, \daleth_n) d(g(\wp), \wp)}{1 + d(\aleph_n, \wp)}$$

$$|d(g(\wp), g(\aleph_n))| \leq \lambda[|d(\aleph_n, \daleth_n) + d(g(\wp), \wp)|] + \frac{\mu |d(\aleph_n, \daleth_n) d(g(\wp), \wp)|}{|1 + d(\aleph_n, \wp)|}$$

as $n \to \infty$, we get $d(g(\wp), \wp) = 0$. Hence $g(\wp) = \wp$. Therefore $\wp$ is a fixed point of g.
If $\hbar$ is another fixed point of g, then $g(\hbar) = \hbar$, $\hbar \in G \cap H$, and hence,

$$d(\wp, \hbar) = d(g(\wp), g(\hbar)) \precsim \lambda[d(\wp, g(\wp)) + d(g(\hbar), \hbar)] + \frac{\mu d(\wp, g(\wp)) d(g(\hbar), \hbar)}{1 + d(\wp, \hbar)}$$

Therefore $d(\wp, \hbar) = 0$ so that $\wp = \hbar$. So g has a UFP. □

If $\mu = 0$ then the above Theorem generalized the Kannan fixed point theorem [3].

Example 3.4. Let G and H be the collections of all singleton and compact subsets of $\mathbb{R}$, respectively. Let $d(\aleph, B) = |\aleph - \inf(B)| + i|\aleph - \sup(B)|$, where $(\aleph, B) \in G \times H$. Then (G, H, d) is a complete CVBMS. Define a contravariant map $g : (G, H, d) \rightleftarrows (G, H, d)$ by $g(B) = \frac{\inf(B) + \sup(B) + 6}{8}$, for all $B \in G \cup H$. Then, g satisfies the inequality $d(g(\daleth), g(\aleph)) \precsim \lambda[d(\aleph, g(\aleph)) + d(g(\daleth), \daleth)] + \frac{\mu d(\aleph, g(\aleph)) d(g(\daleth), \daleth)}{1 + d(\aleph, \daleth)}$ for $\lambda = \frac{1}{3}$ and $\mu = 0$. Thus g has a UFP $\{1\} \in G \cap H$, because of Theorem 3.3.

Theorem 3.5. *Let (G, H, d) be a complete CVBMS. If a contravariant map $g : (G, H, d) \rightleftarrows (G, H, d)$ satisfies $d(g(\daleth), g(\aleph)) \precsim \lambda[d(\aleph, \daleth) + d(\aleph, g(\aleph)) + d(g(\daleth), \daleth)] + \frac{\mu d(\aleph, g(\aleph)) d(g(\daleth), \daleth)}{1 + d(\aleph, g(\aleph)) + d(g(\daleth), \daleth)}$, whenever $(\aleph, \daleth) \in G \times H$, for some $\lambda, \mu \in (0,1)$ with $3\lambda + \mu < 1$, then the function $g : G \cup H \to G \cup H$ has a UFP.*

Proof. Let $\aleph_0 \in G$, $\daleth_0 = g(\aleph_0) \in H$, and $\aleph_1 = g(\daleth_0)$. Suppose, $\daleth_n = g(\aleph_n)$ and $\aleph_{n+1} = g(\daleth_n)$, $\forall\, n \in \mathbb{N}$. Then $(\aleph_n, \daleth_n)$ is a bisequence on (G, H, d). For every $n \in \mathbb{N}$, from

