Available online at http://rgnaa.com
Commun. Nonlinear Anal. 2 (2023), 1–15
Research Article

Communications in Nonlinear Analysis
Publisher
Research Group of Nonlinear Analysis and Applications

Convergence Analysis Monotone Hybrid Algorithms for Countable Family of Generalized Nonexpansive Mappings and Maximal Monotone Operators

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Abstract

A new countable family of generalized nonexpansive mappings is introduced and a new monotone hybrid algorithm is presented in the framework of Banach spaces. Some new results are obtained for the class of generalized nonexpansive mappings and countable family of generalized nonexpansive mappings. The study exhibits the procedure for obtaining a common element of the zero point set of a maximal monotone operator and the newly introduced countable family of generalized nonexpansive mappings.

Keywords: Generalized, Maximal monotone, Nonexpansive, Retraction, Algorithm.
2010 MSC: 47H05, 47H10, 47H09, 47J25, 47J05.

1. Introduction

Let E be a real Banach space and let C be a nonempty closed convex subset of E. The dual of E will be denoted by E^*. Let $\mathbb{N}$ and $\mathbb{R}$, respectively be the sets of all positive integers and real numbers. A self mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tu - Tv\| \leq \|u - v\|, \text{ for all } u, v \in C,$$

and a mapping $T : C \to E$ is said to be generalized nonexpansive provided $F(T) \neq \emptyset$ and

$$\phi(p, Tu) \leq \phi(p, u) \text{ for all } u \in C \text{ and } p \in F(T),$$

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Received 2023-01-30

where $F(T) := \{u : Tu = u\}$, is the set of fixed points of T. An eminent class of the class of nonlinear operators is the class of nonexpansive mappings. The iterative processes on the class of nonexpansive mappings have been successfully applied in several areas such as signal processing and image restoration (see, e.g., [5, 4]). Let $T : C \to C$ be a nonexpansive self-mapping in a Hilbert space H. In 2008, Qin and Su [6] presented a monotone hybrid method, defined as

$$\begin{cases} u_1 = u \in C, C_0 = Q_0 = C, \\ x_n = \beta_n u_n + (1-\beta_n)Tu_n, \\ C_n = \{x \in C_{n-1} \cap Q_{n-1} : \|x - x_n\| \leq \|x - u_n\|\} \\ Q_n = \{x \in C_{n-1} \cap Q_{n-1} : \langle u_n - x, u - u_n\rangle \geq 0\} \\ u_{n+1} = P_{C_n \cap Q_n} u, n \in \mathbb{N}, \end{cases}$$

where $P_C : H \to C$ be a metric projection from H onto C and showed that the sequence $\{u_n\}$ converges strongly under suitable conditions. Monotone hybrid iterative method has also been considered in a uniformly smooth and uniformly convex Banach space E for a family of generalized nonexpansive mappings, say $\{S_n\}$ which satisfies NST-condition. Klin-eam et al. [11] defined $\{S_n\}$ from a generalized nonexpansive mapping $T : C \to E$ by

$$S_n u = \alpha_n u + (1-\alpha_n)\, Tu$$

and also from the generalized nonexpansive mappings T and $G : C \to E$ by

$$S_n u = \alpha_n Tu + (1-\alpha_n)\, Gu,$$

for all $u \in C$ and $\{\alpha_n\} \subset [0, 1]$. Then the iterative sequence $\{u_n\}$ is defined by

$$\begin{cases} u_1 = u \in C, C_0 = Q_0 = C, \\ x_n = \beta_n u_n + (1-\beta_n)S_n u_n, \\ C_n = \{x \in C_{n-1} \cap Q_{n-1} : \phi(x, x_n) \leq \phi(x, u_n)\} \\ Q_n = \{x \in C_{n-1} \cap Q_{n-1} : \langle u_n - x, Ju - Ju_n\rangle \geq 0\} \\ u_{n+1} = R_{C_n \cap Q_n} u, \end{cases}$$

where J denotes the duality mapping on E, $R_{C_n \cap Q_n}$ is the sunny nonexpansive retraction from C onto $C_n \cap Q_n$. Let $N \in \mathbb{N}$, Alizadeh and Moradlou [2], considered the iterative sequence which is given by

$$\begin{cases} u_1 = u \in C, \;\; C_0 = Q_0 = C, \\ x_n = \lambda_n u_n + (1-\lambda_n)S_n u_n, \\ y_n = \beta_n x_n + (1-\beta_n)S_n u_n, \\ C_n = \{x \in C_{n-1} \cap Q_{n-1} : \varphi(x, y_n) \leq \varphi(x, u_n)\} \\ Q_n = \{x \in K_{n-1} \cap Q_{n-1} : \langle u_n - x, Ju - Ju_n\rangle \geq 0\} \\ u_{n+1} = R_{C_n \cap Q_n} u, \end{cases}$$

and defined $\{S_n\}$ by

$$S_n u = \sum_{k=1}^{N} \alpha_{kn} G_k u \;\forall\; u \in C,$$

where $G_1, G_2, G_3, ..., G_N$ are generalized nonexpansive mappings of C into E such that $\bigcap_{n=1}^{\infty} F(G_n) \neq \emptyset$ and the real sequences $\{\lambda_n\}, \{\beta_n\}$ and $\{\alpha_{kn}\}_{k=1}^{N}$ in $[0, 1]$ satisfying $\liminf_{n\to\infty} (1-\lambda_n) > 0$, $\lim_{n\to\infty} \beta_n = 1$,

$\sum_{k=1}^{N} \alpha_{kn} = 1$ for all $n \in \mathbb{N}$ and $\liminf_{n\to\infty} \alpha_{in}\alpha_{jn} > 0$ for all $i, j \in \{1, 2, ..., N\}$ with $i < j$. Let $A \subset E \times E^*$ be a maximal monotone operator. If $0 \in Au$, then u is called a zero of A. An absolutely stunning problems in mathematical analysis that is associated with convex analysis and mathematical optimization is finding a zero of a maximal monotone operator (See, e.g., [9, 10, 12, 19]). Such problems have connection with variational inequality problems and their solutions have applications in several fields, such as economics, science and engineering. It is well known that the variational inequalities are equivalent to the fixed point problem (See, e.g., [16, 3, 17, 20]).

In this paper, a new countable family of generalized nonexpansive mappings which satisfies the NST-condition is introduced and a new monotone hybrid algorithm is presented. This study displays how to find a common element of the zero point set of a maximal monotone operator and the newly introduced countable family of generalized nonexpansive mappings which satisfies NST-condition. The conditions are established for a strong convergence of the proposed algorithm and the results are true in a general Banach space.

