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Existence and Stability for Ambartsumian equation with Ξ- Hilfer generalized proportional fractional derivative

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Abstract

The main objective of this paper is to study the Ambartsumian equation in the sense of Ξ-Hilfer Generalized proportional fractional derivative(HGPFD). The existence and stability properties of solution are studied. The technique used for study is fixed point theorem and Gronwall inequality. Ulam-Hyers-Rassias stability of the solution is also investigated.

Keywords: Ambartsumian equation, Proportional fractional derivative, Existence, Fixed point theorem, Ulam-Hyers-Rassias stability.
2010 MSC: 34A12, 47H09, 47H30.

1. Introduction

The fractional calculus was originated in 1695 as a generalization of the integer order calculus. Fractional calculus and its applications to the sciences and engineering are recent foci of interest to many researchers, see [1, 12, 13] and references therein.

In [11, 22], the authors proposed a fractional integral operator with respect to another function ψ, obtaining a general operator, in the sense that it is enough to choose a function ψ with certain properties to obtain the most of the existing fractional integral operators. Attempting to unify several definitions of fractional derivatives into a single one, the concept of fractional derivative of a function with respect to another function was recently introduced. In [10], authors proposed a new fractional derivative called ψ-Caputo that generalizes a class of fractional derivatives in the Caputo sense. The same idea can be adapted to define the ψ-Riemann-Liouville fractional derivative. In 2018, Sousa and Oliveira [24] unified both

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definitions using Hilfer's idea of interpolating between Riemann-Liouville and Caputo fractional derivatives by introducing a two-parameter family of fractional derivatives of order $\alpha > 0$ and type $\mu \in [0,1]$ which depends on an arbitrary function ψ and called it the ψ-Hilfer fractional derivative. For more details see the papers [2, 6, 20, 23].

Motivated by [17, 22], we study a proportional fractional derivatives and provide a generalization of the operator defined in [9] and named it was ψ-HGPFD of a function with respect to another function.

Stability is the most relevant property of dynamical systems. The study of Ulam stability was initiated due to an interesting problem posed in the year 1940, by Ulam [25], regarding the stability for the equation of group homomorphisms. An answer was given by Hyers [8], in 1941, in the framework of Banach spaces, for the additive Cauchy equation. In the following years, many mathematicians [15, 26] were concerned with this problem, also for the case of differential equations, integral equations, and partial differential equations.

The original integer order Ambartsumian equation was introduced in the theory of surface brightness in the Milky Way. The authors in [3, 4, 5, 16, 19, 21] studied the Ambartsumian equation in different aspects.

In this work, we analyse the existence and stability results of the nonlocal fractional order Ambartsumian equation with Ξ-HGPFD

$$\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathcal{A}(t) = \mathbb{Q}\left(t, \mathcal{A}\left(t\right), \mathcal{A}\left(\frac{t}{\eta}\right)\right), \quad t \in [a,b], \quad \eta > 1, \quad b > a \geq 0, \tag{1.1}$$

$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho,\Xi}\mathcal{A}(t) = \sum_{i=1}^{m} \mu_i \mathcal{A}(\tau_i), \quad \mu_i \in \mathbb{R}, \quad \tau_i \in (a,b), \tag{1.2}$$

where, $\mathbb{Q}\left(t, \mathcal{A}\left(t\right), \mathcal{A}\left(\frac{t}{\eta}\right)\right) = \frac{1}{\eta}\mathcal{A}\left(\frac{t}{\eta}\right) - \mathcal{A}(t)$, $\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}$ is the Ξ-HGPFD, $\mathcal{I}_{a^+}^{1-\vartheta,\varrho,\Xi}$ is the Hilfer proportional fractional integral with $0 < p < 1, 0 \leq q \leq 1, \vartheta = p + q(1-p)$ and $\mathbb{Q} : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The above problem (1.1)-(1.2) is equivalent to the Volterra integral equation,

$$\mathcal{A}(t) = \begin{cases} \frac{\wedge}{\varrho^p\Gamma(p)} e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))} \left(\Xi(t) - \Xi(a)\right)^{\vartheta-1} \\ \times \sum_{i=1}^{m} \mu_i \int_{a^+}^{\tau_i} e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(s))} \left(\Xi(\tau_i) - \Xi(s)\right)^{p-1} \Xi'(s)\mathbb{Q}\left(s, \mathcal{A}(s), \mathcal{A}\left(\frac{s}{\eta}\right)\right) ds \\ + \frac{1}{\varrho^p\Gamma(p)} \int_{a^+}^{t} e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))} \left(\Xi(t) - \Xi(s)\right)^{p-1} \Xi'(s)\mathbb{Q}\left(s, \mathcal{A}(s), \mathcal{A}\left(\frac{s}{\eta}\right)\right) ds, \end{cases} \tag{1.3}$$

where,

$$\wedge = \frac{1}{\varrho^{\vartheta-1}\Gamma(\vartheta) - \sum_{i=1}^{m} \mu_i e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(a))} \left(\Xi(\tau_i) - \Xi(a)\right)^{\vartheta-1}}.$$

2. Preliminaries

In this section, we gives some definitions and lemmas useful in our subsequent discussion. Let $0 \leq a < b < \infty$, $J = [a,b]$ be a finite interval and ϑ be a parameter such that $n-1 \leq \vartheta < n$.

$C[a,b]$ be the space of the continuous functions $\mathbb{Q}$ on J with the norm defined by

$$\|\mathbb{Q}\|_{C[a,b]} = \max_{t\in J} |\mathbb{Q}(t)|,$$

and $AC^n[a,b]$ be the space of n times absolutely continuous differentiable functions given by,

$$AC^n[a,b] = \left\{\mathbb{Q} : [a,b] \to \mathbb{R}; \mathbb{Q}^{n-1} \in AC[a,b]\right\}.$$

The weighted space $C_{\vartheta,\Xi}[a,b]$ of functions $\mathbb{Q}$ on $(a,b]$ is defined by

$$C_{\vartheta,\Xi}[a,b] = \left\{\mathbb{Q} : (a,b] \to \mathbb{R}; \left(\Xi(t) - \Xi(a)\right)^{\vartheta} \mathbb{Q}(t) \in C[a,b]\right\},$$

with the norm defined by

$$\|\mathbb{Q}\|_{C_{\vartheta,\Xi}[a,b]} = \left\|(\Xi(t)-\Xi(a))^{\vartheta}\right\|_{C[a,b]} = \max_{t\in J}\left|(\Xi(t)-\Xi(a))^{\vartheta}\right|.$$

The weighted space $C^{n}_{\vartheta,\Xi}[a,b]$ of functions $\mathbb{Q}$ on $[a,b]$ is defined by

$$C^{n}_{\vartheta,\Xi}[a,b] = \left\{\mathbb{Q} : [a,b] \to \mathbb{R}; \mathbb{Q}(t) \in C^{n-1}[a,b]; \mathbb{Q}^{n}(t) \in C_{\vartheta,\Xi}[a,b]\right\},$$

with the norm defined by,

$$\|\mathbb{Q}\|_{C^{n}_{\vartheta,\Xi}[a,b]} = \sum_{k=0}^{n-1}\left\|\mathbb{Q}^{k}\right\|_{C[a,b]} + \|\mathbb{Q}^{n}\|_{C_{\vartheta,\Xi}[a,b]}.$$

Clearly,

(i) $C^{0}_{\vartheta,\Xi}[a,b] = C_{\vartheta,\Xi}[a,b], \quad for \quad n=0.$

(ii) For $n-1 \le \vartheta_1 < \vartheta_2 < n, \quad C_{\vartheta_1,\Xi}[a,b] \subset C_{\vartheta_2,\Xi}[a,b].$

Definition 2.1. [9, 11] Let $\varphi_0, \varphi_1 : [0,1]\times\mathbb{R} \to [0,\infty)$ be two continuous functions such that forall $t\in\mathbb{R}$ and $\varrho\in[0,1]$, we have,

$$\lim_{\varrho\to 0^{+}}\varphi_0(\varrho,t)=0, \quad \lim_{\varrho\to 0^{+}}\varphi_1(\varrho,t)=1, \quad \lim_{\varrho\to 1^{-}}\varphi_0(\varrho,t)=1, \quad \lim_{\varrho\to 1^{-}}\varphi_1(\varrho,t)=0,$$

and $\varphi_0(\varrho,t)\neq 0, \varrho\in(0,1]; \varphi_1(\varrho,t)\neq 0, \varrho\in(0,1]$. Also let $\Xi(t)$ be a strictly positive increasing continuous function. Then,

$$\mathcal{D}^{\varrho,\Xi}\mathbb{Q}(t) = \varphi_1(\varrho,t)\mathbb{Q}(t) + \varphi_0(\varrho,t)\frac{\mathbb{Q}'(t)}{\Xi'(t)}, \tag{2.1}$$

gives the proportional differential operator of order ϱ with respect to the function $\Xi(t)$ of a function $\mathbb{Q}(t)$. If $\varphi_0(\varrho,t)=\varrho$ and $\varphi_1(\varrho,t)=1-\varrho$, then $\mathcal{D}^{\varrho,\Xi}$ becomes