$$\begin{aligned}
& d(\aleph_n, \daleth_n) \\
= \;& d(g(\daleth_{n-1}), g(\aleph_n)) \\
\precsim \;& \lambda[d(\aleph_n, \daleth_{n-1}) + d(\aleph_n, g(\aleph_n)) + d(g(\daleth_{n-1}), \daleth_{n-1})] + \frac{\mu d(\aleph_n, g(\aleph_n)) d(g(\daleth_{n-1}), \daleth_{n-1})}{1 + d(\aleph_n, g(\aleph_n)) + d(g(\daleth_{n-1}), \daleth_{n-1})} \\
= \;& \lambda[d(\aleph_n, \daleth_{n-1}) + d(\aleph_n, \daleth_n) + d(\aleph_n, \daleth_{n-1})] + \frac{\mu d(\aleph_n, \daleth_n) d(\aleph_n, \daleth_{n-1})}{1 + d(\aleph_n, \daleth_n) + d(\aleph_n, \daleth_{n-1})} \\
\precsim \;& \lambda[d(\aleph_n, \daleth_{n-1}) + d(\aleph_n, \daleth_n) + d(\aleph_n, \daleth_{n-1})] + \mu d(\aleph_n, \daleth_n)
\end{aligned}$$

we conclude that

$$d(\aleph_n, \daleth_n) \precsim \frac{2\lambda}{1 - \lambda - \mu} d(\aleph_n, \daleth_{n-1}),$$

and

$$d(\aleph_n, \daleth_{n-1}) = d(f(\daleth_{n-1}), f(\aleph_{n-1}))$$

$$\begin{aligned}
&\precsim \lambda[d(\aleph_{n-1}, \daleth_{n-1}) + d(\aleph_{n-1}, g(\aleph_{n-1})) + d(g(\daleth_{n-1}), \daleth_{n-1})] \\
&\qquad + \frac{\mu d(\aleph_{n-1}, g(\aleph_{n-1})) d(g(\daleth_{n-1}), \daleth_{n-1})}{1 + d(\aleph_{n-1}, g(\aleph_{n-1})) + d(g(\daleth_{n-1}), \daleth_{n-1})}
\end{aligned}$$

$$= \lambda[d(\aleph_{n-1}, \daleth_{n-1}) + d(\aleph_{n-1}, \daleth_{n-1}) + d(\aleph_n, \daleth_{n-1})] + \frac{\mu d(\aleph_{n-1}, \daleth_{n-1}) d(\aleph_n, \daleth_{n-1})}{1 + d(\aleph_{n-1}, g(\aleph_{n-1})) + d(g(\daleth_{n-1}), \daleth_{n-1})}$$

$$\precsim \lambda[d(\aleph_{n-1}, \daleth_{n-1}) + d(\aleph_{n-1}, \daleth_{n-1}) + d(\aleph_n, \daleth_{n-1})] + \mu d(\aleph_n, \daleth_{n-1})$$

so that

$$d(\aleph_n, \daleth_{n-1}) \precsim \frac{2\lambda}{1-\lambda-\mu} d(\aleph_{n-1}, \daleth_{n-1}),$$

Therefore, by putting $\alpha = \frac{2\lambda}{1-\lambda-\mu}$, we have

$$d(\aleph_n, \daleth_n) \precsim \alpha^{2n} d(\aleph_0, \daleth_0)$$

and

$$d(\aleph_n, \daleth_{n-1}) \precsim \alpha^{2n-1} d(\aleph_0, \daleth_0).$$

For every $m, n \in \mathbb{N}$,

$$\begin{aligned} d(\aleph_n, \daleth_m) &\precsim d(\aleph_n, \daleth_n) + d(\aleph_{n+1}, \daleth_n) + d(\aleph_{n+1}, \daleth_m) \\ &\precsim (\alpha^{2n} + \alpha^{2n+1}) d(\aleph_0, \daleth_0) + d(\aleph_{n+1}, \daleth_m) \\ &\precsim ... \\ &\precsim (\alpha^{2n} + \alpha^{2n+1} + ... + \alpha^{2m-1}) d(\aleph_0, \daleth_0) + d(\aleph_m, \daleth_m) \\ &\precsim (\alpha^{2n} + \alpha^{2n+1} + ... + \alpha^{2m}) d(\aleph_0, \daleth_0), \text{ if } m > n, \end{aligned}$$