2. Preliminaries

Let E and E^*, respectively denote a real Banach space and its dual space. A unit sphere will be denoted by $S(E) := \{u \in E : \|u\| = 1\}$. Given taht the limit

$$\lim_{t\to 0} \frac{\|u + tv\| - \|u\|}{t} \tag{2.1}$$

exists for all $u, v \in S(E)$ with $\|u\| = \|v\| = 1$, the norm $\|.\|$ of E is said to be Gâteaux differentiable and E is said to be smooth in such a case. Given that E is smooth and the limit (2.1) is attained uniformly for each $u, v \in S(E)$, then E is said to be uniformly smooth. The modulus of convexity of a Banach space E, $\delta_E : (0, 2] \to [0, 1]$ is defined by

$$\delta_E(\epsilon) = \inf \left\{1 - \frac{\|u + v\|}{2} : \|u\| = \|v\| = 1, \|u - v\| > \epsilon\right\}.$$

Given that $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$, then E is uniformly convex. A Banach space E is said to be strictly convex if $\|u + v\| < 2$ for all $u, v \in E$ whenever $\|u\| = \|v\| = 1$ and $u \neq v$. It is well known that a space E is uniformly smooth if and only if E^* is uniformly convex. Let $J : E \to 2^{E^*}$ be defined by

$$Ju = \{u^* \in E^* : \langle u, u^*\rangle = \|u\|\|u^*\|, \|u^*\| = \|u\|\} \ \forall\, u \in E.$$

Then J is called a normalized duality mapping it is known to be uniformly norm-to-norm continuous on bounded sets of E if E is uniformly smooth. Let E be a given Banach space and let $A \subset E \times E^*$ be a multi-valued operator. Given that for all $(u, u^*), (v, v^*) \in A$,

$$\langle u - v, u^* - v^*\rangle \geq 0,$$

then A is said to be monotone and it is said to be maximal monotone if it is monotone and its graph is not properly contained in the graph of any other monotone mapping. For a maximal monotone operator A, the set $A^{-1}(0) := \{u \in E : Au = 0\}$ is closed and convex. According to a result of Rockafellar [18], in a given strictly convex, smooth and reflexive Banach space E, A is said to be maximum monotone if it is monotone and the range of $(J + rA)$ is all of E^* for all $r > 0$.

Definition 2.1. Let E be a given smooth Banach space and define the function $\varphi : E \times E \to \mathbb{R}$ by

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2,$$

for all $u, v \in E$. In a framework of Hilbert space, it is expressed as $\phi(u, v) = \|u - v\|^2 \geq 0$. For all $u, v, w \in E$,

(i) $(\|u\| - \|v\|)^2 \leq \phi(u, v) \leq (\|u\| + \|v\|)^2$,

(ii) $\phi(u, v) = \phi(u, w) + \phi(w, v) + 2\langle u - w, Jw - Jv\rangle$,

(iii) $\phi(u, v) = \langle u, Ju - Jv\rangle + \langle u - v, Jv\rangle \leq \|u\|\|Ju - Jv\| + \|u - v\|\|v\|$.

Definition 2.2. Resolvent: Let E be a given Banach space, which is strictly convex, smooth, and reflexive and let $A \subset E \times E^*$ a maximal monotone mapping. Given $r > 0$ and $u \in E$, then there exists a unique $u_r \in D(A)$ such that $Ju \in Ju_r + rAu_r$. Therefore, a single-valued mapping $J_r : E \to D(A)$ can be defined as

$$J_r u = \{w \in D(A) : Ju \in Jw + rAw\},$$

which is called the resolvent of A. $J_r u$ is known to consist of one point and for all $r > 0$, $A^{-1}(0) = F(J_r)$, where $F(J_r)$ is the set of fixed points of J_r. Also, for all $r > 0$ and $u \in E$, the Yosida approximation $A_r : C \to E^*$ is defined by

$$A_r u = \frac{1}{r}(J - JJ_r)u.$$

For all $r > 0$ and $u \in E$, the following hold (See, for example [14, 7])

(i) $\phi(p, J_r u) + \phi(J_r u, u) \leq \phi(p, u)$ for all $p \in A^{-1}(0)$.

(ii) $(J_r u, A_r u) \in A$.

Definition 2.3. Metric projection: Let H be a Hilbert space and C be a nonempty closed convex subset of H. A mapping $P_C : H \to C$ of H onto C which satisfies

$$\|u - P_C u\| = \min_{v \in C} \|u - v\|,$$

is called the metric projection. This is known to be a singleton. An important property of the metric projection states that for $u \in H$ and $u_0 \in C, u_0 = P_C u$ if and only if

$$\langle u - u_0, u_0 - v\rangle \geq 0 \ \forall\ v \in C.$$

Definition 2.4. Retraction: Let C be nonempty subset of a Banach space E. A mapping $R : E \to C$ is called sunny if

$$R(Ru + \alpha(u - Ru)) = Ru,$$

for all $u \in E$ and all $\alpha \geq 0$. If $Ru = u$ for all $u \in C$, it is also called a retraction. A retraction which is also sunny and nonexpansive is called a sunny nonexpansive retraction. If E is a smooth Banach space, the sunny nonexpansive retraction of E onto C is denoted by R_C. C is said to be a sunny generalized nonexpansive retract of E provided that there exists a sunny generalized nonexpansive retraction R from E onto C.

Some important results on sunny generalized nonexpansive retraction which will be needed are stated here. For their proof, see [7, 13].

Lemma 2.5. *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E. Let R_C be a retraction of E onto C. Then R_C is sunny and generalized nonexpansive if and only if*

$$\langle u - R_C u, JR_C u - Jv\rangle \geq 0$$

for each $u \in E$ and $v \in C$.

Lemma 2.6. *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(u, w) \in E \times C$. Then the following hold:*

(i) $w = Ru$ *if and only if* $\langle u - w, Jv - Jw\rangle \leq 0$ *for all* $v \in C$;

(ii) $\phi(u, R_C v) + \phi(R_C v, v) \leq \phi(u, v)$.

Lemma 2.7. *Let* E *be a smooth, strictly convex and reflexive Banach space and let* C *be a nonempty closed subset of* E. *Then the following are equivalent:*

(i) C *is a sunny generalized nonexpansive retract of* E;

(ii) C *is a generalized nonexpansive retract of* E;

(iii) JC *is closed and convex.*

Definition 2.8. NST-Condition: Let C be a closed subset of a real Banach space E. Suppose that $\{S_n\}$ and Γ are two families of the generalized nonexpansive mappings of C into E such that $\bigcap_{n=1}^{\infty} F(S_n) = F(\Gamma) \neq \emptyset$, where $F(\Gamma)$ is the set of all common fixed points of Γ. The sequence $\{S_n\}$ is said to satisfy the NST-condition with Γ if

$$\lim_{n\to\infty} \|u_n - S_n u_n\| = \lim_{n\to\infty} \|u_n - S u_n\|$$

for all $S \in \Gamma$ and all bounded sequence $\{u_n\}$ in C [15]. If Γ possesses one element, i.e., $\Gamma = \{S\}$, then $\{S_n\}$ satisfies the NST-condition with $\{S\}$. If we put $S_n = S$ for all $n \in \mathbb{N}$, then $\{S_n\}$ satisfies the NST-condition with $\{S\}$.

