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Generalization of the Riesz-Markov-Kakutani Representation Theorem and Weak Spectral Families

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Abstract

Firstly, we generalize some definitions such as the definitions of the weak spectral family, the solitary operator, and the construction of the functional calculus. Secondly, we prove that for a functional calculus Φ on the measurable space $(Z,\, \Sigma)$ exists a measurable space $(\Omega, \mathrm{F}, \mu)$, an operator $U\; :\; X \to L^p\,(\Omega, \mathrm{F}, \mu)$, and a continuous $*$ -homomorphism $F\;\;:\;\; M\,(Z,\,\Sigma) \to M\,(\Omega, \mathrm{F})$, such that $M_{F(f)} = U^{-1}\Phi\,(f)\,U$ for all $f\,\in\, M\,(Z,\,\Sigma)$. Thirdly, we establish the correlation between the well-bounded operators and the weak spectral families. It has been proven that for a linear well-bounded operator $A \in L\,(X)$ there is a weak spectral family $\{E\,(\lambda) \in L\,(X^*)\,,\quad \lambda \in \mathbb{R}\}$ on a compact interval $[a,\, b]$ such that an integral representation $\langle A\,(x)\,,\; y^*\rangle = b\,\langle x,\; y^*\rangle - \int_{[a,\, b]} \langle x,\; E\,(\lambda)\,y^*\rangle\, d\lambda$ holds for all $x \in X$, $y^* \in X^*$, where equivalence is understood in the weak topology.

Keywords: functional calculus, Riesz–Kakutani theorem, spectral theorem, projection-valued measure, well-bounded operator, singular integral, representation theorem, spectral family, Stieltjes integral.
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1. Introduction

The important tool of quantum physics and its generalizations are differ-integral operators in a certain Banach space, for instance, the pseudodifferential operators, which emerge from the theory of partial differential equations, usually these equations describe the evolution of the quantum system[1, 2, 3, 4, 18, 19, 20]. If we define the self-adjoint operators on a Hilbert space then Neumann spectral theorem implies the existence of a Borel functional calculus corresponded with this operator, which defines homomorphism from the space of the Borel-measurable functions of a real variable into the space of linear operators on the Hilbert space[6, 17].

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Another approach to the self-adjoint operators and their generalization, well-bounded operators, gives the Riesz–Markov–Kakutani representation theorem and its generalization considered in the present article. The main idea of the Riesz–Markov–Kakutani representation theorem is to establish a one-to-one correspondence between the space of continuous linear functional on the set of the continuous functions, vanishing at infinity, on the locally compact Hausdorff space and space of regular countably additive complex Borel measures on that Hausdorff space, the norm of the functional coincides with the total variation of the measure.

The main idea of the spectral theorem is that every self-adjoint operator A can be expressed in form of the integral $\int \lambda \, dE(\lambda)$ with respect to projection-valued measure E, this representation of A presents a possibility to define a functional calculus as a pair (Φ, X), where X is a Banach space and Φ is a mapping from a functional space in the space of linear operators $X \to X$ such isomorphism can be defined by $f(A) = \int f(\lambda) \; dE(\lambda)$, and $\Phi(f) = f(A)$ or $f \mapsto f(A)$. From this definition, we can restore important functions of linear operators:

resolvent $R(\lambda) = (A - \lambda I)^{-1} = \int (s - \lambda)^{-1} dE(s), \quad Im\lambda = 0;$

semigroup $\exp(-i\lambda A) = \int \exp(-i\lambda s) \, dE(s), \quad \lambda \in (-\infty, \infty)$.

The functional classes $R(\lambda)$, $\exp(-i\lambda\cdot)$ and $E(\lambda)$, which is called functional triplet, can be defined axiomatically by their properties as follows:

- The resolvent must satisfy the equality

$$R(\lambda) - R(\mu) = (\lambda - \mu) R(\lambda) R(\mu);$$

- The semigroup

$$\exp(-i\lambda\cdot)\exp(-i\mu\cdot) = \exp(-i(\lambda+\mu)\cdot), \quad \exp(-i0\cdot) = I;$$

- The projection-valued measure $E(\lambda)$ must satisfy the equalities

$$E(\lambda) E(\mu) = E(\min\{\lambda, \mu\}), \lim_{\lambda \to -\infty} E(\lambda) = 0$$

and

$$\lim_{\lambda \to \infty} E(\lambda) E(\infty) = I.$$

The concept of functional calculus presents a convenient apparatus for studying these functional classes from a consistent cohesive perspective, the isomorphism Φ can be defined based not only on $E(\lambda)$ as above but on either of the A representation, namely, $R(\lambda)$, $\exp(-i\lambda\cdot)$ or $E(\lambda)$. The functional calculus allows generalization of the definitions of each function of the triplet and establishes further their properties such as ergodic theorem, behavior on infinity, and in particular that the null space $N(R(\lambda)) = N(\exp(-i0\cdot)) = E(\infty) = I$.