$$\mathcal{D}^{\varrho,\Xi}\mathbb{Q}(t) = (1-\varrho)\mathbb{Q}(t) + \varrho\frac{\mathbb{Q}'(t)}{\Xi'(t)}, \tag{2.2}$$

and the integral corresponding to the proportional derivative which is defined in Eq.(2.2) is given by,

$$\mathcal{I}^{1,\varrho,\Xi}_{a}\mathbb{Q}(t) = \frac{1}{\varrho}\int_{a}^{t} e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}\mathbb{Q}(s)\Xi'(s)ds, \tag{2.3}$$

where, $\mathcal{I}^{0,\varrho,\Xi}_{a}\mathbb{Q}(t)=\mathbb{Q}(t)$.
The generalized proportional integral of order n corresponding to the proportional fractional derivative $\mathcal{D}^{n,\varrho,\Xi}\mathbb{Q}(t)$ is defined by,

$$\left(\mathcal{I}^{n,\varrho,\Xi}_{a}\mathbb{Q}\right)(t) = \frac{1}{\varrho^{n}\Gamma(n)}\int_{a}^{t} e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{n-1}\Xi'(s)\mathbb{Q}(s)ds, \tag{2.4}$$

where, $\mathcal{D}^{n,\varrho,\Xi} = \mathcal{D}^{\varrho,\Xi}.\mathcal{D}^{\varrho,\Xi}.\mathcal{D}^{\varrho,\Xi}...\mathcal{D}^{\varrho,\Xi}$.

Definition 2.2. [9, 11] If $\varrho\in(0,1]$ and $p\in\mathbb{C}$ with $Re(p)>0$. Then the left-sided generalized proportional fractional integral of order p of the function $\mathbb{Q}$ with respect to the another function Ξ is defined by,

$$\left(\mathcal{I}^{p,\varrho,\Xi}_{a^{+}}\mathbb{Q}\right)(t) = \frac{1}{\varrho^{p}\Gamma(p)}\int_{a^{+}}^{t} e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi'(s)\mathbb{Q}(s)ds, \quad t>a. \tag{2.5}$$

Definition 2.3. [9, 11] If $\varrho \in (0,1]$ and $p \in \mathbb{C}$ with $Re(p) \geq 0$ and $\Xi \in C[a,b]$ where $\Xi^{'}(s) > 0$. Then the left-sided generalized proportional fractional derivative of order p of the function $\mathbb{Q}$ with respect to the another function Ξ is defined by,

$$\left(\mathcal{D}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t) = \frac{\mathcal{D}_t^{n,\varrho,\Xi}}{\varrho^{n-p}\Gamma(n-p)} \int_{a^+}^{t} e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{n-p-1}\Xi^{'}(s)\mathbb{Q}(s)ds,$$

where $\Gamma(.)$ is the gamma function and $n = [Re(p)] + 1$.

Proposition 2.4. *[9, 11] If $p, q \in \mathbb{C}$ such that $Re(p) \geq 0$ and $Re(q) > 0$, then for any $\varrho > 0$, we have*

(i) $\left(\mathcal{I}_{a^+}^{p,\varrho,\Xi} e^{\frac{\varrho-1}{\varrho}(\Xi(s)-\Xi(a))}(\Xi(s)-\Xi(a))^{q-1}\right)(t) = \frac{\Gamma(q)}{\varrho^p\Gamma(p+q)} e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{p+q-1}$,

(ii) $\left(\mathcal{D}_{a^+}^{p,\varrho,\Xi} e^{\frac{\varrho-1}{\varrho}(\Xi(s)-\Xi(a))}(\Xi(s)-\Xi(a))^{q-1}\right)(t) = \frac{\varrho^p\Gamma(q)}{\Gamma(q-p)} e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{q-p-1}$.

Theorem 2.5. *[9, 11] Suppose $\varrho \in (0,1], Re(p) > 0$ and $Re(q) > 0$. Then, if $\mathbb{Q}$ is continuous and defined for $t \geq a$, we have*

$$\mathcal{I}_{a^+}^{p,\varrho,\Xi}\left(\mathcal{I}_{a^+}^{q,\varrho,\Xi}\mathbb{Q}\right)(t) = \mathcal{I}_{a^+}^{q,\varrho,\Xi}\left(\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t) = \left(\mathcal{I}_{a^+}^{p+q,\varrho,\Xi}\mathbb{Q}\right)(t).$$

Theorem 2.6. *[9, 11] Suppose $\varrho \in (0,1], 0 \leq n < [Re(p)] + 1$ with $n \in \mathbb{N}$. If $\mathbb{Q} \in \mathcal{L}_1(a,b)$, then*

$$\mathcal{D}_{a^+}^{n,\varrho,\Xi}\left(\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t) = \left(\mathcal{I}_{a^+}^{p-n,\varrho,\Xi}\mathbb{Q}\right)(t). \tag{2.6}$$

In particular, for n=1, hence by using the Leibnitz rule, we have

$$\mathcal{D}_{a^+}^{1,\varrho,\Xi}\left(\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t) = \frac{p-1}{\varrho^{p-1}\Gamma(p)} \int_{a^+}^{t} e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-2}\Xi^{'}(s)\mathbb{Q}(s)ds. \tag{2.7}$$

Corollary 2.7. *[9, 11] If $0 < Re(q) < Re(p)$ and $n-1 < Re(q) \leq n, n \in \mathbb{N}$. Then we have*

$$\mathcal{D}_{a^+}^{q,\varrho,\Xi}\left(\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t) = \left(\mathcal{I}_{a^+}^{p-q,\varrho,\Xi}\mathbb{Q}\right)(t). \tag{2.8}$$

Theorem 2.8. *[9, 11] Suppose $\mathbb{Q} \in \mathcal{L}_1(a,b)$ and $Re(p) > 0, \varrho \in (0,1], n = [Re(p)] + 1$. Then,*

$$\mathcal{D}_{a^+}^{p,\varrho,\Xi}\left(\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t) = \mathbb{Q}(t), \quad t \geq a. \tag{2.9}$$

Theorem 2.9. *[11] If $p > n, \varrho \in (0,1]$ and n is a positive integer, then we have* $\left(\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathcal{D}_{a^+}^{n,\varrho,\Xi}\mathbb{Q}\right)(t) = \left(\mathcal{D}_{a^+}^{n,\varrho,\Xi}\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t)$

$$-\sum_{k=0}^{n-1} \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{p-n+k}}{\varrho^{p-n+k}\Gamma(p+k-n+1)}\left(\mathcal{D}^{k,\varrho,\Xi}\mathbb{Q}\right)(a). \tag{2.10}$$

Theorem 2.10. *[11] Assume that $Re(p) > 0$, $n = -[-Re(p)]$, $\mathbb{Q} \in \mathcal{L}_1(a,b)$ and $\left(\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t) \in AC^n[a,b]$. Then,*

$$\left(\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathcal{D}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t) = \mathbb{Q}(t) - \sum_{j=1}^{n} \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{p-j}}{\varrho^{p-j}\Gamma(p-j+1)}\left(\mathcal{I}_{a^+}^{j-p,\varrho,\Xi}\mathbb{Q}\right)(a). \tag{2.11}$$

Definition 2.11. [9, 11] If $\varrho \in (0,1]$ and $p \in \mathbb{C}$ with $Re(p) \geq 0$, then the generalized left Caputo proportional fractional derivative of function $\mathbb{Q}$ with respect to a another function Ξ is defined by,

$$\left({}^C\mathcal{D}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t) = \mathcal{I}_{a^+}^{n-p,\varrho,\Xi}\left(\mathcal{D}^{n,\varrho,\Xi}\mathbb{Q}\right)(t),$$

$$= \frac{1}{\varrho^{n-p}\Gamma(n-p)}\int_{a^+}^{t} e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{n-p-1}\Xi'(s)\left(\mathcal{D}_s^{n,\varrho,\Xi}\mathbb{Q}(s)ds\right), \tag{2.12}$$

where, $n = [Re(p)] + 1$.