The following are well known results and will be applied to establish the main results.

Lemma 2.9. *Let* E *be a uniformly convex and smooth Banach space and let* $\{u_n\}$ *and* $\{v_n\}$ *be two sequences in* E *such that either* $\{u_n\}$ *or* $\{v_n\}$ *is bounded. If* $\lim_{n\to\infty} \varphi(u_n, v_n) = 0$, *then* $\lim_{n\to\infty} \|u_n - v_n\| = 0$ *(See, for example [9]).*

Lemma 2.10. *Let* E *be a uniformly convex and smooth Banach space and let* $d > 0$. *Then there exists a strictly increasing, continuous and convex function* $g : [0, \infty) \to [0, \infty)$ *such that* $g(0) = 0$ *and*

$$g(\|u - v\|) \leq \phi(u, v)$$

for all $u, v \in B_d(0)$, *where* $B_d(0) = \{w \in E : \|w\| \leq d\}$ *(See, for example [9]).*

Lemma 2.11. *Let* E *be a uniformly convex Banach space and let* $d > 0$. *Then there exists a strictly increasing, continuous and convex function* $g : [0, \infty) \to [0, \infty)$ *such that* $g(0) = 0$ *and*

$$\|\alpha u + (1 - \alpha)v\|^2 \leq \alpha\|u\|^2 + (1 - \alpha)\|v\|^2 - \alpha(1 - \alpha)g(\|u - v\|)$$

for all $u, v \in B_d(0)$ *and* $\alpha \in [0, 1]$, *where* $B_d(0) = \{w \in E : \|w\| \leq d\}$ *(See, for example [21]).*

Lemma 2.12. *Let* E *be a smooth and strictly convex Banach space, let* $p \in E$ *and let* $\{\alpha_i\}_i^m \subset (0, 1)$ *with* $\sum_i^m \alpha_i = 1$. *If* $\{\alpha_i\}_i^m$ *is a finite sequence in* E *such that*

$$\phi\left(p, J^{-1}\left(\sum_i^m \alpha_i J w_i\right)\right) = \phi(p, w_i),$$

then $w_1 = w_2 = ... = w_m$*(See, for example [8]).*

3. Main Results

Lemma 3.1. *Let E be a strictly convex, smooth, and reflexive Banach space and let $A \subset E \times E^*$ be a maximal monotone mapping with $A^{-1}(0) \neq \emptyset$. For each $\lambda > 0$, let $J_\lambda : E \to E$ be the resolvent of A associated with λ. Then J_λ is a generalized nonexpansive mapping.*

Proof. The proof is given in [1]. However, to make this manuscript a complete paper, the sketch of the proof is given here.
Let $u \in E, v \in F(J_\lambda)$ and $\lambda > 0$. Recall that $A^{-1}(0) = F(J_\lambda)$ since A is a maximal monotone operator. Wherefore by Definition 2.2 (i),

$$\phi(v, J_\lambda u) + \phi(J_\lambda u, u) \leq \phi(v, u) \text{ for all } v \in A^{-1}(0).$$

It is known by Definition 2.1 (i) that $\phi(J_\lambda u, u) \geq 0$. Hence,

$$\phi(v, J_\lambda u) \leq \phi(v, u).$$

□

Assumption 3.2. Let $N \in \mathbb{N}$, the real sequences $\{\alpha_{kn}\}_{k=1}^N$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $[0, 1]$ are assumed to satisfy the following conditions:

(i) $\displaystyle\sum_{k=1}^N \alpha_{kn} = 1$ for all $n \in \mathbb{N}$;

(ii) $\liminf_{n\to\infty} \alpha_{in}\alpha_{jn} > 0$ for all $i, j \in \{1, 2, ..., N\}$ with $i < j$;

(iii) $\liminf_{n\to\infty} (1 - \beta_n) > 0$;

(iv) $\lim_{n\to\infty} \gamma = 1$.

Lemma 3.3. *Let E be a uniformly convex and uniformly smooth Banach space E, C be a nonempty closed convex subset of E and $R_C : E \to C$ be a sunny and generalized nonexpansive retraction from E onto C. Let $G_1, G_2, ..., G_N$ are be generalized nonexpansive mappings of C into E such that $\displaystyle\bigcap_{k=1}^N F(G_k) \neq \emptyset$. Suppose that for each $n \in \mathbb{N}$, the mapping $S_n : C \to E$ is defined by*

$$S_n u = J^{-1}\left(\sum_{k=1}^N \alpha_{kn} J G_k u\right) \ \forall\ u \in C, \tag{3.1}$$

where $\{\alpha_{kn}\}_{k=1}^N$ is a sequence in $[0, 1]$ satisfying the Assumption 3.2 (i) and (ii). Then, the countable family of generalized nonexpansive mappings $\{S_n\}$ satisfies NST-condition with $\Gamma = \{G_1, G_2, ..., G_N\}$.

Proof. **Step 1**: Firstly, we show that for each $n \in \mathbb{N}$, S_n is a generalized nonexpansive mapping and that $\displaystyle\bigcap_{n=1}^\infty F(S_n) = F(\Gamma)$. Notice that

$$F(\Gamma) = \bigcap_{k=1}^N F(G_k) \subset \bigcap_{n=1}^\infty F(S_n). \tag{3.2}$$

Therefore, for $p \in F(\Gamma)$ and $x \in C$,

$$\begin{aligned}
\phi(p, S_n x) &= \phi\left(p, J^{-1}\left(\sum_{k=1}^{N} \alpha_{kn} J G_k x\right)\right) \\
&= \|p\|^2 - 2\left\langle p, \sum_{k=1}^{N} \alpha_{kn} J G_k x \right\rangle + \left\| \sum_{k=1}^{N} \alpha_{kn} G_k x \right\|^2 \\
&\leq \|p\|^2 - 2\sum_{k=1}^{N} \alpha_{kn} \langle p, J G_k x\rangle + \sum_{k=1}^{N} \alpha_{kn} \|G_k x\|^2 \\
&= \sum_{k=1}^{N} \alpha_{kn} \phi(p, G_k x) \\
&\leq \sum_{k=1}^{N} \alpha_{kn} \phi(p, x) \\
&= \phi(p, x).
\end{aligned}$$

Thus, S_n is a generalized nonexpansive mapping. Moreover, for $\zeta \in \bigcap_{n=1}^{\infty} F(S_n)$,

$$\begin{aligned}
\phi(p, \zeta) &= \phi(p, S_n \zeta) \\
&= \phi\left(p, J^{-1}\left(\sum_{k=1}^{N} \alpha_{kn} J G_k \zeta\right)\right) \\
&= \|p\|^2 - 2\left\langle p, \sum_{k=1}^{N} \alpha_{kn} J G_k \zeta \right\rangle + \left\| \sum_{k=1}^{N} \alpha_{kn} G_k \zeta \right\|^2 \\
&\leq \|p\|^2 - 2\sum_{k=1}^{N} \alpha_{kn} \langle p, J G_k \zeta\rangle + \sum_{k=1}^{N} \alpha_{kn} \|G_k \zeta\|^2 \\
&= \sum_{k=1}^{N} \alpha_{kn} \phi(p, G_k \zeta) \\
&\leq \sum_{k=1}^{N} \alpha_{kn} \phi(p, \zeta) \\
&= \phi(p, \zeta),
\end{aligned}$$

which indicates that

$$\phi\left(p, J^{-1}\left(\sum_{k=1}^{N} \alpha_{kn} J G_k \zeta\right)\right) = \sum_{k=1}^{N} \alpha_{kn} \phi(p, G_k \zeta) = \phi(p, \zeta).$$

By applying Lemma 2.12, one gets that $\zeta = S_n \zeta = G_1 \zeta = G_2 \zeta = ... = G_N \zeta$. Thus, $F(S_n) \subset F(\Gamma)$ for all $n \in \mathbb{N}$ and hence, $\bigcap_{n=1}^{\infty} F(S_n) = F(\Gamma)$.