The definition of well-bounded operators on the reflexive Banach spaces generalizes the concept of self-adjoint operators on the Hilbert spaces, the operator A is said to be well-bounded if for every polynomial P there is a compact interval $[a, b]$ and a positive constant c such that

$$\|P(A)\| \le c \left(\sup\{|P(\lambda)|, \quad \lambda \in [a, b]\} + \underset{[a,\, b]}{var} P \right).$$

The well-bounded operators are similar to the self-adjoint operators in the sense that the well-bounded operators have conditional integral representations with respect to the spectral decomposition, so well-bounded operators yield a functional calculus for the absolutely continuous functions on the $[a, b]$.

In this paper, we have shown that for a functional calculus (Φ, X) on the measurable space (Z, Σ), exist a measurable space $(\Omega, \mathrm{F}, \mu)$, a mapping $U : X \to L^p(\Omega, \mathrm{F}, \mu)$, and a continuous $*$ -homomorphism $F : M(Z, \Sigma) \to M(\Omega, \mathrm{F})$, such that for all $f \in M(Z, \Sigma)$, we have the following equivalence $M_{F(f)} = U^{-1}\Phi(f) U$.

Also, it has been proven that for a linear well-bounded operator $A \in L(X)$ exists a weak spectral family $\{E(\lambda) \in L(X^*), \quad \lambda \in \mathbb{R}\}$ on $[a,\ b]$ such that a weak integral representation $\langle A(x),\ y^* \rangle = b\langle x,\ y^* \rangle - \int_{[a,\ b]} \langle x,\ E(\lambda) y^* \rangle \, d\lambda$ holds for all $x \in X$, $y^* \in X^*$.

Since Borel measure μ can be decomposed as $\mu = \mu_p \oplus \mu_{ac} \oplus \mu_{\sin g}$ we have proven that The Banach space X can be represented as $X = X_p \oplus X_{ac} \oplus X_{\sin g}$.

The correlation between the well-bounded operators and the weak spectral families according is established by

$$\langle A(x),\ y^* \rangle = b\langle x,\ y^* \rangle - \int_R \langle x,\ E(\lambda) y^* \rangle \, d\lambda,$$

where equivalence is understood in the weak topology.

2. Definitions and Preliminaries

Definition 2.1. An operator-function $E(\lambda)$ is called a spectral decomposition of the operator A if the set $\{E(\lambda), \quad \lambda \in \mathbb{R}\} \subset LB(X, X)$ satisfies the following conditions:

1. $E(\lambda) E(\mu) = E(\mu) E(\lambda) = E(\lambda)$ for $\lambda \le \mu$; and $\sup_{\lambda} \|E(\lambda)\| < \infty$;
2. $E(\lambda) = strong - \lim_{\lambda < \mu, \quad \mu \to \lambda} E(\mu)$;
3. $strong - \lim_{\lambda \to -\infty} E(\lambda) = O$ and $strong - \lim_{\lambda \to \infty} E(\lambda) = I$;
4. $A = \int_{\mathbb{R}} \lambda \, dE(\lambda) = strong - \lim_{N \to \infty} \int_{[-N,\ N]} \lambda \, dE(\lambda)$,

Where the integral is an operator-valued Riemann-Stieltjes integral in the topology of the operator norm. We are going to consider the integral $\int_{[a,\ b]} f(\lambda) \ dE(\lambda)$ as an operator-valued Riemann-Stieltjes integral. Let P be a partition of the interval $[a,\ b]$ as follows $a = \lambda_0 < \lambda_1 < ... < \lambda_n = b$ and the direction of the partition $|P| = \max_{i=1,\dots,n} |\lambda_i - \lambda_{i-1}|$ then if for any chosen set $\{\xi_i\}_{1,\dots,n}$ of points $\xi_i \in [\lambda_{i-1},\ \lambda_i]$ there is a limit

$$\lim_{|P| \to 0} \sum_{i=1,\dots,n} f(\xi_i) \left(E(\lambda_i) - E(\lambda_{i-1})\right),$$

and this limit is independent of the specifics of the partitions, this limit is called the Riemann-Stieltjes integral of the continuous function f, and can be written as

$$\int_{[a,\ b]} f(\lambda) \ dE(\lambda) = \lim_{|P| \to 0} \sum_{i=1,\dots,n} f(\xi_i) \left(E(\lambda_i) - E(\lambda_{i-1})\right).$$