Corollary 2.12. *[11] Let $p \in \mathbb{C}$ with $Re(p) > 0$ and $\varrho \in (0,1], n = [Re(p)] + 1$. If $\mathbb{Q} \in C^n[a,b]$ then,*

$$\left({}^C\mathcal{D}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}\right)(t) = \mathcal{D}_{a^+}^{p,\varrho,\Xi}\left[\mathbb{Q}(t) - \sum_{k=0}^{n-1}\frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^k}{\varrho^k.k!}\left(\mathcal{D}_{a^+}^{k,\varrho,\Xi}\mathbb{Q}\right)(a)\right]. \tag{2.13}$$

Proposition 2.13. *[9, 11] If $p, q \in \mathbb{C}$ with $Re(p) > 0, Re(q) > 0$ then for any $\varrho > 0$ and $n = [Re(p)] + 1$, we obtained as follows*

$$\left({}^C\mathcal{D}_{a^+}^{p,\varrho,\Xi}e^{\frac{\varrho-1}{\varrho}(\Xi(s)-\Xi(a))}(\Xi(s)-\Xi(a))^{q-1}\right)(t) = \frac{\varrho^p\Gamma(q)}{\Gamma(q-p)}e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{q-p-1}.$$

For k=0,1,2,........,n-1, we have

$$\left({}^C\mathcal{D}_{a^+}^{p,\varrho,\Xi}e^{\frac{\varrho-1}{\varrho}(\Xi(s)-\Xi(a))}(\Xi(s)-\Xi(a))^k\right)(t) = 0.$$

In particular, $\left({}^C\mathcal{D}_{a^+}^{p,\varrho,\Xi}e^{\frac{\varrho-1}{\varrho}(\Xi(s)-\Xi(a))}\right)(t) = 0.$

3. Main Result

Definition 3.1. [17] Let $J = [a,b]$, where $-\infty \leq a < b \leq \infty$ be an interval and $\mathbb{Q}, \Xi \in C^n[a,b]$ be two functions such that Ξ is positive strictly increasing and $\Xi'(t) \neq 0, \forall t \in [a,b]$. The Ξ-HGPFD of order p and type q of $\mathbb{Q}$ with respect to the another function Ξ are defined by

$$\left(\mathcal{D}_{a^\pm}^{p,q,\varrho,\Xi}\mathbb{Q}\right)(t) = \left(\mathcal{I}_{a^\pm}^{q(n-p),\varrho,\Xi}\left(\mathcal{D}^{n,\varrho,\Xi}\right)\mathcal{I}_{a^\pm}^{(1-q)(n-p),\varrho,\Xi}\mathbb{Q}\right)(t), \tag{3.1}$$

where $n-1 < p < n, 0 \leq q \leq 1$ with $n \in \mathbb{N}$ and $\varrho \in (0,1]$. Also $\mathcal{D}^{\varrho,\Xi}\mathbb{Q}(t) = (1-\varrho)\mathbb{Q}(t) + \varrho\frac{\mathbb{Q}'(t)}{\Xi'(t)}$ and $\mathcal{I}$ is the generalized proportional fractional integral defined in Eq.(2.5).
In particular, if $n = 1$, then $0 < p < 1$ and $0 \leq q \leq 1$, so Eq.(3.1) becomes,

$$\left(\mathcal{D}_{a^\pm}^{p,q,\varrho,\Xi}\mathbb{Q}\right)(t) = \left(\mathcal{I}_{a^\pm}^{q(1-p),\varrho,\Xi}\left(\mathcal{D}^{1,\varrho,\Xi}\right)\mathcal{I}_{a^\pm}^{(1-q)(1-p),\varrho,\Xi}\mathbb{Q}\right)(t).$$

Remark 3.2. [17] From the Definition 3.1, we can view the operator $\mathcal{D}_{a^\pm}^{p,q,\varrho,\Xi}$ is the interpolate between the Riemann-Liouville and Caputo generalized proportional fractional derivatives respectively, since

$$\mathcal{D}_{a^\pm}^{p,q,\varrho,\Xi}\mathbb{Q} = \begin{cases}\mathcal{D}^{n,\varrho,\Xi}\mathcal{I}_{a^\pm}^{n-p,\varrho,\Xi}\mathbb{Q}, & if \quad q = 0,\\ \mathcal{I}_{a^\pm}^{q(n-p),\varrho,\Xi}\mathcal{D}^{n,\varrho,\Xi}\mathbb{Q}, & if \quad q = 1.\end{cases}$$

Property 3.3. [17] The Ξ-HGPFD $\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathbb{Q}$ is equivalent to

$$\left(\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathbb{Q}\right)(t) = \left(\mathcal{I}_{a^+}^{q(n-p),\varrho,\Xi}\left(\mathcal{D}^{n,\varrho,\Xi}\right)\mathcal{I}_{a^+}^{(1-q)(n-p),\varrho,\Xi}\mathbb{Q}\right)(t) = \left(\mathcal{I}_{a^+}^{q(n-p),\varrho,\Xi}\mathcal{D}_{a^+}^{\vartheta,\varrho,\Xi}\mathbb{Q}\right)(t),$$

where, $\vartheta = p + q(n-p)$.

Theorem 3.4. *[17] Let $n-1<p<n$ with $n\in\mathbb{N}, 0\leq q\leq 1, \varrho\in(0,1]$ and $\vartheta=p+q(n-p)$. For $\alpha\in\mathbb{R}$ such that $\alpha>n$ then the image of the function $\mathbb{Q}(t)=e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\alpha-1}$ under the operator $\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}$ is defined by*

$$\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathbb{Q}(t)=\frac{\varrho^p\Gamma(\alpha)}{\Gamma(\alpha-p)}e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\alpha-p-1}.$$

Lemma 3.5. *[17] Let $n-1<p<n$ with $n\in\mathbb{N}, 0\leq q\leq 1, \varrho\in(0,1]$ and $\vartheta=p+q(n-p)$. For $\theta>0$, consider the function $\mathbb{Q}(t)=e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}E_\alpha(\theta\,(\Xi(t)-\Xi(a)))^p$, where $E_\alpha(.)$ is the Mittag-Leffler function with one parameter. Then,*

$$\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathbb{Q}(t)=\theta\varrho^p\mathbb{Q}(t).$$

Property 3.6. [17] Assume that p, q, ϑ satisfying the relations as

$$\vartheta=p+q(n-p), n-1<p,\vartheta\leq n, 0\leq q\leq 1,$$

and

$$\vartheta\geq p, \vartheta>q, n-\vartheta<n-q(n-p).$$

Let us consider the weighted spaces of continuous functions on (a,b] follows as,

$$C_{n-\vartheta,\Xi}^{p,q}[a,b]=\left\{\mathbb{Q}\in C_{n-\vartheta,\Xi}[a,b], \mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathbb{Q}\in C_{\vartheta,\Xi}[a,b]\right\},$$

and

$$C_{n-\vartheta,\Xi}^{p,q}[a,b]=\left\{\mathbb{Q}\in C_{n-\vartheta,\Xi}[a,b], \mathcal{D}_{a^+}^{\vartheta,\varrho,\Xi}\mathbb{Q}\in C_{n-\vartheta,\Xi}[a,b]\right\}.$$

By the Property 3.3, we say that

$$C_{n-\vartheta,\Xi}^{\vartheta}[a,b]\subset C_{n-\vartheta,\Xi}^{p,q}[a,b].$$

Lemma 3.7. *[17] Let $n-1\leq\vartheta<n, n-1<p<n$ with $n\in\mathbb{N}, \varrho\in(0,1]$. If $\mathbb{Q}\in C_\vartheta[a,b]$ then*

$$\mathcal{I}_{a^+}^{p,q,\Xi}\mathbb{Q}(a)=\lim_{t\to a^+}\mathcal{I}_{a^+}^{p,q,\Xi}\mathbb{Q}(t)=0,\quad n-1\leq\vartheta<p. \tag{3.2}$$

Lemma 3.8. *[17] Let $n-1<p<n$ with $n\in\mathbb{N}, 0\leq q\leq 1, \varrho\in(0,1]$ and $\vartheta=p+q(n-p)$. If $\mathbb{Q}\in C_{n-\vartheta}^{\vartheta}[a,b]$ then*

$$\mathcal{I}_{a^+}^{\vartheta,\varrho,\Xi}\mathcal{D}_{a^+}^{\vartheta,\varrho,\Xi}\mathbb{Q}=\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathbb{Q},$$

and

$$\mathcal{D}_{a^+}^{\vartheta,\varrho,\Xi}\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}=\mathcal{D}_{a^+}^{q(n-p),\varrho,\Xi}\mathbb{Q}.$$

Lemma 3.9. *[17] Let $\mathbb{Q}\in\mathcal{L}_1(a,b)$. If $\mathcal{D}^{q(n-p),\varrho,\Xi}\mathbb{Q}$ exists in $\mathcal{L}_1(a,b)$, then*

$$\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}=\mathcal{I}_{a^+}^{q(n-p),\varrho,\Xi}\mathcal{D}_{a^+}^{q(n-p),\varrho,\Xi}\mathbb{Q}.$$

Lemma 3.10. *[17] Let $n-1<p<n$ with $n\in\mathbb{N}, 0\leq q\leq 1, \varrho\in(0,1]$ and $\vartheta=p+q(n-p)$. If $\mathbb{Q}\in C_{n-\vartheta}^{\vartheta}[a,b]$ and $\mathcal{I}_{a^+}^{n-q(n-p),\varrho,\Xi}\mathbb{Q}\in C_{n-\vartheta,\Xi}^{n}[a,b]$ then $\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}$ exists in (a,b] and*

$$\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}(t)=\mathbb{Q}(t),\quad t\in(a,b].$$

Theorem 3.11. *[17] Let $n-1<p<n$ with $n\in\mathbb{N}, 0\le q\le 1, \varrho\in(0,1]$. If $\mathbb{Q}\in C^n[a,b]$, then*

$$\left(\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathbb{Q}\right)(t)=\mathcal{D}_{a^+}^{n-q(n-p),\varrho,\Xi}\left[\mathcal{I}_{a^+}^{n-\vartheta,\varrho,\Xi}\mathbb{Q}(t)-\sum_{k=0}^{n-1}\frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^k}{\varrho^k k!}\left(\mathcal{D}_{a^+}^{\vartheta,\varrho,\Xi}\mathbb{Q}\right)(t)\right],$$

where $\vartheta=p+q(n-p)$.