Step 2: To prove that $\{S_n\}$ satisfies NST-condition with $G_1, G_2, ..., G_N$, an arbitrary bounded sequence $\{\omega_n\}$ in C is assumed given such that $\lim_{n\to\infty} \|\omega_n - S_n \omega_n\| = 0$. Consequently, the norm-to-norm uniform continuouity of the duality mapping J on bounded sets gives that

$$\lim_{n\to\infty} \|J\omega_n - JS_n\omega_n\| = 0. \tag{3.3}$$

It is given that $\{\omega_n\}$ is bounded, therefore $\{G_k\omega_n\}$ are bounded for $k = 1, 2, ..., N$. Let $r = \max\{\sup_n \|\omega_n\|, \sup_n \|G_k\omega_n\|\}$. Thus, there exists $r > 0$ with $B_r(0) = \{z \in E : \|z\| \le r\}$ and $\{\omega_n\}, \{G_k\omega_n\} \subset B_r(0)$. Then by Lemma 2.11, there exists a function $g : [0, \infty) \to [0, \infty)$ which is strictly increasing, continuous and convex with $g(0) = 0$ such that for $p \in \bigcap_{n=1}^{\infty} F(S_n)$,

$$\begin{aligned}
\phi(p, S_n\omega_n) &= \phi\left(p, J^{-1}\left(\sum_{k=1}^{N} \alpha_{kn} JG_k\omega_n\right)\right) \\
&= \|p\|^2 - 2\left\langle p, \sum_{k=1}^{N} \alpha_{kn} JG_k\omega_n\right\rangle + \|\sum_{k=1}^{N} \alpha_{kn} G_k\omega_n\|^2 \\
&\le \|p\|^2 - 2\sum_{k=1}^{N} \alpha_{kn}\langle p, JG_k\omega_n\rangle + \sum_{k=1}^{N} \alpha_{kn}\|G_k\omega_n\|^2 - \alpha_{in}\alpha_{jn} g(\|G_i\omega_n - G_j\omega_n\|) \\
&= \sum_{k=1}^{N} \alpha_{kn}\phi(p, G_k\omega_n) - \alpha_{in}\alpha_{jn} g(\|G_i\omega_n - G_j\omega_n\|) \\
&\le \sum_{k=1}^{N} \alpha_{kn}\phi(p, \omega_n) - \alpha_{in}\alpha_{jn} g(\|G_i\omega_n - G_j\omega_n\|) \\
&= \phi(p, \omega_n) - \alpha_{in}\alpha_{jn} g(\|G_i\omega_n - G_j\omega_n\|).
\end{aligned}$$

Consequently,

$$\alpha_{in}\alpha_{jn} g(\|G_i\omega_n - G_j\omega_n\|) \le \phi(p, \omega_n) - \phi(p, S_n\omega_n). \tag{3.4}$$

Let $\{\|G_i\omega_{n_k} - G_j\omega_{n_k}\|\}$ be an arbitrary subsequence of $\{\|G_i\omega_n - G_j\omega_n\|\}$. The boundedness of $\{\omega_{n_k}\}$ guarantees the existence of a subsequence $\{\omega_{n'_m}\}$ of $\{\omega_{n_k}\}$ such that

$$\lim_{m\to\infty} \phi(p, \omega_{n'_m}) = \limsup_{k\to\infty} \phi(p, \omega_{n_k}) = 0.$$

Applying Definition 2.1 ((ii) and (iii)) result in

$$\begin{aligned}
\phi(p, \omega_{n'_m}) &= \phi(p, S_{n'_m}\omega_{n'_m}) + \phi(S_{n'_m}\omega_{n'_m}, \omega_{n'_m}) \\
&\quad + 2\langle p - S_{n'_m}\omega_{n'_m}, JS_{n'_m}\omega_{n'_m} - J\omega_{n'_m}\rangle \\
&\le \phi(p, S_{n'_m}\omega_{n'_m}) + \|S_{n'_m}\omega_{n'_m}\|\|JS_{n'_m}\omega_{n'_m} - J\omega_{n'_m}\| \\
&\quad + \|S_{n'_m}\omega_{n'_m} - \omega_{n'_m}\|\|\omega_{n'_m}\| + 2\|p - S_{n'_m}\omega_{n'_m}\|\|JS_{n'_m}\omega_{n'_m} - J\omega_{n'_m}\|.
\end{aligned} \tag{3.5}$$

By taking the limit inferior on both sides of (3.5) while taking note of (3.17) and (3.3) results in

$$c = \liminf_{m\to\infty} \phi(p, \omega_{n'_m}) = \liminf_{m\to\infty} \phi(p, S_{n'_m}\omega_{n'_m}).$$

Alternatively, $\varphi(p, S_n\omega_n) \le \varphi(p, \omega_n)$ leads to

$$\limsup_{m\to\infty} \phi(p, S_{n'_m}\omega_{n'_m}) = \limsup_{m\to\infty} \phi(p, \omega_{n'_m}) = c,$$

wherefore

$$\lim_{m\to\infty} \phi(p, \omega_{n'_m}) = \lim_{m\to\infty} \phi(p, S_{n'_m}\omega_{n'_m}) = c.$$

Thus, it can be deduced from (3.4) that

$$\lim_{m\to\infty} g(\|G_1\omega_{n'_m} - G_2\omega_{n'_m}\|) = 0,$$

since $\liminf_{n\to\infty} \alpha_{1m}\alpha_{2m} > 0$. Therefore, by the properties of the function g, $\lim_{m\to\infty} \|G_1\omega_{n'_m} - G_2\omega_{n'_m}\| = 0$, and thus

$$\lim_{n\to\infty} \|G_1\omega_n - G_2\omega_n\| = 0.$$

Using similar analysis, it can be shown that $\lim_{n\to\infty} \|G_1\omega_n - G_j\omega_n\| = 0$, for $j = 3, 4, 5, ..., N$. Considering the fact that

$$\begin{aligned} \|\omega_n - G_1\omega_n\| &= \|\omega_n - S_n\omega_n\| + \|S_n\omega_n - G_1\omega_n\| \\ &\leq \|\omega_n - S_n\omega_n\| + \sum_{k=2}^{N} \alpha_{kn}\|G_1\omega_n - G_k\omega_n\|, \end{aligned}$$

which results in $\lim_{n\to\infty} \|\omega_n - G_1\omega_n\| = 0$. Similar analysis yields that $\lim_{n\to\infty} \|\omega_n - G_j\omega_n\| = 0$, for $j = 2, 3, 4, ..., N$. Therefore,

$$\lim_{n\to\infty} \|\omega_n - S\omega_n\| = 0 \ \forall\ S \in \Gamma. \tag{3.6}$$