Definition 2.2. Let function $f \in BV([a,b])$ and let $\{E(\lambda), \quad \lambda \in R\}$ be a spectral decomposition concentrated on the compact interval $[a,b]$, the integral of the function $f \in BV([a,b])$ with respect to the spectral decomposition $\{E(\lambda)\}$ is

$$\int_{[a,b]}^{\oplus} f(\lambda) \, dE(\lambda) =$$

$$= strong - \lim_{\lambda \in \Pi} \left(f(a) E(a) + \sum_{i=1,2,\dots,n} f(\lambda_i) \left(E(\lambda_i) - E(\lambda_{i-1})\right) \right).$$

This definition is correct since assume $\{E(\lambda), \quad \lambda \in R\}$ is a spectral decomposition concentrated on the compact interval $[a,b]$ and $f \in BV([a,b])$, we have

$$f(a) E(a) + \sum_{i=1,2,\dots,n} f(\lambda_i) \left(E(\lambda_i) - E(\lambda_{i-1})\right) =$$
$$= f(b) E(b) - \sum_{i=1,2,\dots,n} \left(f(\lambda_i) - f(\lambda_{i-1})\right) E(\lambda_{i-1})$$

for any finite partition

$$\Pi(\lambda) = \{a = \lambda_0 < \lambda_1 < \lambda_2 < ... < \lambda_{n-1} < \lambda_n = b\}$$

of the compact interval $[a, b]$.

Let X be a reflexive Banach space then an operator $A \in L(X,\ X)$ can be defined by the operator-valued Riemann-Stieltjes integral

$$A = \int_{\sigma(A)} \lambda \, dE(\lambda),$$

where $E()$ is a spectral family of the operator A.

Let us assume that function f is absolutely continuous on the interval $[a,\ b]$ and let a spectral family $E(\lambda)$ be concentrated on the interval $[a,\ b]$, then the mapping $\Phi\ :\ AC([a,\ b]) \to L(X,\ X)$, that is prescribed according to formula

$$\Phi(f) = f(a)\ E(a) + \int_{[a,\ b]} f(\lambda)\ dE(\lambda),$$

is a functional calculus for the operator $A \in L(X,\ X)$. The integral in the last formula is understood in the Riemann-Stieltjes sense, this integral construction is correctly defined if $\|E(\lambda)\| \in BV([a,\ b])$ as a function of the parameter λ.

Example 2.3. Let H be Hilbert space and let $A \in L(H,\ H)$ be self-adjoint operator, then we can define the following function of operator A:

1. The resolvent $R(\lambda, A) = (\lambda - A)^{-1} = \int_{(-\infty,\ \infty)} (\lambda - \xi)^{-1}\ dE(\xi)$,
$Im\lambda \neq 0$;
2. The continuous semigroup $\exp(-i\lambda A) = \int_{(-\infty,\ \infty)} \exp(-i\xi\lambda)\ dE(\xi)$.

Definition 2.4. A solitary operator is a bounded linear surjective operator $U\ :\ X \to X$ on a Banach space, which for all $x \in X$ and $y \in X^*$ satisfies the following equality

$$\langle Ux, U^*y\rangle = \langle x, y\rangle,$$

where $U^* :\ X^* \to X^*$.

3. Functional calculus on the measurable spaces

One of the main ideas of the spectral theorem is that every bounded self-adjoint operator A on a Hilbert space H can be represented as the multiplication operator M_φ, so, there is a measurable space $(Z,\ \Sigma)$ and an essentially bounded measurable function φ on this space $(Z,\ \Sigma,\ \mu)$, and a unitary operator $U\ :\ H \to L^2(Z,\ \Sigma,\ \mu)$ that transforms the Hilbert space in the Lebesgue space such that $U^*M_\varphi U = A$.

Generalizing this idea, we are obtaining the following result.

Theorem 3.1. *Assume $(\Phi,\ X)$ is a measurable functional calculus on the measurable space $(Z,\ \Sigma)$, where Σ is a sigma-algebra. Then there are a semi-finite measure space $(\Omega, \mathrm{F}, \mu)$, a solitary operator $U\ :\ X \to L^p(\Omega, \mathrm{F}, \mu)$, and an injective pointwise continuous $*$ -homomorphism $F\ :\ M(Z,\ \Sigma) \to M(\Omega, \mathrm{F})$, such that the following equality*

$$M_{F(f)} = U^{-1}\Phi(f)\,U$$

holds for all functions $f \in M(Z,\ \Sigma)$, where M_{Ff} is the operator of the multiplication by f.