$$|d(\aleph_n, \daleth_m)| \leq (\alpha^{2n} + \alpha^{2n+1} + ... + \alpha^{2m}) |d(\aleph_0, \daleth_0)|, \text{ if } m > n,$$

and similarly, if $m < n$, then

$$d(\aleph_n, \daleth_m) \precsim (\alpha^{2m+1} + \alpha^{2m+2} + ... + \alpha^{2n+1}) d(\aleph_0, \daleth_0),$$

$$|d(\aleph_n, \daleth_m)| \leq (\alpha^{2m+1} + \alpha^{2m+2} + ... + \alpha^{2n+1}) |d(\aleph_0, \daleth_0)|.$$

By $\alpha \in (0,1)$, $|d(\aleph_n, \daleth_m)| \to 0$, as $n, m \to \infty$, we get to the conclusion that $(\aleph_n, \daleth_n)$ is a Cauchy bisequence. Also, $(\aleph_n, \daleth_n)$ converges, and biconverges to a point $\wp \in G \cap H$, because (G, H, d) is complete. Hence, $g(\aleph_n) = \daleth_n \to \wp \in G \cap H$ as $n \to \infty$ implies $d(g(\wp), g(\aleph_n)) \to d(g(\wp), \wp)$ as $n \to \infty$, because of Lemma 2.9. Also by taking the limit from

$$d(g(\wp), g(\aleph_n)) \precsim \lambda[d(\aleph_n, \wp) + d(\aleph_n, \daleth_n) + d(g(\wp), \wp)] + \frac{\mu d(\aleph_n, \daleth_n) d(g(\wp), \wp)}{1 + d(\aleph_n, \daleth_n) + d(g(\wp), \wp)}$$

$$|d(g(\wp), g(\aleph_n))| \leq \lambda[|d(\aleph_n, \wp) + d(\aleph_n, \daleth_n) + d(g(\wp), \wp)|] + \frac{\mu |d(\aleph_n, \daleth_n) d(g(\wp), \wp)|}{|1 + d(\aleph_n, \daleth_n) + d(g(\wp), \wp)|}$$

as $n \to \infty$, we get $d(g(\wp), \wp) = 0$. Therefore $g(\wp) = \wp$, and $\wp$ is a fixed point of g.
If $\hbar$ is another fixed point of g, then $g(\hbar) = \hbar$, $\hbar \in G \cap H$, and hence,

$$d(\wp, \hbar) = d(g(\wp), g(\hbar)) \precsim \lambda[d(\wp, \hbar) + d(\wp, g(\wp)) + d(g(\hbar), \hbar)] + \frac{\mu d(\wp, g(\wp)) d(g(\hbar), \hbar)}{1 + d(\wp, g(\wp)) + d(g(\hbar), \hbar)}$$

Therefore $d(\wp, \hbar) = 0$ so that $\wp = \hbar$. So g has a UFP. □

If $\mu = 0$ in the previous Theorem, then we get next Corollary.

Corollary 3.6. *Let (G, H, d) be a complete CVBMS. If a contravariant map $g : (G, H, d) \rightleftarrows (G, H, d)$ satisfies $d(g(\daleth), g(\aleph)) \precsim \lambda[d(\aleph, \daleth) + d(\aleph, g(\aleph)) + d(g(\daleth), \daleth)]$, whenever $(\aleph, \daleth) \in G \times H$, for some $\lambda \in (0,1)$ with $3\lambda < 1$, then the function $g : G \cup H \to G \cup H$ has a UFP.*

4. Conclusions

It is possible to think of all fixed point theorems in complex valued bipolar metric spaces as generalisations of those in complex valued metric spaces. A generalisation of fixed point theorems in metric spaces can be made of all fixed point theorems in complex valued metric spaces. Therefore, it is crucial to study fixed point outcomes in complex valued bipolar metric spaces.

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Conflict of Interest

The authors have no conflict of interest regarding the publication of this article.

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