□

Theorem 3.4. *Let E be a uniformly convex and uniformly smooth Banach space E, C be a nonempty closed convex subset of E and $R_C : E \to C$ be a sunny and generalized nonexpansive retraction from E onto C. The resolvent associated with a maximal monotone operator $A \subset E \times E^*$ will be denoted by $J_\lambda : E \to E$ for all $\lambda > 0$ and $G_1, G_2, ..., G_N$ are closed generalized nonexpansive mappings of C into E with $\Gamma = \{G_1, G_2, ..., G_N\}$ such that $F(\Gamma) \cap A^{-1}(0) \neq \emptyset$. Suppose that for each $n \in \mathbb{N}$, the sequence $\{u_n\}$ is defined by*

$$\begin{cases} u_1 = u \in C, \ C_0 = Q_0 = C, \\ x_n = J^{-1}\left(\beta_n Ju_n + (1-\beta_n)JS_nR_C\left(J_{\lambda_n}u_n\right)\right), \\ y_n = J^{-1}\left(\gamma_n Jx_n + (1-\gamma_n)JS_nR_C\left(J_{\lambda_n}u_n\right)\right), \\ C_n = \{x \in C_{n-1} \cap Q_{n-1} : \phi(x, y_n) \leq \phi(x, u_n)\}, \\ Q_n = \{x \in C_{n-1} \cap Q_{n-1} : \langle u_n - x, Ju - Ju_n\rangle \geq 0\}, \\ u_{n+1} = R_{C_n\cap Q_n}u, \end{cases}$$

where J is the duality mapping on E, real sequences $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy Assumption 3.2 (iii) and (iv), respectively, and $\{S_n\}$ is a countable family of generalized nonexpansive mappings that is given by (3.1). Then the sequence $\{u_n\}$ converges strongly to $R_{F(\Gamma)\cap A^{-1}(0)}u$, where $R_{F(\Gamma)\cap A^{-1}(0)}$ is the sunny nonexpansive retraction from C onto $F(\Gamma) \cap A^{-1}(0)$.

Proof. **Step 1**: To show that C_n and Q_n are closed and convex for all $n \in \mathbb{N}$. Closedness of C_n is obvious from its definition and it can also be seen from the definition of Q_n that it is closed and convex for each $n \in \mathbb{N}$. To show that C_n is convex, observe from the definition of C_n that

$$\phi(x, y_n) \leq \phi(x, u_n)$$

implies that for all $x \in C_n$,

$$\|u_n\|^2 - \|y_n\|^2 - 2\langle x, Ju_n - Jy_n\rangle \geq 0,$$

which is affine in x, and thus C_n is convex. Hence, the closedness and convexity of $C_n \cap Q_n \subset E$ for all $n \in \mathbb{N}$ is established.

Step 2: To establish that $F(\Gamma)\cap A^{-1}(0)\subset C_n\cap Q_n$. Setting $\omega_n = R_C\left(J_{r_n}u_n\right)$ and for $p\in F(\Gamma)\cap A^{-1}(0)$,

$$\begin{aligned}
\phi\left(p,x_n\right) &= \phi\left(p,J^{-1}\left(\beta_n Ju_n+(1-\beta_n)JS_n\omega_n\right)\right)\\
&= \|p\|^2-2\left\langle p,\beta_n Ju_n+(1-\beta_n)JS_n\omega_n\right\rangle+\|\beta_n Ju_n+(1-\beta_n)JS_n\omega_n\|^2\\
&\le \|p\|^2-2\beta_n\left\langle p,Ju_n\right\rangle-2(1-\beta_n)\left\langle p,JS_n\omega_n\right\rangle+\beta_n\|u_n\|^2+(1-\beta_n)\|S_n\omega_n\|^2\\
&= \beta_n\phi\left(p,u_n\right)+(1-\beta_n)\phi\left(p,S_n\omega_n\right)\\
&\le \beta_n\phi\left(p,u_n\right)+(1-\beta_n)\phi\left(p,\omega_n\right)\ \text{(by generalized nonexpansive property of } S_n\text{)} \qquad (3.7)\\
&= \beta_n\phi\left(p,u_n\right)+(1-\beta_n)\phi\left(p,R_K\left(J_{r_n}u_n\right)\right)\\
&\le \beta_n\phi\left(p,u_n\right)+(1-\beta_n)\phi\left(p,J_{r_n}u_n\right)\ \text{(by the property of } R_C\text{)}\\
&\le \beta_n\phi\left(p,u_n\right)+(1-\beta_n)\phi\left(p,u_n\right)\ \text{(by generalized nonexpansive property of } J_{r_n}\text{)}\\
&= \varphi\left(p,u_n\right).
\end{aligned}$$

Wherefore,

$$\begin{aligned}
\phi\left(p,y_n\right) &= \phi\left(p,J^{-1}\left(\gamma_n Jx_n+(1-\gamma_n)JS_n\omega_n\right)\right)\\
&= \|p\|^2-2\left\langle p,\gamma_n Jx_n+(1-\gamma_n)JS_n\omega_n\right\rangle+\|\gamma_n Jx_n+(1-\gamma_n)JS_n\omega_n\|^2\\
&\le \|p\|^2-2\gamma_n\left\langle p,Jx_n\right\rangle-2(1-\gamma_n)\left\langle p,JS_n\omega_n\right\rangle+\gamma_n\|x_n\|^2+(1-\gamma_n)\|S_n\omega_n\|^2\\
&= \gamma_n\phi\left(p,x_n\right)+(1-\gamma_n)\varphi\left(p,S_n\omega_n\right)\\
&\le \gamma_n\phi\left(p,x_n\right)+(1-\gamma_n)\phi\left(p,\omega_n\right)\\
&= \gamma_n\phi\left(p,x_n\right)+(1-\gamma_n)\phi\left(p,R_C\left(J_{r_n}x_n\right)\right)\\
&\le \gamma_n\phi\left(p,x_n\right)+(1-\gamma_n)\phi\left(p,J_{r_n}x_n\right)\\
&\le \gamma_n\phi\left(p,x_n\right)+(1-\gamma_n)\phi\left(p,x_n\right)\\
&\le \gamma_n\phi\left(p,u_n\right)+(1-\gamma_n)\phi\left(p,u_n\right)\\
&= \varphi\left(p,u_n\right).
\end{aligned}$$