Proof. For arbitrary set $A \in \Sigma$, we define measure $\mu_x(A) = \langle\Phi(\chi_A)\,x, x^*\rangle$ as a function of $x \in X$, so there is equality $\langle\Phi(f)\,x, x^*\rangle = \langle\Phi(f)\rangle_{\mu_x}$ for every bounded f. Next, for every bounded f, we define the space $B_x = [\{\Phi(f)\,x\ ,\ f \in M_b(Z,\ \Sigma)\}]$, thus there is a solitary operator
$W_x\ :\ L^p(Z, \Sigma, \mu_x) \to B_x$ as an extension of mappings $M_b(Z,\ \Sigma) \to B_x$ and $f \to \Phi(f)\,x$.

Let $\{x_i\}$ and $\{x_i^*\}$ be two sets of unit vectors in X and X^* spaces, respectively, with properties

$$\langle x_k, x_k^* \rangle = \|x_k\| \, \|x_k^*\|_* = 1 \quad \forall k \in N$$

and

$$\langle x_i, x_k^* \rangle = 0$$

for every $i \neq k$.

For every k, we can define the set $Z_k = Z \times \{k\}$ as an exemplar of Z then the set Ω can be represented as the disjoint union $\bigcup_k Z_k$. Let

Let us define an additive set function μ by the formula

$$\mu(A) = \sum_k \mu_{x_k}(A \cap Z_k) \quad \forall A \in \mathrm{F}.$$

The additive set function μ is the measure on the maximal sigma-algebra F on Ω, which includes all measurable mapping $Z_k = Z \times \{k\}$ into Ω.

The operator W_{x_k} is correctly defined on $L^p(Z_k, \Sigma, \mu_{x_k})$ and maps $W_{x_k} : L^p(Z_{x_k}, \Sigma, \mu_{x_k}) \to B_{x_k}$, so, we define the operator $U : X \to L^p(\Omega, \mathrm{F}, \mu)$ by the condition $U^{-1} = W_{x_k}$on $L^p(Z_k, \Sigma, \mu_{x_k}) \subseteq L^p(\Omega, \mathrm{F}, \mu)$.

Then the $*$-homomorphism $F : M(Z, \Sigma) \to M(\Omega, \mathrm{F})$, we introduce by the formula

$$(F\ f)(x, k) = f(x), \quad x \in X.$$

For all $f \in (Z, \Sigma)$, we define the multiplication operator calculus as $M_{Ff} = U\Phi(f)U^{-1}$, so the theorem has been proven. □

Theorem 3.2. *Let X be a reflexive Banach space and let the operator $A \in L(X)$ be well-bounded then there is a unique spectral family $E(\cdot)$ in X such that*

$$A = a\,E(a) + \int_{[a,\,b]} \lambda\, dE(\lambda).$$

In theorem 2, the spectral family $E(\cdot)$ is concentrated on a compact interval $[a, b]$.

Proof. Let us define a functional calculus $\Upsilon : AC([a, b]) \to LB(X)$. We define a set $F(\lambda, \eta)$ of all real-valued absolutely continuous functions $f \in AC([a, b])$ such that

$$f = \begin{cases} 1 \quad \text{on} \quad [a, \lambda] \\ \text{decreasing} \quad on \quad [\lambda, \lambda+\eta] \\ 0 \quad \text{on} \quad [\lambda+\eta, b] \end{cases}$$

for all $\lambda \in [a, b)$ and $0 < \eta < (b-\lambda)$. Next, we have $\|f\|_{Bound} \leq 1$ for any $f \in F(\lambda, \eta)$. The class $K(\lambda, \eta)$ can be defined as a closure in the weak topology

$$K(\lambda, \eta) = weak\ cl\,\{\Upsilon(f) : f \in F(\lambda, \eta)\} \subset LB(X^*)$$

For $\eta_1 < \eta_2$ we obtain $K(\lambda, \eta_1) \subset K(\lambda, \eta_2)$ and it can be deduced that set $K(\lambda) = \bigcap_{\eta>0} K(\lambda, \eta)$ is a weakly compact uniformly bounded set.