Lemma 3.12. *[17] Let $n-1<p<n$ with $n\in\mathbb{N}, 0\le q\le 1, \varrho\in(0,1]$ with $\vartheta=p+q(n-p)$ such that $n-1<\vartheta<n$. If $\mathbb{Q}\in C_\vartheta[a,b]$, and $\mathcal{I}_{a^+}^{n-\vartheta,\varrho,\Xi}\mathbb{Q}\in C^n_{\vartheta,\Xi}[a,b]$, then*

$$\left(\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathbb{Q}\right)(t)=\mathbb{Q}(t)-\left[\sum_{k=1}^{n}\frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-k}}{\varrho^{\vartheta-k}\Gamma(\vartheta-k+1)}\left(\mathcal{I}_{a^+}^{k-\vartheta,\varrho,\Xi}\mathbb{Q}\right)(a)\right]. \tag{3.3}$$

Lemma 3.13. *Let $0<p<1, 0\le q\le 1, \vartheta=p+q(1-p)$ and assume that $\mathbb{Q}\left(.,\mathcal{A}(.),\mathcal{A}\left(\frac{.}{\eta}\right)\right)\in C_{1-\vartheta}[a,b]$ for any $\mathcal{A}\in C_{1-\vartheta}[a,b]$ where $\mathbb{Q}:(a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a function. If $\mathcal{A}\in C^\vartheta_{1-\vartheta}[a,b]$ then $\mathcal{A}$ satisfies our proposed problem (1.1)-(1.2) if and only if $\mathcal{A}$ satisfies Eq.(1.3).*

Proof. Let us consider $\mathcal{A}\in C^\vartheta_{1-\vartheta}[a,b]$ is a solution of our proposed problem (1.1)-(1.2). Now we have to prove that $\mathcal{A}$ is a solution of Eq. (1.3). From the Lemma 3.12 with $n=1$, we get

$$\left(\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathcal{D}_{a^+}^{p,q,\varrho,\Xi}\mathcal{A}\right)(t)=\mathcal{A}(t)-\frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}}{\varrho^{\vartheta-1}\Gamma(\vartheta)}\left(\mathcal{I}_{a^+}^{1-\vartheta,\varrho,\Xi}\mathcal{A}\right)(a),$$

which gives,

$$\begin{aligned}\mathcal{A}(t)&=\frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}}{\varrho^{\vartheta-1}\Gamma(\vartheta)}\left(\mathcal{I}_{a^+}^{1-\vartheta,\varrho,\Xi}\mathcal{A}\right)(a)\\&+\frac{1}{\varrho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds.\end{aligned} \tag{3.4}$$

Let us take $t=\tau_i$ and multiplying μ_i on both sides of Eq. 3.4, we get

$$\begin{aligned}\mu_i\mathcal{A}(\tau_i)&=\frac{\mu_i e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(a))}(\Xi(\tau_i)-\Xi(a))^{\vartheta-1}}{\varrho^{\vartheta-1}\Gamma(\vartheta)}\left(\mathcal{I}_{a^+}^{1-\vartheta,\varrho,\Xi}\mathcal{A}\right)(a)\\&+\frac{1}{\varrho^p\Gamma(p)}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds,\end{aligned}$$

which implies that

$$\begin{aligned}\sum_{i=1}^{m}\mu_i\mathcal{A}(\tau_i)&=\frac{1}{\varrho^{\vartheta-1}\Gamma(\vartheta)}\sum_{i=1}^{m}\mu_i e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(a))}(\Xi(\tau_i)-\Xi(a))^{\vartheta-1}\left(\mathcal{I}_{a^+}^{1-\vartheta,\varrho,\Xi}\mathcal{A}\right)(a)\\&+\frac{1}{\varrho^p\Gamma(p)}\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds,\end{aligned} \tag{3.5}$$

where $\tau_i>a$.
Hence by Eq.(1.2), we get

$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho,\Xi}\mathcal{A}(a)=\frac{\varrho^{\vartheta-1}\Gamma(\vartheta)}{\varrho^p\Gamma(p)}\wedge\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds. \tag{3.6}$$

After substituting (3.6) in (3.4), we get the required result.
Hence $\mathcal{A}(t)$ satisfies (1.3).
Conversely, suppose that $\mathcal{A} \in C^{\vartheta}_{1-\vartheta}[a,b]$ satisfies (1.3), we have to show that $\mathcal{A}(t)$ also satisfies (1.1)-(1.2). Now applying the operator $\mathcal{D}^{\vartheta,\varrho,\Xi}_{a^+}$ on both sides of Eq. (1.3) we get,

$$\begin{aligned}\mathcal{D}^{\vartheta,\varrho,\Xi}_{a^+}\mathcal{A}(t) = \mathcal{D}^{\vartheta,\varrho,\Xi}_{a^+}&\left(\frac{\wedge}{\varrho^p\Gamma(p)}e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}\left(\Xi(t)-\Xi(a)\right)^{\vartheta-1}\right.\\ &\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(s))}\left(\Xi(\tau_i)-\Xi(s)\right)^{p-1}\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds\Bigg)\\ &+\mathcal{D}^{\vartheta,\varrho,\Xi}_{a^+}\left(\frac{1}{\varrho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}\left(\Xi(t)-\Xi(s)\right)^{p-1}\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds\right). \end{aligned} \tag{3.7}$$

By using the Proposition 2.4 and Lemma 3.8, we get

$$\mathcal{D}^{\vartheta,\varrho,\Xi}_{a^+}\mathcal{A}(t) = \mathcal{D}^{q(1-p),\varrho,\Xi}_{a^+}\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)(t).$$

Since $\mathcal{A} \in C^{\vartheta}_{1-\vartheta}[a,b]$ and by the definition of weighted space $C^{\vartheta}_{1-\vartheta}[a,b]$ we get $\mathcal{D}^{\vartheta,\varrho,\Xi}_{a^+}\mathcal{A} \in C_{1-\vartheta}[a,b]$ and by the Eq.(3.7)we get

$$\mathcal{D}^{q(1-p),\varrho,\Xi}_{a^+}\mathbb{Q} = \mathcal{D}^{1,\varrho,\Xi}\mathcal{I}^{1-q(1-p),\varrho,\Xi}\mathbb{Q} \in C_{1-\vartheta,\Xi}[a,b].$$

And for $\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)(t) \in C_{1-\vartheta}[a,b]$ and by the Theorem 2.8, it gives that

$$\mathcal{I}^{1-q(1-p),\varrho,\Xi}_{a^+}\mathbb{Q} \in C_{1-\vartheta,\Xi}[a,b],$$

and from the definition of $C^{n}_{1-\vartheta,\Xi}[a,b]$, that

$$\mathcal{I}^{1-q(1-p),\varrho,\Xi}_{a^+}\mathbb{Q} \in C^{1}_{1-\vartheta,\Xi}[a,b].$$

Now applying the operator $\mathcal{I}^{q(1-p),\varrho,\Xi}_{a^+}$ on both sides of Eq.(3.7) and by the Theorem 2.10 and the Lemma 3.7,we get

$$\begin{aligned}\mathcal{I}^{q(1-p),\varrho,\Xi}_{a^+}\mathcal{D}^{\vartheta,\varrho,\Xi}_{a^+}\mathcal{A}(t) &= \mathcal{I}^{q(1-p),\varrho,\Xi}_{a^+}\mathcal{D}^{q(1-p),\varrho,\Xi}_{a^+}\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)(t)\\ &= \mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right) - \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}\mathcal{I}^{1-q(1-p),\varrho,\Xi}_{a^+}\mathbb{Q}(a)}{\varrho^{q(1-p)-1}\Gamma(q(1-p))}\left(\Xi(t)-\Xi(s)\right)^{q(1-p)-1}\\ &= \mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right).\end{aligned} \tag{3.8}$$

Hence,

$$\mathcal{I}^{q(1-p),\varrho,\Xi}_{a^+}\mathcal{D}^{\vartheta,\varrho,\Xi}_{a^+}\mathcal{A}(t) = \mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right), \quad t \in J.$$

Now, we have to prove that Eq.(1.2) holds. To prove this, we have to applying the operator $\mathcal{I}^{1-\vartheta,\varrho,\Xi}_{a^+}$ on the both sides of Eq.(1.3), we get

$$\begin{aligned}\mathcal{I}^{1-\vartheta,\varrho,\Xi}_{a^+}\mathcal{A}(t) = \mathcal{I}^{1-\vartheta,\varrho,\Xi}_{a^+}&\left(\frac{\wedge}{\varrho^p\Gamma(p)}e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}\left(\Xi(t)-\Xi(a)\right)^{\vartheta-1}\right.\\ &\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(s))}\left(\Xi(\tau_i)-\Xi(s)\right)^{p-1}\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds\Bigg)\\ &+\mathcal{I}^{1-\vartheta,\varrho,\Xi}_{a^+}\left(\frac{1}{\varrho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}\left(\Xi(t)-\Xi(s)\right)^{p-1}\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds\right).\end{aligned}$$