This justifies that $p\in C_n$ for all $n\in\mathbb{N}$ and thus $F(\Gamma)\cap A^{-1}(0)\subset C_n$. Induction will be used to show that $F(\Gamma)\cap A^{-1}(0)\subset Q_n$ for all $n\in\mathbb{N}$. By definition, for $n=1, F(\Gamma)\cap A^{-1}(0)\subset C=C_0\cap Q_0$. Recall that J is one-to-one. Wherefore $J\left(C_n\cap Q_n\right)=JC_n\cap JQ_n$ and it known to be closed convex (See e.g., [2]). Lemma 2.7 gives that $C_n\cap Q_n$ is a sunny generalized nonexpansive retract of E. Assume that $F(\Gamma)\cap A^{-1}(0)\subset C_{j-1}\cap Q_{j-1}$ for some $j\in\mathbb{N}$. Given that $x_j=R_{C_{j-1}\cap Q_{j-1}}$, application of Lemma 2.5 gives

$$\langle u-u_j, Ju_j-Jv\rangle\ge 0,$$

for all $v\in C_{j-1}\cap Q_{j-1}$. This implies that

$$\langle u-u_j, Ju_j-Jv\rangle\ge 0,\ \forall\ v\in F(\Gamma)\cap A^{-1}(0) \qquad (3.8)$$

since $F(\Gamma)\cap A^{-1}(0)\subset C_{j-1}\cap Q_{j-1}$. It can be deduced from the inequality (3.8) and by the definition of Q_n that $F(T)\cap A^{-1}(0)\subset Q_i$ and consequently $F(\Gamma)\cap A^{-1}(0)\subset Q_n$ for all $n\in\mathbb{N}$. Thus, $F(\Gamma)\cap A^{-1}(0)\subset C_n\cap Q_n$ for all $n\in\mathbb{N}$ and this confirms that $\{u_n\}$ is well defined.

Step 3: This is to show that as $n\to\infty$, $u_n\to R_{F(\Gamma)\cap A^{-1}(0)}u$. It can be obtained from the definition of Q_n that $x_n=R_{Q_n}x$. Applying Lemma 2.6 (ii) gives,

$$\phi(u,u_n)=\phi(u,R_{Q_n}u)\le\phi(u,x)-\phi(R_{Q_n}u,x)\le\phi(u,x),$$

for all $F(\Gamma)\cap A^{-1}(0)\subset Q_n$. Thus, boundedness of $\{\phi(u,u_n)\}$ is established. Moreover, it can be deduced from the definition of ϕ that $\{u_n\},\{x_n\}$ and $\{y_n\}$ are bounded. Therefore, the limit of $\{\varphi(u,u_n)\}$ exists. One can have from $x_n=R_{Q_n}x$ for each $n\in\mathbb{N}$ such that

$$\varphi\left(u_n,u_{n+k}\right)=\phi\left(R_{Q_n}u,u_{n+k}\right)\le\phi\left(u,u_{n+k}\right)-\phi\left(u,R_{Q_n}u\right)\le\phi\left(u,u_{n+k}\right)-\phi\left(u,u_n\right),$$

for a given a positive integer k, which leads to

$$\lim_{n\to\infty} \phi(u_n, u_{n+k}) = 0. \tag{3.9}$$

Lemma 2.10 gives that there exists a strictly increasing, convex and continuous function $g : [0, 2r] \to [0, \infty)$, such that for $j, k \in \mathbb{N}$ with $k > j$,

$$g(\|u_k - u_j\|) \leq \phi(u_k, u_j) \leq \phi(u_k, u_0) - \phi(u_j, u_0).$$

The property of g leads to the deduction that $\{u_n\}$ is Cauchy. Thus, there exists $v \in C$ such that $u_n \to v$. For $u_{n+1} = R_{C_n \cap Q_n} u \in C_n$, it can be obtained from the definition of C_n,

$$\phi(u_{n+1}, u_n) - \phi(u_{n+1}, y_n) \geq 0, \ \forall\, n \in \mathbb{N}. \tag{3.10}$$

From (3.9) and (3.10), one can deduce that $\lim_{n\to\infty} \phi(u_{n+1}, u_n) = \lim_{n\to\infty} \phi(u_{n+1}, y_n) = 0$. Apply Lemma 2.9 since E is uniformly convex and smooth to obtain

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = \lim_{n\to\infty} \|u_{n+1} - y_n\| = 0, \tag{3.11}$$

which yields

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.12}$$

Recall that the duality mapping J is norm-to-norm uniform continuous on bounded sets. Therefore

$$\lim_{n\to\infty} \|Ju_{n+1} - Ju_n\| = \lim_{n\to\infty} \|Ju_{n+1} - Jy_n\| = \|Ju_n - Jy_n\| = 0. \tag{3.13}$$

From (3.7), observe that

$$\phi(p, \omega_n) \geq \frac{1}{(1-\beta_n)} \left(\phi(p, x_n) - \beta_n \phi(p, u_n)\right).$$

Since $\omega_n := R_C(J_{r_n} u_n)$, thus,

$$\begin{aligned}
\phi(\omega_n, u_n) &= \phi(R_C(J_{r_n} u_n), u_n) \leq \phi(p, u_n) - \phi(p, \omega_n) \ \text{(by Lemma 2.6 (ii),)} \\
&\leq \phi(p, u_n) - \frac{1}{(1-\beta_n)} \left(\phi(p, x_n) - \beta_n \phi(p, u_n)\right) \\
&= \frac{1}{(1-\beta_n)} \left(\phi(p, u_n) - \phi(p, x_n)\right) \\
&= \frac{1}{(1-\beta_n)} \left(\|u_n\|^2 - \|x_n\|^2 - 2\langle p, Ju_n - Jx_n\rangle\right) \\
&\leq \frac{1}{(1-\beta_n)} \left(|\|u_n\|^2 - \|x_n\|^2| + 2|\langle p, Ju_n - Jx_n\rangle|\right) \\
&\leq \frac{1}{(1-\beta_n)} \left(|\|u_n\| - \|x_n\|| (\|u_n\| + \|x_n\|) + 2\|p\|\|Ju_n - Jx_n\|\right) \\
&\leq \frac{1}{(1-\beta_n)} \left(\|u_n - x_n\| (\|u_n\| + \|x_n\|) + 2\|p\|\|Ju_n - Jx_n\|\right).
\end{aligned}$$

By (3.12) and (3.13), one can have that $\lim_{n\to\infty} \phi(\omega_n, u_n) = 0$. So Lemma 2.9 gives that

$$\lim_{n\to\infty} \|\omega_n - u_n\| = 0. \tag{3.14}$$

Moreover,

$$\begin{aligned}\|Ju_{n+1} - Jx_n\| &= \|Ju_{n+1} - \beta_n Ju_n - (1-\beta_n)JS_n\omega_n\| \\ &= \|(1-\beta_n)(Ju_{n+1} - JS_n\omega_n) - \beta_n (Ju_n - Ju_{n+1})\| \\ &\geq (1-\beta_n)\|Ju_{n+1} - JS_n\omega_n\| - \beta_n\|Ju_n - Ju_{n+1}\|.\end{aligned}$$