The set Z is a subset of the reflexive Banach space defined by the formula

$$Z(\lambda) = \left\{ x \in X : \Upsilon(f)\,x = 0, \quad \text{for} \quad \text{all} \quad f \in \bigcup_{\eta>0} (1 - F(\lambda, \eta)) \right\}.$$

Let $y \in Z(\lambda) \in K(\lambda) = \bigcap_{\eta>0} K(\lambda,\ \eta)$ then there is a net $\{g_\alpha\}_{\alpha\in\Lambda} \subset K(\lambda,\ \eta)$ with the following property

$$\langle Ex,\ y^*\rangle = \lim_{\alpha\in\Lambda} \langle \Upsilon(g_\alpha)\, x,\ y^*\rangle = \lim_{\alpha\in\Lambda} \langle (1-\Upsilon(1-f_\alpha))\, x,\ y^*\rangle$$

for all $x \in X$. Since $\langle Ex,\ y^*\rangle = \langle x,\ y^*\rangle$ we have $x \in\ Rang\,(E)$thus set $Z(\lambda)$ is the range of each $K(\lambda) = \bigcap_{\eta>0} K(\lambda,\ \eta)$.

For any $\theta > 0$, there is $\eta_0 > 0$ such that $0 \le f(t) \le \frac{\theta}{2}$ for all $t \in [\lambda,\ \lambda+\eta_0]$, so for $E \in K(\lambda,\ \eta_0)$ there is a net $\{g_\alpha\}_{\alpha\in\Lambda} \subset F(\lambda,\ \eta_0)$ with the property $weak - \lim\limits_{\alpha\in\Lambda} \Upsilon(g_\alpha) = E$.

Now, we are going to apply the fourth condition of the definition

$$\begin{aligned}&\int_{[a,\ b]} \left|(fg_\alpha)'\right| = \int_{[a,\ b]} \left|f' g_\alpha + f\ g_\alpha{}'\right| \le\\ &\le \int_{[\lambda,\ \lambda+\eta_0]} \left|f' g_\alpha\right| + \int_{[\lambda,\ \lambda+\eta_0]} \left|+f\ g_\alpha{}'\right| \le\\ &\le \frac{\theta}{2} + \frac{\theta}{2} = \theta,\end{aligned}$$

so

$$\begin{aligned}&|\langle\ \Upsilon(f)\, x,\ x^*\rangle| \le |\langle \Upsilon(f)\ \ x,\ y^*\rangle| = |\langle\ \Upsilon(f)\, Ex,\ y^*\rangle| =\\ &= |\langle Ex, (\Upsilon(f))^*\, y^*\rangle| = \left|\lim_{\alpha\in\Lambda} \langle \Upsilon(g_\alpha)\, x,\ (\Upsilon(f))^*\, y^*\rangle\right| =\\ &= \left|\lim_{\alpha\in\Lambda} \langle \Upsilon(f\ g_\alpha)\, x,\ y^*\rangle\right| \le \underset{\alpha\in\Lambda}{sub} \|\Upsilon(f\ g_\alpha)\|\ \ \|x\|\ \ \|y^*\|\end{aligned}$$

for all $y^* \in X^*$ so $E \in K(\lambda) = \bigcap_{\eta>0} K(\lambda,\ \eta)$. Thus, from the inequality $|\langle\ \Upsilon(f)\, x,\ y^*\rangle| \le \theta\, \|\Upsilon\|\ \ \|x\|\ \ \|y^*\|$ follows $\Upsilon(f)\, x = 0$, so the range of E coincides with $Z(\lambda)$; the set E is a projection.

Let us establish that $K(\lambda,\ \eta)$ is a commutative multiplicative semigroup. Let $\widehat{K},\ \breve{K} \in K(\lambda,\ \eta)$, we have that there are nets $\{g_\alpha\}_{\alpha\in\Lambda},\ \{h_\beta\}_{\beta\in\mathrm{B}} \in F(\lambda,\ \eta)$ such that