And then using Proposition 2.4 and Theorem 2.5, we get

$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho,\Xi}\mathcal{A}(t) = \frac{\varrho^{\vartheta-1}\Gamma(\vartheta)}{\varrho^p\Gamma(p)} \wedge e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i} e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(s))} (\Xi(\tau_i)-\Xi(s))^{p-1}\,\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds + \mathcal{I}_{a^+}^{1-q(1-p),\varrho,\Xi}\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)(t). \quad (3.9)$$

Since $1-\vartheta < q(1-p)$, so taking the limit as $t \to a^+$ and using Lemma 3.7 in the Eq.(3.9), we get $\mathcal{I}_{a^+}^{1-\vartheta,\varrho,\Xi}\mathcal{A}(a^+) =$

$$\frac{\varrho^{\vartheta-1}\Gamma(\vartheta)}{\varrho^p\Gamma(p)} \wedge \sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i} e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(s))} (\Xi(\tau_i)-\Xi(s))^{p-1}\,\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds. \quad (3.10)$$

Next, substituting $t=\tau_i$ and multiply μ_i on both sides of Eq. (1.3), we get

$$\begin{aligned}\mu_i\mathcal{A}(\tau_i) &= \frac{\wedge}{\varrho^p\Gamma(p)}\mu_i e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(a))}(\Xi(\tau_i)-\Xi(a))^{\vartheta-1}\\ &\quad\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i} e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(s))} (\Xi(\tau_i)-\Xi(s))^{p-1}\,\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds\\ &\quad+\frac{\mu_i}{\varrho^p\Gamma(p)}\int_{a^+}^{\tau_i} e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(s))} (\Xi(\tau_i)-\Xi(s))^{p-1}\,\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds,\end{aligned}$$

which gives that,

$$\begin{aligned}\sum_{i=1}^{m}\mu_i\mathcal{A}(\tau_i) &= \wedge\sum_{i=1}^{m}\mu_i\left(\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right)(\tau_i)\sum_{i=1}^{m}\mu_i e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(a))}(\Xi(\tau_i)-\Xi(a))^{\vartheta-1}\\ &\quad+\sum_{i=1}^{m}\mu_i\left(\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right)(\tau_i)\\ &=\sum_{i=1}^{m}\mu_i\left(\mathcal{I}_{a^+}^{p,\varrho,\Xi}\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right)(\tau_i)\left(1+\wedge\sum_{i=1}^{m}\mu_i e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(a))}(\Xi(\tau_i)-\Xi(a))^{\vartheta-1}\right).\end{aligned}$$

Thus,

$$\sum_{i=1}^{m}\mu_i\mathcal{A}(\tau_i) = \frac{\varrho^{\vartheta-1}\Gamma(\vartheta)}{\varrho^p\Gamma(p)} \wedge \sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i} e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(s))} (\Xi(\tau_i)-\Xi(s))^{p-1}\,\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds. \quad (3.11)$$

Hence from (3.10) and (3.11), we get

$$\mathcal{I}_{a^+}^{1-\vartheta,\varrho,\Xi}\mathcal{A}(a^+) = \sum_{i=1}^{m}\mu_i\mathcal{A}(\tau_i), \quad (3.12)$$

this complete the proof. □

4. Existence of solution

Now in our next theorem, we prove the existence of solution of Eq.(1.1)-(1.2) in the weighted space $C_{1-\vartheta,\Xi}^{p,q}$ by the concepts of Krasnoselskii's fixed point theorem.

Theorem 4.1. *(Krasnoselskii's fixed point theorem)[7] Let B be a non empty bounded closed convex subset of a Banach space X. Let $N, M : B \to X$ be two continuos operators satisfying:*

- *$Nx + My \in B$ whenever $x, y \in B$,*
- *N is compact and continuous,*
- *M is contraction mapping,*

then, there exist $u \in B$ such that $u = Nu + Mu$.

Let us consider the following hypotheses:
(H_1) : Let $\mathbb{Q} : (a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $\mathbb{Q} \in C^{q(1-p)}_{1-\vartheta,\Xi}[a, b]$ for any $\mathcal{A} \in C^{\vartheta}_{1-\vartheta,\Xi}[a, b]$.
(H_2) : There exists a constant $k > 0$ such that

$$|\mathbb{Q}(t, u, v) - \mathbb{Q}(t, \overline{u}, \overline{v})| \le k \{|u - v| + |\overline{u} - \overline{v}|\}, \quad \forall u, v, \overline{u}, \overline{v} \in \mathbb{R} \quad and \quad t \in J.$$

(H_3) : Let us assume that

$$k\phi < 1,$$

where,

$$\phi = \frac{\beta(\vartheta, p)}{\varrho^p \Gamma(p)} \left(|\wedge| \sum_{i=1}^{m} \mu_i (\Xi(\tau_i) - \Xi(a))^{p+\vartheta-1} + (\Xi(b) - \Xi(a))^p \right), \tag{4.1}$$

and

$$\beta(\vartheta, p) = \int_0^1 t^{\vartheta-1}(1 - t)^{p-1} dt, \quad Re(\vartheta), \quad Re(p) > 0,$$

is the beta function.
(H_4) :Also let

$$k\triangle < 1,$$

where,

$$\triangle = \frac{\beta(\vartheta, p)}{\varrho^p \Gamma(p)} |\wedge| \sum_{i=1}^{m} \mu_i (\Xi(\tau_i) - \Xi(a))^{p+\vartheta-1}. \tag{4.2}$$

Theorem 4.2. *Let $0 < p < 1, 0 \le q \le 1$ and $\vartheta = p+q(1-p)$. Suppose that the assumptions $(H_1), (H_2), (H_4)$ hold. Then the problem (1.1)-(1.2) has at least one solution in the space $C^{\vartheta}_{1-\vartheta}[a, b]$.*

Proof. Given that $\|X\|_{C_{1-\vartheta,\Xi}[a,b]} = \sup_{t\in J} \left| (\Xi(t) - \Xi(a))^{1-\vartheta} X(t) \right|$ and choose $k \ge M \|X\|_{C_{1-\vartheta,\Xi}[a,b]}$, where

$$M = \frac{\beta(\vartheta, p)}{\varrho^p \Gamma(p)} \left(|\wedge| \sum_{i=1}^{m} \mu_i (\Xi(\tau_i) - \Xi(a))^{p+\vartheta-1} + (\Xi(b) - \Xi(a))^p \right), \tag{4.3}$$

also consider $B_k = \left\{ \mathcal{A} \in C[a, b] : \|\mathcal{A}\|_{C_{1-\vartheta}[a,b]} \le k \right\}$. Thus $\forall t \in [a, b]$ consider the operators $\mathcal{G}$ and $\mathcal{H}$ defined on B_k by

$$(\mathcal{H}\mathcal{A})(t) = \frac{1}{\varrho^p \Gamma(p)} \int_{a^+}^{t} e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))} (\Xi(t) - \Xi(s))^{p-1} \Xi'(s) \mathbb{Q}\left(s, \mathcal{A}(s), \mathcal{A}\left(\frac{s}{\eta}\right)\right) ds,$$

$$(\mathcal{G}\mathcal{A})(t)=\frac{\wedge}{\varrho^{p}\Gamma(p)}e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds.$$

Step1. For all $\mathcal{A},\overline{\mathcal{A}}\in B_k$, yields
$\left|\left(\mathcal{H}\mathcal{A}(t)+\mathcal{G}\overline{\mathcal{A}}(t)\right)(\Xi(t)-\Xi(a))^{1-\vartheta}\right|$

$$\begin{aligned}
&\le\frac{(\Xi(t)-\Xi(a))^{1-\vartheta}}{\varrho^{p}\Gamma(p)}\int_{a^+}^{t}(\Xi(t)-\Xi(s))^{p-1}(\Xi(s)-\Xi(a))^{\vartheta-1}\Xi'(s)\\
&\left|\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)(\Xi(s)-\Xi(a))^{\vartheta-1}\right|ds\\
&+\frac{\wedge}{\varrho^{p}\Gamma(p)}\sum_{i=1}^{m}\mu_i\frac{(\Xi(t)-\Xi(a))^{1-\vartheta}}{\varrho^{p}\Gamma(p)}\int_{a^+}^{\tau_i}(\Xi(t)-\Xi(s))^{p-1}(\Xi(s)-\Xi(a))^{\vartheta-1}\\
&\left|\mathbb{Q}\left(s,\overline{\mathcal{A}}(s),\overline{\mathcal{A}}\left(\frac{s}{\eta}\right)\right)(\Xi(\tau_i)-\Xi(a))^{\vartheta-1}\right|ds\\
&\le\|X\|\left[\frac{\beta(\vartheta,p)}{\varrho^{p}\Gamma(p)}\left(|\wedge|\sum_{i=1}^{m}\mu_i(\Xi(\tau_i)-\Xi(a))^{p+\vartheta-1}+(\Xi(b)-\Xi(a))^{p}\right)\right]\\
&\le\|X\|M\\
&\le\varrho<\infty,
\end{aligned}$$