Thus

$$\|Ju_{n+1} - JS_n\omega_n\| \leq \frac{1}{(1-\beta_n)}\left(\|Ju_{n+1} - Jx_n\| + \beta_n\|Ju_n - Ju_{n+1}\|\right).$$

It is already given that $\liminf_{n\to\infty}(1-\beta_n) > 0$. Considering (3.12) leads to

$$\lim_{n\to\infty} \|Ju_{n+1} - JS_n\omega_n\| = 0.$$

By the norm-to-norm uniform continuity of J^{-1} on bounded sets, one can have that

$$\lim_{n\to\infty} \|u_{n+1} - S_n\omega_n\| = 0. \tag{3.15}$$

Also observe that

$$\|u_n - S_n\omega_n\| \leq \|u_n - u_{n+1}\| + \|u_{n+1} - S_n\omega_n\|.$$

Apply (3.11) and (3.15) to obtain

$$\lim_{n\to\infty} \|u_n - S_n\omega_n\| = 0. \tag{3.16}$$

In a similar manner, observe that

$$\|\omega_n - S_n\omega_n\| \leq \|\omega_n - u_n\| + \|u_n - S_n\omega_n\|,$$

which by (3.14) and (3.16) leads to

$$\lim_{n\to\infty} \|\omega_n - S_n\omega_n\| = 0. \tag{3.17}$$

Since $\{S_n\}$ satisfies the NST-condition with Γ, one can have that

$$\lim_{n\to\infty} \|\omega_n - S\omega_n\| = 0 \ \forall \ S \in \Gamma. \tag{3.18}$$

From (3.14) and (3.18), it can be obtained that

$$\|u_n - S\omega_n\| \leq \|u_n - \omega_n\| + \|\omega_n - S\omega_n\| \to 0 \text{ as } n \to \infty.$$

Since it is known that $u_n \to v$, one can deduce by (3.14) that $\omega_n \to v$. The elements of the set Γ are known to be closed. Therefore S is closed as it belongs to Γ. Furthermore, $\omega_n \to v$, hence v is a fixed point of S.

It is necessary to show that $v \in A^{-1}(0)$. Since E is uniformly smooth, one can have from (3.14) that

$$\lim_{n\to\infty} \|Ju_n - J\omega_n\| = 0.$$

For $\lambda_n \geq a$, it is obtained that

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\|Ju_n - J\omega_n\| = 0.$$

Accordingly,

$$\lim_{n\to\infty} \|A_{\lambda_n} u_n\| = \lim_{n\to\infty} \frac{1}{\lambda_n}\|Ju_n - J\omega_n\| = 0.$$

Since A is monotone, for $(\rho, \rho^*) \in A$,

$$\langle \rho - v_n, \rho^* - A_{\lambda_n} u_n\rangle \geq 0 \text{ for all } n \in \mathbb{N}.$$

Consequently, as $n \to \infty$,

$$\langle \rho - v, \rho^* \rangle \geq 0.$$

The maximal monotone property of A gives that $v \in A^{-1}(0)$. Finally, it is necessary to show that $v = R_{F(\Gamma)\cap A^{-1}(0)}u$. By Lemma 2.6,

$$\phi\left(v, R_{F(\Gamma)\cap A^{-1}(0)}u\right) + \phi\left(R_{F(\Gamma)\cap A^{-1}(0)}u, u\right) \leq \phi\left(v, u\right).$$

Since $u_{n+1} = R_{K_n \cap Q_n}u$ and $v \in F(\Gamma) \cap A^{-1}(0) \subset C_n \cap Q_n$, applying Lemma 2.6 gives,

$$\phi\left(R_{F(\Gamma)\cap A^{-1}(0)}u, u_{n+1}\right) + \phi\left(u_{n+1}, u\right) \leq \phi\left(R_{F(\Phi)\cap A^{-1}(0)}u, u\right).$$

By the definition of ϕ, it can be obtained that $\phi\left(v, u\right) \leq \phi\left(R_{F(\Gamma)\cap A^{-1}(0)}u, u\right)$ and $\phi\left(v, u\right) \geq \varphi\left(R_{F(\Gamma)\cap A^{-1}(0)}u, u\right)$, therefore, $\phi\left(v, u\right) = \phi\left(R_{F(\Gamma)\cap A^{-1}(0)}u, u\right)$. Thus, since $R_{F(T)\cap A^{-1}(0)}u$ is unique, then $v = R_{F(\Gamma)\cap A^{-1}(0)}u$. □

The following results can be deduced from Theorem 3.4.

Corollary 3.5. *Let E be a uniformly convex and uniformly smooth Banach space E, C be a nonempty closed convex subset of E and $R_C : E \to C$ be a sunny and generalized nonexpansive retraction from E onto C. The resolvent associated with a maximal monotone operator $A \subset E \times E^*$ will be denoted by $J_\lambda : E \to E$ for all $\lambda > 0$. Let G_1 and G_2 be closed generalized nonexpansive mappings of C into E with $\Gamma = \{G_1, G_2\}$ such that $F(\Gamma) \cap A^{-1}(0) \neq \emptyset$. Suppose that for each $n \in \mathbb{N}$, the sequence $\{u_n\}$ is defined by*

$$\begin{cases} u_1 = u \in C, \;\; C_0 = Q_0 = C, \\ x_n = J^{-1}\left(\beta_n J u_n + (1-\beta_n) J S_n R_C\left(J_{\lambda_n} u_n\right)\right), \\ y_n = J^{-1}\left(\gamma_n J x_n + (1-\gamma_n) J S_n R_C\left(J_{\lambda_n} u_n\right)\right), \\ C_n = \left\{x \in C_{n-1} \cap Q_{n-1} : \phi(x, y_n) \leq \phi(x, u_n)\right\}, \\ Q_n = \left\{x \in C_{n-1} \cap Q_{n-1} : \langle u_n - x, Ju - Ju_n \rangle \geq 0\right\}, \\ u_{n+1} = R_{C_n \cap Q_n} u, \end{cases}$$

where J is the duality mapping on E and $\{S_n\}$ is a countable family of generalized nonexpansive mappings such that the mapping S_n of C into E is given by

$$S_n u = J^{-1}\left(\alpha_n J G_1 u + (1-\alpha_n) J G_2 u\right) \;\; \forall \; u \in C. \tag{3.19}$$

Then the sequence $\{u_n\}$ converges strongly to $R_{F(\Gamma)\cap A^{-1}(0)}u$, where $R_{F(\Gamma)\cap A^{-1}(0)}$ is the sunny nonexpansive retraction from C onto $F(\Gamma) \cap A^{-1}(0)$.