$$\widehat{K} = weak - \lim_{\alpha\in\Lambda} \Upsilon(g_\alpha)$$

and

$$\breve{K} = weak - \lim_{\beta\in\mathrm{B}} \Upsilon(h_\beta)\,.$$

For all $x \in X$, we have

$$\begin{aligned}&\left\langle \widehat{K}\ \ \breve{K}x,\ y^*\right\rangle = \lim_{\alpha\in\Lambda} \left\langle \Upsilon(g_\alpha)\ \ \breve{K}x,\ y^*\right\rangle =\\ &= \lim_{\alpha\in\Lambda} \left\langle \breve{K}x,\ (\Upsilon(g_\alpha))^*\, y^*\right\rangle = \lim_{\alpha\in\Lambda} \left\{\lim_{\beta\in\mathrm{B}} \langle \Upsilon(h_\beta)\ \ x,\ (\Upsilon(g_\alpha))^*\, y^*\rangle\right\} =\\ &= \lim_{\alpha\in\Lambda} \left\{\lim_{\beta\in\mathrm{B}} \langle \Upsilon(g_\alpha h_\beta)\, x,\ y^*\rangle\right\} = \lim_{\alpha\in\Lambda} \left\{\lim_{\beta\in\mathrm{B}} \langle\ \Upsilon(h_\beta)\, \Upsilon(g_\alpha)\, x, y^*\rangle\right\} =\\ &= \lim_{\alpha\in\Lambda} \left\langle \breve{K}\Upsilon(g_\alpha)\, x,\ \ y^*\right\rangle = \lim_{\alpha\in\Lambda} \left\langle \Upsilon(g_\alpha)\, x,\ \left(\breve{K}\right)^* y^*\right\rangle =\\ &= \left\langle \widehat{K}\ \ x,\ \left(\breve{K}\right)^* y^*\right\rangle = \left\langle\ \breve{K}\ \widehat{K}\ x,\ \ y^*\right\rangle,\end{aligned}$$

so $\widehat{K}\ \ \breve{K} = \breve{K}\ \widehat{K}$, thus $E(\lambda) \in K(\lambda) = \bigcap_{\eta>0} K(\lambda,\ \eta)$, uniqueness is following from the properties of the projections. We define the set of the projection $\{E(\lambda)\}_{\lambda\in[a,b]}$ on X by presuming $E(\lambda) = O$ for $\lambda < a$ and $E(\lambda) = I$ for $\lambda > b$.

Now, let us establish the properties of $\{E(\lambda)\}_{\lambda\in[a,b]}$. Assuming that $a \le \lambda < \mu < b$, and assuming η is large enough, we are going to obtain that from $E(\lambda),\ E(\mu) \in K(\lambda,\ \eta)$ follows $E(\lambda)\, E(\mu) = E(\mu)\, E(\lambda) = E(\lambda)$. If $\eta = \mu - \lambda$, then from $E(\lambda) \in K(\lambda,\ \eta)$ follows existence of the nets $\{g_\alpha\}_{\alpha\in\Lambda} \ \in F(\lambda,\ \eta)$ and

$\{h_\beta\}_{\beta\in \mathrm{B}} \in F(\lambda,\ \eta)$ with the properties $weak - \lim\limits_{\alpha\in\Lambda} \Upsilon(g_\alpha) = E(\lambda)$ and $weak - \lim\limits_{\beta\in\mathrm{B}} \Upsilon(h_\beta) = E(\mu)$. Next, since $g_\alpha h_\beta = g_\alpha$ we have

$$\begin{aligned}
&\langle E(\lambda)\ E(\mu)x,\ y^*\rangle = \lim_{\alpha\in\Lambda} \langle \Upsilon(g_\alpha)\ E(\mu)x,\ y^*\rangle = \\
&= \lim_{\alpha\in\Lambda} \langle\ E(\mu)x,\ (\Upsilon(g_\alpha))^* y^*\rangle = \lim_{\alpha\in\Lambda} \left\{ \lim_{\beta\in\mathrm{B}} \langle\ \Upsilon(h_\beta)x,\ (\Upsilon(g_\alpha))^* y^*\rangle \right\} = \\
&= \lim_{\alpha\in\Lambda} \left\{ \lim_{\beta\in\mathrm{B}} \langle\ \Upsilon(g_\alpha)\Upsilon(h_\beta)x,\ y^*\rangle \right\} = \\
&= \lim_{\alpha\in\Lambda} \left\{ \lim_{\beta\in\mathrm{B}} \langle\ \Upsilon(g_\alpha h_\beta)x,\ y^*\rangle \right\} = \lim_{\alpha\in\Lambda} \left\{ \lim_{\beta\in\mathrm{B}} \langle\ \Upsilon(g_\alpha)x, y^*\rangle \right\}
\end{aligned}$$

for all $x \in X,\ y^* \in X^*$. Thus, it has been obtained $\langle E(\lambda)\ E(\mu)x,\ y^*\rangle = \langle E(\lambda)x,\ y^*\rangle$ and so equality of projection

$$E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\lambda)$$

holds for all $a \le \lambda < \mu < b$.

Since $strong - \lim\limits_{\mu\to\lambda+0} E(\mu) = E(\lambda+0)$ we have $E(\lambda+0) \in K(\lambda)$.