this implies that $\mathcal{H}\mathcal{A}+\mathcal{H}\overline{\mathcal{A}}\in B_k$.
Step2. We show that M is a contraction. Let $\mathcal{A},\overline{\mathcal{A}}\in C_{1-\vartheta}[a,b]$ and $t\in J$ then
$\left|\left(\mathcal{G}\mathcal{A}(t)-\mathcal{G}\overline{\mathcal{A}}(t)\right)(\Xi(t)-\Xi(a))^{1-\vartheta}\right|$

$$\begin{aligned}
&=\left|\wedge e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}\sum_{i=1}^{m}\mu_i\mathcal{I}_{a^+}^{1-q(1-p),\varrho,\Xi}\left(\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)-\mathbb{Q}\left(s,\overline{\mathcal{A}}(s),\overline{\mathcal{A}}\left(\frac{s}{\eta}\right)\right)\right)\right|ds\\
&\le\frac{k\wedge}{\varrho^{p}\Gamma(p)}\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}(\Xi(\tau_i)-\Xi(s))^{p-1}(\Xi(s)-\Xi(a))^{\vartheta-1}\Xi'(s)\left|\mathcal{A}(s)-\overline{\mathcal{A}}(s)\right|ds\\
&\le\left[\frac{k\wedge}{\varrho^{p}\Gamma(p)}\beta(\vartheta,p)\sum_{i=1}^{m}\mu_i(\Xi(\tau_i)-\Xi(s))^{p+\vartheta-1}\right]\left\|\mathcal{A}-\overline{\mathcal{A}}\right\|_{C_{1-\vartheta,\Xi}[a,b]}
\end{aligned}$$

$$\left|\left(\mathcal{G}\mathcal{A}(t)-\mathcal{G}\overline{\mathcal{A}}(t)\right)(\Xi(t)-\Xi(a))^{1-\vartheta}\right|\le k\triangle\left\|\mathcal{A}-\overline{\mathcal{A}}\right\|_{C_{1-\vartheta,\Xi}[a,b]}. \tag{4.4}$$

Hence by (H_4) and Eq.(4.4), we can say M is a contraction.
Step3. Now we have to verify that the operator N is continuous and compact.
Since the function $\mathbb{Q}$ is continuous, so the operator N is also continuous.
Hence, for any $\mathcal{A}\in C_{1-\vartheta}[a,b]$, we get

$$\|\mathcal{H}\mathcal{A}\|\le\|X\|\frac{\beta(\vartheta,p)}{\varrho^{p}\Gamma(p)}(\Xi(b)-\Xi(a))^{p}<\infty.$$

This shows that $\mathcal{H}$ is uniformly bounded on B_k. Therefore it remains to prove that the operator $\mathcal{H}$ is compact. Denoting $\sup_{(t,\mathcal{A})\in J\times B_k}\left|\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right|=\delta<\infty$ and for any $a<\tau_1<\tau_2<b$,

$$
\begin{aligned}
&\left|(\Xi(\tau_2)-\Xi(a))^{1-\vartheta}(\mathcal{HA}(\tau_2))+(\Xi(\tau_1)-\Xi(a))^{1-\vartheta}(\mathcal{HA}(\tau_1))\right| \\
&\quad=\left|\frac{(\Xi(\tau_2)-\Xi(a))^{1-\vartheta}}{\varrho^p\Gamma(p)}\int_{a^+}^{\tau_2} e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_2)-\Xi(s))}(\Xi(\tau_2)-\Xi(s))^{p-1}\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds\right. \\
&\quad\left.+\frac{(\Xi(\tau_1)-\Xi(a))^{1-\vartheta}}{\varrho^p\Gamma(p)}\int_{a^+}^{\tau_1} e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_1)-\Xi(s))}(\Xi(\tau_1)-\Xi(s))^{p-1}\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds\right| \\
&\quad\le\frac{1}{\vartheta^p\Gamma(p)}\int_{\tau_1}^{\tau_2}\left[(\Xi(\tau_2)-\Xi(a))^{1-\vartheta}(\Xi(\tau_2)-\Xi(s))^{p-1}(\Xi(\tau_1)-\Xi(a))^{1-\vartheta}(\Xi(\tau_1)-\Xi(s))^{p-1}\right] \\
&\quad\times\Xi'(s)\left|\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right|ds \\
&\quad+\frac{1}{\vartheta^p\Gamma(p)}\int_{\tau_1}^{\tau_2}(\Xi(\tau_1)-\Xi(a))^{1-\vartheta}(\Xi(\tau_1)-\Xi(s))^{p-1}\Xi'(s)\left|\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right|ds
\end{aligned}
$$

$$
\left|(\Xi(\tau_2)-\Xi(a))^{1-\vartheta}(\mathcal{HA}(\tau_2))+(\Xi(\tau_1)-\Xi(a))^{1-\vartheta}(\mathcal{HA}(\tau_1))\right|\to 0\quad as\quad \tau_2\to\tau_1.
$$

As a consequence of Arzela-Ascoli theorem $\mathcal{H}$ is compact on B_k. Thus, as a result of our proposed problem (1.1)-(1.2) has at least one solution. □

5. Uniqueness of solution

In this section, we prove the uniqueness of solution of (1.1)-(1.2) in the weighted space $C^{\vartheta}_{1-\vartheta,\Xi}$.

Theorem 5.1. *Let $0<p<1$, $0\le q\le 1$ and $\vartheta=p+q(1-p)$. Suppose that the assumptions $(H_2)-(H_3)$ hold, then the problem (1.1)-(1.2) has a unique solution in the space $C^{\vartheta}_{1-\vartheta,\Xi}[a,b]$.*

Proof. Consider the fractional operator $\mathcal{T}: C_{1-\vartheta,\Xi}[a,b]\to C_{1-\vartheta,\Xi}[a,b]$ defined by:

$$
(\mathcal{TA})(t)=\begin{cases}\frac{\wedge}{\varrho^p\Gamma(p)}e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1} \\ \times\sum_{i=1}^m\mu_i\int_{a^+}^{\tau_i}e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds \\ +\frac{1}{\varrho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi'(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)ds.\end{cases}\tag{5.1}
$$

Clearly the operator $\mathcal{T}$ is well defined. Now for any $\mathcal{A}_1,\mathcal{A}_2\in C_{1-\vartheta}[a,b]$, $t\in J$ and $\left|e^{\frac{\varrho-1}{\varrho}\Xi(t)}\right|<1$, gives

$$
\begin{aligned}
&\left|((\mathcal{TA}_1)(t)-(\mathcal{TA}_2)(t))(\Xi(t)-\Xi(a))^{1-\vartheta}\right| \\
&\le\frac{\wedge}{\varrho^p\Gamma(p)}\sum_{i=1}^m\mu_i\int_{a^+}^{\tau_i}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi'(s)\left|\mathbb{Q}\left(s,\mathcal{A}_1(s),\mathcal{A}_1\left(\frac{s}{\eta}\right)\right)-\mathbb{Q}\left(s,\mathcal{A}_2(s),\mathcal{A}_2\left(\frac{s}{\eta}\right)\right)\right|ds \\
&+\frac{(\Xi(t)-\Xi(a))^{1-\vartheta}}{\varrho^p\Gamma(p)}\int_{a^+}^{t}(\Xi(t)-\Xi(s))^{p-1}\Xi'(s)\left|\mathbb{Q}\left(s,\mathcal{A}_1(s),\mathcal{A}_1\left(\frac{s}{\eta}\right)\right)-\mathbb{Q}\left(s,\mathcal{A}_2(s),\mathcal{A}_2\left(\frac{s}{\eta}\right)\right)\right|ds \\
&\le\frac{k\wedge}{\varrho^p\Gamma(p)}\left(\sum_{i=1}^m\mu_i\int_{a^+}^{\tau_i}(\Xi(\tau_i)-\Xi(s))^{p-1}(\Xi(s)-\Xi(a))^{\vartheta-1}\Xi'(s)ds\right)\|\mathcal{A}_1-\mathcal{A}_2\|_{C_{1-\vartheta,\Xi}[a,b]} \\
&+\frac{k(\Xi(t)-\Xi(a))^{1-\vartheta}}{\varrho^p\Gamma(p)}\left(\int_{a^+}^{t}(\Xi(t)-\Xi(s))^{p-1}(\Xi(s)-\Xi(a))^{1-\vartheta}\Xi'(s)\right)\|\mathcal{A}_1-\mathcal{A}_2\|_{C_{1-\vartheta,\Xi}[a,b]}. \\
&\le\frac{k\wedge}{\varrho^p\Gamma(p)}\beta(\vartheta,p)\sum_{i=1}^m\mu_i(\Xi(\tau_i)-\Xi(s))^{p+\vartheta-1}\|\mathcal{A}_1-\mathcal{A}_2\|_{C_{1-\vartheta,\Xi}[a,b]} \\
&+\frac{k(\Xi(b)-\Xi(a))^{p}}{\varrho^p\Gamma(p)}\beta(\vartheta,p)\|\mathcal{A}_1-\mathcal{A}_2\|_{C_{1-\vartheta,\Xi}[a,b]}.
\end{aligned}
$$

Hence,
$\|\mathcal{T}\mathcal{A}_1 - \mathcal{T}\mathcal{A}_2\|_{C_{1-\vartheta,\Xi}[a,b]}$

$$\leq \frac{k}{\varrho^p \Gamma(p)} \beta(\vartheta, p) \left(|\wedge| \sum_{i=1}^{m} \mu_i \left(\Xi(\tau_i) - \Xi(s)\right)^{p+\vartheta-1} + \left(\Xi(b) - \Xi(a)\right)^p \right) \|\mathcal{A}_1 - \mathcal{A}_2\|_{C_{1-\vartheta,\Xi}[a,b]}$$

$$\|\mathcal{T}\mathcal{A}_1 - \mathcal{T}\mathcal{A}_2\|_{C_{1-\vartheta,\Xi}[a,b]} \leq k\phi \, \|\mathcal{A}_1 - \mathcal{A}_2\|_{C_{1-\vartheta,\Xi}[a,b]} . \tag{5.2}$$

Thus, from (H_3) and Eq.(5.2), $\mathcal{T}$ is a contraction map. Hence, our proposed problem (1.1)-(1.2) has a unique solution. □

6. Stability theory

In this section, we study the Ulam-Hyers-Rassias(U-H-R) stability results for our proposed problem (1.1)-(1.2).