Proof. By letting $N = 2$ in (3.1), the desired result follows from Theorem 3.4. □

Corollary 3.6. *Let E be a uniformly convex and uniformly smooth Banach space E, C be a nonempty closed convex subset of E and $R_C : E \to C$ be a sunny and generalized nonexpansive retraction from E onto C. The resolvent associated with a maximal monotone operator $A \subset E \times E^*$ will be denoted by $J_\lambda : E \to E$ for all $\lambda > 0$. Let $G_1, G_2, ..., G_N$ be closed generalized nonexpansive mappings of C into E with $\Gamma = \{G_1, G_2, ..., G_N\}$ such that $F(\Gamma) \cap A^{-1}(0) \neq \emptyset$. Suppose that for each $n \in N$, the sequence $\{u_n\}$ is defined by*

$$\begin{cases} u_1 = u \in C, \;\; C_0 = Q_0 = C, \\ x_n = J^{-1}\left(\beta_n J u_n + (1-\beta_n) J S_n R_C\left(J_{\lambda_n} u_n\right)\right), \\ C_n = \left\{x \in C_{n-1} \cap Q_{n-1} : \phi(x, y_n) \leq \phi(x, u_n)\right\}, \\ Q_n = \left\{x \in C_{n-1} \cap Q_{n-1} : \langle u_n - x, Ju - Ju_n \rangle \geq 0\right\}, \\ u_{n+1} = R_{C_n \cap Q_n} u, \end{cases}$$

where J is the duality mapping on E and $\{S_n\}$ is a countable family of generalized nonexpansive mappings such that the mapping S_n of C into E is given by (3.1). Then the sequence $\{u_n\}$ converges strongly to $R_{F(\Gamma)\cap A^{-1}(0)}u$, where $R_{F(\Gamma)\cap A^{-1}(0)}$ is the sunny nonexpansive retraction from C onto $F(\Gamma) \cap A^{-1}(0)$.

Proof. By letting $\gamma_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.4, the desired result follows. □

Corollary 3.7. *Let E be a uniformly convex and uniformly smooth Banach space E, C be a nonempty closed convex subset of E and $R_C : E \to C$ be a sunny and generalized nonexpansive retraction from E onto C. The resolvent associated with a maximal monotone operator $A \subset E \times E^*$ will be denoted by $J_\lambda : E \to E$ for all $\lambda > 0$. Let G are closed generalized nonexpansive mappings of C into E such that $F(G) \cap A^{-1}(0) \neq \emptyset$. Suppose that for each $n \in N$, the sequence $\{u_n\}$ is defined by*

$$\begin{cases} u_1 = u \in C, \; C_0 = Q_0 = C, \\ x_n = J^{-1}\left(\beta_n J u_n + (1-\beta_n) J G R_C\left(J_{\lambda_n} u_n\right)\right), \\ y_n = J^{-1}\left(\gamma_n J x_n + (1-\gamma_n) J G R_C\left(J_{\lambda_n} u_n\right)\right), \\ C_n = \left\{x \in C_{n-1} \cap Q_{n-1} : \phi(x, y_n) \leq \phi(x, u_n)\right\}, \\ Q_n = \left\{x \in C_{n-1} \cap Q_{n-1} : \langle u_n - x, Ju - Ju_n\rangle \geq 0\right\}, \\ u_{n+1} = R_{C_n \cap Q_n} u, \end{cases}$$

where J is the duality mapping on E. Then the sequence $\{u_n\}$ converges strongly to $R_{F(G)\cap A^{-1}(0)}u$, where $R_{F(G)\cap A^{-1}(0)}$ is the sunny nonexpansive retraction from C onto $F(G) \cap A^{-1}(0)$.

Proof. By letting $N = 1$ in (3.1), it is obvious that $\{S_n\} = \{G\}$. Therefore, the desired result follows from Theorem 3.4. □

The result below is in the framework of Hilbert spaces and its proof can be deduced from the main result of this paper.

Corollary 3.8. *Let H be a Hilbert space with C a nonempty closed convex subset of H and $P_C : H \to K$ be a metric projection from H onto C. For all $\lambda > 0$, let $J_\lambda : H \to H$ denote the resolvent which is associated with a maximal monotone mapping $A \subset H \times H$. Let $G_1, G_2, ..., G_N$ be closed generalized nonexpansive mappings of C into H with $\Gamma = \{G_1, G_2, ..., G_N\}$ such that $F(\Gamma) \cap A^{-1}(0) \neq \emptyset$. For each $n \in \mathbb{N}$, define the sequence $\{u_n\}$ by*

$$\begin{cases} u_1 = u \in C, \; C_0 = Q_0 = C, \\ x_n = J^{-1}\left(\beta_n J u_n + (1-\beta_n) J S_n R_C\left(J_{\lambda_n} u_n\right)\right), \\ y_n = J^{-1}\left(\gamma_n J x_n + (1-\gamma_n) J S_n R_C\left(J_{\lambda_n} u_n\right)\right), \\ C_n = \left\{x \in C_{n-1} \cap Q_{n-1} : \phi(x, y_n) \leq \phi(x, u_n)\right\}, \\ Q_n = \left\{x \in C_{n-1} \cap Q_{n-1} : \langle u_n - x, Ju - Ju_n\rangle \geq 0\right\}, \\ u_{n+1} = R_{C_n \cap Q_n} u, \end{cases}$$

where $\{S_n\}$ is a countable family of generalized nonexpansive mappings such that the mapping S_n of C into H is given by

$$S_n u = \sum_{k=1}^{N} \alpha_{kn} G_k u \;\forall\; u \in C,$$

where $\{\alpha_{kn}\}_{k=1}^{N}$ is a sequence in $[0, 1]$ satisfying the Assumption 3.2 (i) and (ii). Then the sequence $\{u_n\}$ converges strongly to $R_{F(\Gamma)\cap A^{-1}(0)}u$, where $R_{F(\Gamma)\cap A^{-1}(0)}$ is the metric projection from C onto $F(\Gamma) \cap A^{-1}(0)$.

Proof. It is generally known that in a Hilbert space, $\phi(u, v) = \|u - v\|^2$ for all $u, v \in H$ and J is the identity mapping. Therefore, the desired result readily follows from Theorem 3.4. □

4. Conclusion

This study has made a significant contribution to the fundamental quest on how to solve some important nonlinear problems. Most important nonlinear problems in mathematics can be reduced to finding the fixed points of a certain operator with contractive type conditions. An eminent class of the nonlinear operators is the class of nonexpansive mappings. Algorithms on the class of nonexpansive mappings have been successfully applied in several areas such as signal processing and image restoration. Also, amny problems can be modelled as contructing zeros of a maximal monotone operator. This study introduced a new countable family of generalized nonexpansive mappings and presented a new monotone hybrid algorithm in the framework of Banach spaces. This study presented the procedures which are easy to follow in obtaining a common element of the zero point set of a maximal monotone operator and the newly introduced countable family of generalized nonexpansive mappings.

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