For any pair $x \in X,\ y^* \in X^*$ and any $f \in AC([a,\ b])$, the morphism $f \mapsto \langle\ \Upsilon(f)x,\ y^*\rangle$ is an element of the dual space to $AC([a,\ b])$ and since $AC([a,\ b])$ is isometric to $L^1([a,\ b]) \oplus C$, from the duality argument, we have that there are $\gamma\langle x,\ y^*\rangle \in L^\infty([a,\ b])\ \tilde{c}\langle x,\ y^*\rangle \in C$, which satisfy the following equality

$$\langle \Upsilon(f)x,\ y^*\rangle = \tilde{c}\langle x,\ y^*\rangle f(b) + \int_{[a,\ b]} f'(t)\gamma\langle x,\ y^*\rangle(t)\,dt$$

for all $f \in AC([a,\ b])$.

For any $\lambda \in [a,\ b)$, we assume $0 < \lambda+\eta < b$ then the function

$$g(\lambda,\eta)(t) = \begin{cases} 1 \quad \text{on} \quad [a,\ \lambda] \\ \text{ecreasing} \quad on \quad [\lambda,\ \lambda+\eta] \\ 0 \quad on \quad [\lambda+\eta,\ b] \end{cases}$$

belongs to $F(\lambda,\ \eta)$ and

$$\langle \Upsilon(g(\lambda,\eta))x,\ y^*\rangle = -\frac{1}{\eta}\int_{[\lambda,\ \lambda+\eta]} \gamma\langle x,\ y^*\rangle(t)\,dt.$$

Thus, there is a weak limit $g(\lambda,\eta) \overset{weak-\eta\to 0+}{\longrightarrow} E(\lambda)$.

So, λ - almost everywhere, we obtain $\gamma\langle x,\ y^*\rangle(\lambda) = -\langle E(\lambda)x,\ y^*\rangle$, and for arbitrary $x \in X,\ y^* \in X^*$, the integral equality

$$\langle \Upsilon(f)x,\ y^*\rangle = \langle x,\ y^*\rangle f(b) - \int_{[a,\ b]} f'(\lambda)\langle E(\lambda)x,\ y^*\rangle\,d\lambda$$

holds for all $f \in AC([a,\ b])$.

Next, we have

$$\begin{aligned}
&\left\langle \left(\textstyle\int_{[a,\ b]}^{\oplus} f\ dE\right)x,\ y^*\right\rangle = \\
&= \lim_{\Lambda\in\Pi} \{\langle E(b)x,\ y^*\rangle f(b) - \langle \textstyle\sum_\Lambda (f(\lambda_i) - f(\lambda_{i-1}))E(\lambda_i)x,\ y^*\rangle\} = \\
&= \langle x,\ y^*\rangle f(b) - \lim_{\Lambda\in\Pi} \{\textstyle\sum_\Lambda (f(\lambda_i) - f(\lambda_{i-1}))\langle E(\lambda_i)x,\ y^*\rangle\} = \\
&= \langle x,\ y^*\rangle f(b) - \textstyle\int_{[a,\ b]} f'(\lambda)\langle E(\lambda)x,\ y^*\rangle\,d\lambda = \langle \Upsilon(f)x,\ y^*\rangle.
\end{aligned}$$

Thus, by taking $f(\lambda) = \lambda$, we have

$$\langle Ax,\ y^*\rangle = b\langle x,\ y^*\rangle - \int_{[a,\ b]} \langle E(\lambda)x,\ y^*\rangle\,d\lambda.$$

□

The projection-valued measure can be thought as a measure whose values are projection on certain Banach space, assuming $x \in X,\ x^* \in X^*$ then the formula

$$\langle Bx,\ \ x^* \rangle = \int_{\sigma(A)} f(\lambda) \langle E(\lambda) x,\ x^* \rangle \, d\lambda$$

represents the mapping $f \mapsto f(A)$ and defines a Borel functional calculus as measurable functional calculus over Borel algebra. Let f be a Borel measurable function on the spectrum of the operator A so the functional calculus can be defined by $\Phi(f) = f(A)$ and $f \mapsto f(A)$ is given by integral representation.

The theorem of this type can be formulated in the weak form as follows.

Definition 3.3. The set $\{E(\lambda) \in L(X^*), \quad \lambda \in \mathbb{R}\}$ of projection operators that satisfies the following conditions

1. $E(\cdot)$ is concentrated on a compact interval $[a,\ b]$;
2. $E(\lambda) E(\mu) = E(\mu) E(\lambda) = E(\lambda)$ for $\lambda \leq \mu$; and $\sup\limits_{\lambda} \|E(\lambda)\| < \infty$;
3. $E(\lambda) = \mathrm{O}$ for all $\lambda < a$ and $E(\lambda) = I$ for all $b < \lambda$;
4. there is $\lim\limits_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{[t,\ t+\varepsilon]} \langle x,\ E(\lambda) y^* \rangle \, d\lambda = \ \langle x,\ E(t) y^* \rangle$ for all $x \in X,\ y^* \in X^*$ and for all $t \in (a,\ b)$

is called a weak spectral family.