Now, we consider the Ulam-Hyers(U-H) stability for the problem

$$\mathcal{D}_{a^+}^{p,q,\varrho,\Xi} \mathcal{A}(t) = \mathbb{Q}\left(t, \mathcal{A}(t), \mathcal{A}\left(\frac{t}{\eta}\right)\right), \quad t \in J.$$

Let $\epsilon > 0$ and $\nu : (a, b] \to [0, \infty)$ be a continuous function.
We consider the following inequality,

$$\left| \mathcal{D}_{a^+}^{p,q,\varrho,\Xi} \mathcal{A}(t) - \mathbb{Q}\left(t, \mathcal{A}(t), \mathcal{A}\left(\frac{t}{\eta}\right)\right) \right| \leq \epsilon \nu(t), \quad t \in J. \tag{6.1}$$

Definition 6.1. [26] Our proposed problem (1.1)-(1.2) is U-H-R stable with respect to ν, if there exists a real number $\alpha_{\mathbb{Q}} > 0$ and for each solution $\mathcal{A} \in C^{\vartheta}_{n-\vartheta,\Xi}[a,b]$ of the inequality (6.1) there exists a solution $\overline{\mathcal{A}} \in C^{\vartheta}_{n-\vartheta,\Xi}[a,b]$ of the problem (1.1)-(1.2) with

$$\left|\mathcal{A}(t) - \overline{\mathcal{A}}(t)\right| \leq \epsilon \alpha_{\mathbb{Q},\nu}(t), \quad t \in J.$$

Remark 6.2. [26] A function $\mathcal{A} \in C^{\vartheta}_{n-\vartheta,\Xi}[a,b]$ is a solution of inequality (6.1) if and only if there exist a function $\sigma \in C^{\vartheta}_{n-\vartheta,\Xi}[a,b]$ such that,

- $|\sigma(t)| < \epsilon \nu(t), \quad t \in J.$
- $\mathcal{D}_{a^+}^{p,q,\varrho,\Xi} \mathcal{A}(t) = \mathbb{Q}\left(t, \mathcal{A}(t), \mathcal{A}\left(\frac{t}{\eta}\right)\right) + \sigma(t), \quad t \in J.$

Lemma 6.3. *[26] Let $0 < p < 1$, $0 \leq q \leq 1$, $\overline{\mathcal{A}} \in C^{\vartheta}_{n-\vartheta,\Xi}[a,b]$ is a solution of the inequality (6.1) then $\overline{\mathcal{A}}$ is a solution of the following integral inequality*

$$\left| \overline{\mathcal{A}}(t) - Y_t - \frac{1}{\varrho^p \Gamma(p)} \int_{a^+}^{t} e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))} \left(\Xi(t) - \Xi(s)\right)^{p-1} \Xi'(s) \mathbb{Q}\left(s, \mathcal{A}(s), \mathcal{A}\left(\frac{s}{\eta}\right)\right) \right|$$
$$\leq \wedge e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))} \left(\Xi(t) - \Xi(a)\right)^{\vartheta-1} \epsilon k_\nu \nu(t),$$

where,

$$Y_t = \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))} \left(\Xi(t) - \Xi(a)\right)^{\vartheta-1}}{\varrho^p \Gamma(p)} \wedge \sum_{i=1}^{m} \mu_i$$
$$\int_{a^+}^{\tau_i} e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(s))} \left(\Xi(\tau_i) - \Xi(s)\right)^{p-1} \Xi'(s) \mathbb{Q}\left(s, \mathcal{A}(s), \mathcal{A}\left(\frac{s}{\eta}\right)\right) ds.$$

Lemma 6.4. *[26] Let $\nu : [0,T] \to [0,\infty)$ be a real function and $w(.)$ is a non negative and locally integrable function on $[0,T]$ and there are constants $\alpha > 0, \quad 0 < p < 1$ such that*

$$\nu(t) \leq w(t) + \alpha \int_0^t \frac{\nu(s)}{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))} (\Xi(t)-\Xi(s))^{p-1} \Xi'(s)} ds.$$

Then there exists a constant k such that

$$\nu(t) \leq w(t) + k\alpha \int_0^t \frac{w(s)}{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))} (\Xi(t)-\Xi(s))^{p-1} \Xi'(s)} ds.$$

(H_5) : There exist a increasing function $\phi(t) \in C^{\vartheta}_{n-\vartheta,\Xi}[a,b]$ there exist $\lambda_\phi > 0$ then,

$$\mathcal{I}^{1-\vartheta,\varrho,\Xi}_{a^+} \phi \leq \lambda_\phi \phi(t).$$

Theorem 6.5. *Let us assume that $(H_1)-(H_5)$ are hold. Then our proposed problem (1.1)-(1.2) is U-H-R stable.*

Proof. Let $\epsilon > 0$, and let $\mathcal{A} \in C^{\vartheta}_{n-\vartheta,\Xi}[a,b]$ be a function which satisfies the inequality

$$\left| \mathcal{D}^{p,q,\varrho,\Xi}_{a^+} \mathcal{A}(t) - \mathbb{Q}\left(t, \mathcal{A}(t), \mathcal{A}\left(\frac{t}{\eta}\right)\right) \right| \leq \epsilon \nu(t), \quad t \in J,$$

and let $\overline{\mathcal{A}} \in C^{\vartheta}_{n-\vartheta,\Xi}[a,b]$ is the unique solution of the problem,

$$\begin{aligned} \mathcal{D}^{p,q,\varrho,\Xi}_{a^+} \mathcal{A}(t) &= \mathbb{Q}\left(t, \mathcal{A}(t), \mathcal{A}\left(\frac{t}{\eta}\right)\right) \\ \mathcal{I}^{1-\vartheta,\varrho,\Xi}_{a^+} \mathcal{A}(t) &= \sum_{i=1}^{m} \mu_i \mathcal{A}(\tau_i), \quad t \in J, \tau_i \in [0,T], \vartheta = p + q(1-p). \end{aligned}$$

where $0 < p < 1$ and $0 \leq q \leq 1$.
By the Lemma 3.13, we get

$$\mathcal{A}(t) = Y_t + \frac{1}{\varrho^p \Gamma(p)} \int_{a^+}^t e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))} (\Xi(t)-\Xi(s))^{p-1} \Xi'(s) \mathbb{Q}\left(s, \mathcal{A}(s), \mathcal{A}\left(\frac{s}{\eta}\right)\right),$$

where,

$$\begin{aligned} Y_t = & \frac{e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}}{\varrho^p \Gamma(p)} \wedge \sum_{i=1}^{m} \mu_i \\ & \times \int_{a^+}^{\tau_i} e^{\frac{\varrho-1}{\varrho}(\Xi(\tau_i)-\Xi(s))} (\Xi(\tau_i)-\Xi(s))^{p-1} \Xi'(s) \mathbb{Q}\left(s, \mathcal{A}(s), \mathcal{A}\left(\frac{s}{\eta}\right)\right) ds. \end{aligned}$$