Theorem 3.4. *Let $A \in L(X)$ be linear well-bounded operator then there is unique weak spectral family $\{E(\lambda) \in L(X^*), \quad \lambda \in \mathbb{R}\}$ concentrated on a compact interval $[a,\ b]$ such that the equality*

$$\langle A(x),\ y^* \rangle = b \langle x,\ y^* \rangle - \int_{[a,\ b]} \langle x,\ E(\lambda) y^* \rangle \, d\lambda$$

holds for all $x \in X,\ y^* \in X^*$.

The proof the theorem 3 is analogous to the proof of theorem 2 and can be found in our previous works.

Now, we have two statements: first, let A be a well-bounded operator on a reflexive Banach space X then there is a corresponding unique spectral family $E(\lambda)$, projection-valued measure, in X such that $A = \int_R \lambda \, dE(\lambda)$, and the functional calculus $\Phi\ :\ AC \to L_C(X, X)$ can be defined as $f(A) = \int_R f(\lambda)\ dE(\lambda)$; second, let $(\Phi,\ X)$ be a measurable functional calculus on the measurable space $(Z,\ \Sigma)$, then there is continuous $*$-homomorphism F from $M(Z,\ \Sigma)$ in $M(\Omega, \mathrm{F})$ with the property $M_{F(f)} = U^{-1} \Phi(f) U$, where $U\ :\ X \to L^p(\Omega, \mathrm{F}, \mu)$. These two statements can be combined into one theorem.

Thus, the properties of the measurable functional calculi correlate with the properties of the multiplication operators.

Next, we are going to consider the weak generalization of the Riesz-Markov-Kakutani theorem. Let $x \in X,\ x^* \in X^*$ then the mapping $f \mapsto \langle x,\ f(A) x^* \rangle$ is a positive linear functional over the spectrum of the operator A. Then, according to Riesz–Markov–Kakutani representation theorem, there is a measure $\mu_{x,\,x^*} = \mu(x,\ x^*)$ defined on the compact set of the spectrum of the operator A such that $\langle x,\ f(A) x^* \rangle = \int_{\sigma(A)} f(\lambda)\ d\mu_{x,\,x^*}$.

Since any Borel measure μ can be presented as the sum of a pure point measure μ_p, absolutely continuous measure μ_{ac} with the respect to the Lebesgue measure, and singular measure $\mu_{\sin g}$ with the respect to the Lebesgue measure, then space $L^p(\Omega, \mathrm{F}, \mu)$ can be represented as a direct sum $L^p(\Omega, \mathrm{F}, \mu) = L^p(\Omega, \mathrm{F}, \mu_p) \oplus L^p(\Omega, \mathrm{F}, \mu_{ac}) \oplus L^p(\Omega, \mathrm{F}, \mu_{\sin g})$.

By x^*, we denote $y \in X^*$ such that $\langle x,\ y \rangle = \|x\| \|y\|$. So, we are obtaining the following theorem.

Theorem 3.5. *The Banach space X can be presented as a direct sum*

$$X = X_p \oplus X_{ac} \oplus X_{\sin g},$$

where $X_p = \{x \ : \ \mu(x,\, x^*(x)) = \mu_p\}$, $X_{ac} = \{x \ : \ \mu(x,\, x^*(x)) = \mu_{ac}\}$ *and* $X_{\sin g} = \{x \ : \ \mu(x,\, x^*(x)) = \mu_{\sin g}\}$. *The restriction of the operator* A *on the subspace* X_p *has a complete set of the eigenvectors, the restriction of* A *on the subspace* X_{ac} *has only absolutely continuous measures and the restriction of* A *on the subspace* $X_{\sin g}$ *has only singular spectral measures.*

Theorem 3 gives us the invertible correspondence between the linear well-bounded operators and the weak spectral families according to the following equivalence

$$\langle A(x)\,,\ y^* \rangle = b \langle x,\ y^* \rangle - \int_R \langle x,\ E(\lambda)\, y^* \rangle \, d\lambda$$

that holds in the weak topology.

Conflict of Interest

The authors have no conflict of interest regarding the publication of this article.

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