Hence, by the integration of the inequality (6.1) and by the Lemma 6.3, we get

$$\begin{aligned} & \left| \overline{\mathcal{A}}(t) - Y_t - \frac{1}{\varrho^p \Gamma(p)} \int_{a^+}^t e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))} (\Xi(t)-\Xi(s))^{p-1} \Xi'(s) \mathbb{Q}\left(s, \mathcal{A}(s), \mathcal{A}\left(\frac{s}{\eta}\right)\right) \right| \\ & \qquad \leq |\wedge| \, e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))} (\Xi(t)-\Xi(a))^{\vartheta-1} \epsilon k_\nu \nu(t). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \left|\overline{\mathcal{A}}(t) - \mathcal{A}(t)\right| \leq & \left| \overline{\mathcal{A}}(t) - Y_t - \frac{1}{\varrho^p \Gamma(p)} \int_{a^+}^t e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))} (\Xi(t)-\Xi(s))^{p-1} \Xi'(s) \mathbb{Q}\left(s, \mathcal{A}(s), \mathcal{A}\left(\frac{s}{\eta}\right)\right) \right| \\ & + k \frac{1}{\varrho^p \Gamma(p)} \int_{a^+}^t e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))} (\Xi(t)-\Xi(s))^{p-1} \Xi'(s) \left(\mathbb{Q}\left(s, \overline{\mathcal{A}}(s), \overline{\mathcal{A}}\left(\frac{s}{\eta}\right)\right) - \mathbb{Q}\left(s, \mathcal{A}(s), \mathcal{A}\left(\frac{s}{\eta}\right)\right) \right) \end{aligned}$$

$$\left|\overline{\mathcal{A}}(t)-\mathcal{A}(t)\right| \leq |\wedge|\, e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}\left(\Xi(t)-\Xi(a)\right)^{\vartheta-1}\epsilon k_{\nu}\nu(t)$$
$$+k\frac{1}{\varrho^{p}\Gamma(p)}\int_{a^{+}}^{t} e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(s))}\left(\Xi(t)-\Xi(s)\right)^{p-1}\Xi'(s)\left|\mathbb{Q}\left(s,\overline{\mathcal{A}}(s),\overline{\mathcal{A}}\left(\frac{s}{\eta}\right)\right)-\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right|$$

By the Lemma 6.4 and (H_5), we get,

$$\left|\overline{\mathcal{A}}(t)-\mathcal{A}(t)\right| \leq \left[|\wedge|\, e^{\frac{\varrho-1}{\varrho}(\Xi(t)-\Xi(a))}\left(\Xi(t)-\Xi(a)\right)^{\vartheta-1}\left((1+k)\alpha k_{\nu}\right)k_{\nu}\right]\epsilon k_{\nu}\nu(t)$$
$$\left|\overline{\mathcal{A}}(t)-\mathcal{A}(t)\right| \leq \lambda_{\mathbb{Q}}\epsilon\phi(t),$$

which completes the proof of the theorem. □

7. Examples

Example 7.1. Let consider the following fractional order Ambartsumian equation with GHPFD

$$\begin{cases}\mathcal{D}_{0^{+}}^{\frac{1}{3},\frac{1}{7},\frac{2}{3},\Xi}\mathcal{A}(t) &= \frac{1}{8}\mathcal{A}\left(\frac{t}{8}\right)-\mathcal{A}(t), \quad t\in[0,1],\\ \mathcal{I}_{0^{+}}^{1-\vartheta,\frac{2}{3},\Xi}\mathcal{A}(0) &= 5\mathcal{A}\left(\frac{1}{3}\right)+\sqrt{3}\mathcal{A}\left(\frac{3}{5}\right).\end{cases} \tag{7.1}$$

Now comparing Eq.(7.1) with our proposed problem (1.1)-(1.2), we get
$p=\frac{1}{3},\quad q=\frac{1}{7},\quad \varrho=\frac{2}{3},\quad \vartheta=\frac{3}{7},\quad \eta=8,\quad a=0,\quad b=1,\quad \mu_1=5,\quad \mu_2=\sqrt{3}$ as $m=2,\quad \tau_1=\frac{2}{5}$,
$\tau_2=\frac{3}{7}\in[0,1]$.
Also,$\mathbb{Q}:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. is a function defined by

$$\mathbb{Q}\left(t,\mathcal{A}(t),\mathcal{A}\left(\frac{t}{\eta}\right)\right)=\frac{1}{8}\mathcal{A}\left(\frac{t}{8}\right)-\mathcal{A}(t), \quad t\in[0,1].$$

Clearly, $\mathbb{Q}$ is continuous function and for $u,v,\overline{u},\overline{v}\in\mathbb{R}$,

$$\left|\mathbb{Q}\left(t,u,v\right)-\mathbb{Q}\left(t,\overline{u},\overline{v}\right)\right| \leq \frac{1}{8}\left\{|u-v|+|\overline{u}-\overline{v}|\right\}.$$

Hence the hypotheses $(H_1),(H_2)$ hold with $k=\frac{1}{8}$.
Now choose $\Xi(t)=t^2+1$, then it implies that $\Xi(t)$ is positive increasing and continuous in $[0,1]$.
Next substituting the values that we mentioned above in $|\wedge|$.

$$|\wedge|=\left|\frac{1}{\left(\frac{2}{3}\right)^{\frac{3}{7}-1}\Gamma\left(\frac{3}{7}\right)-\left(5e^{\left(\frac{-1}{18}\right)}\left(\frac{1}{9}\right)^{\left(\frac{2}{3}-1\right)}+\sqrt{3}e^{\left(\frac{-9}{50}\right)}\left(\frac{9}{25}\right)^{\left(\frac{3}{7}-1\right)}\right)}\right|\approx 0.1,$$

and

$$\phi=\frac{\beta\left(\frac{3}{7},\frac{1}{3}\right)}{\left(\frac{2}{3}\right)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)}\left\{|\wedge|\left(5\left(\frac{1}{9}\right)+\sqrt{3}\left(\frac{9}{25}\right)\right)+1\right\}\approx 2.04.$$

This implies that $k\phi<1$, which is (H_3).

Further more $\triangle\approx 0.46$ and $k<1$, which means that the assumption (H_4) is also satisfied. Hence by Theorem 4.2 and Theorem 5.1, defined problem has at least one solution and hence is unique on J.Here all the conditions of Theorem 6.5 are satisfied, hence our proposed problem (7.1) is U-H-R stable.

Example 7.2. Let consider the following problem given by

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{1}{3},\frac{1}{7},\frac{2}{3},\Xi}\mathcal{A}(t) & = \frac{1}{5}\mathcal{A}\left(\frac{t}{5}\right) - \mathcal{A}(t), \quad t \in [0,1], \\ \mathcal{I}_{0^+}^{1-\vartheta,\frac{2}{3},\Xi}\mathcal{A}(0) & = 5\mathcal{A}\left(\frac{1}{3}\right) + \sqrt{3}\mathcal{A}\left(\frac{3}{5}\right) + \sqrt{5}\left(\frac{6}{7}\right). \end{cases} \tag{7.2}$$

Now comparing Eq.(7.2) with our proposed problem (1.1)-(1.2), we get
$p = \frac{1}{3}$, $q = \frac{1}{7}$, $\varrho = \frac{2}{3}$, $\vartheta = \frac{3}{7}$, $\eta = 5$, $a = 0$, $b = 1$, $\mu_1 = 5$, $\mu_2 = \sqrt{3}$, $\mu_3 = \sqrt{5}$ as $m = 3$, $\tau_1 = \frac{2}{5}, \tau_2 = \frac{3}{7}$, $\tau_3 = \frac{6}{7} \in [0,1]$.
Also,$\mathbb{Q} : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. is a function defined by

$$\mathbb{Q}\left(t, \mathcal{A}(t), \mathcal{A}\left(\frac{t}{\eta}\right)\right) = \frac{1}{5}\mathcal{A}\left(\frac{t}{5}\right) - \mathcal{A}(t), \quad t \in= [0,1].$$

Clearly, $\mathbb{Q}$ is continuous function and for $u, v, \overline{u}, \overline{v} \in \mathbb{R}$,

$$|\mathbb{Q}(t,u,v) - \mathbb{Q}(t,\overline{u},\overline{v})| \leq \frac{1}{5}\{|u-v| + |\overline{u}-\overline{v}|\}.$$

Hence the hypotheses $(H_1), (H_2)$ hold with $k = \frac{1}{5}$.
Now choose $\Xi(t) = t^2 + 1$, then it implies that $\Xi(t)$ is positive increasing and continuous in $[0,1]$.
Next substituting the values that we mentioned above in $|\wedge|$.

$$|\wedge| = \left| \frac{1}{\left(\frac{2}{3}\right)^{\frac{3}{7}-1}\Gamma\left(\frac{3}{7}\right) - \left(5e^{\left(\frac{-1}{18}\right)}\left(\frac{1}{9}\right)^{\left(\frac{2}{3}-1\right)} + \sqrt{3}e^{\left(\frac{-9}{50}\right)}\left(\frac{9}{25}\right)^{\left(\frac{3}{7}-1\right)} + \sqrt{7}e^{\left(\frac{-18}{49}\right)}\left(\frac{36}{49}\right)^{\frac{2}{3}-1}\right)} \right| \approx 0.03,$$

and

$$\phi = \frac{\beta\left(\frac{3}{7},\frac{1}{3}\right)}{\left(\frac{2}{3}\right)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)}\left\{|\wedge|\left(5\left(\frac{1}{9}\right) + \sqrt{3}\left(\frac{9}{25}\right) + \sqrt{7}\left(\frac{36}{49}\right)\right) + 1\right\} \approx 2.14.$$

This implies that $k\phi < 1$, which is (H_3).

Further more $\triangle \approx 0.183$ and $k < 1$, which means that the assumption (H_4) is also satisfied. Hence by Theorem 4.2 and Theorem 5.1, defined problem has at least one solution and hence is unique on J. Here all the conditions of Theorem 6.5 are satisfied, hence our proposed problem (7.2) is U-H-R stable

Conflict of Interest

The authors have no conflict of interest regarding the publication of this article.

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