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A Monotone Hybrid Algorithm for Maximal Monotone Operators and A Family of Generalized Nonexpansive Mappings

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Abstract

In this paper, a new monotone hybrid method is introduced in the framework of Banach spaces for finding a common element of the set of zeros of a maximum monotone operator and the fixed point set of a family of generalized nonexpansive mappings. The prove is given in the framework of Banach spaces for the strong convergence of a sequence of iteration to a common element of the set of zeros of a maximum monotone operator and the fixed point set of a family of generalized nonexpansive mappings. New convergence results are obtained for resolvents of maximal monotone operators and a family of generalized nonexpansive mappings in a Banach space.

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1. Introduction

Let E be a real Banach space and K a nonempty closed convex subset of E. A mapping $T : K \to K$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in K.$$

Throughout this paper, the set of fixed points of T will be denoted by $F(T) := \{x : Tx = x\}$. A mapping $T : K \to E$ is called generalized nonexpansive whenever $F(T) \neq \emptyset$ and

$$\varphi(p, Tx) \leq \varphi(p, x) \text{ for all } x \in K \text{ and } p \in F(T).$$

The class of nonexpansive mappings constitutes an important part of nonlinear operators and which numerous authors have considered. Studies on the iterative processes for such maps are gaining the attention

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of the researchers. The results on the algorithms for the class of nonexpansive mappings have been applied in several areas, such as signal processing and image restoration (see, e.g., [1, 2]). To obtain strong convergence results for the class of nonexpansive mappings, research efforts have been on the modification of some existing iterative processes such as Picard's sequence (see, e.g., [3]) and Mann's iteration process (see, e.g., [4]). Given a real sequence $\{\beta_n\} \subset [0,1]$ and with an initial guess $x_1 \in K$, which is chosen arbitrarily, the Manns iteration process is defined as

$$x_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \quad n \in \mathbb{N}. \tag{1.1}$$

It is known that even in a Hilbert space, Mann's iteration has only weak convergence.

In 2003, by using hybrid method in mathematical programming, a modification of (1.1) was proposed as

$$\begin{cases} x_1 = x \in K, \\ u_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ K_n = \{y \in K : \|y-u_n\| \le \|y-x_n\|\} \\ Q_n = \{y \in K : \langle x_n - y, x - x_n\rangle \ge 0\} \\ x_{n+1} = P_{K_n \cap Q_n} x, \end{cases} \tag{1.2}$$

for a nonexpansive mapping T in a Hilbert space H and for all $n \in \mathbb{N}$, where $P_{K_n\cap Q_n}$ is the metric projection of H onto $K_n \cap Q_n$ (see, e.g., [5]). Under suitable control condition on the sequence $\{\beta_n\}$, the strong convergence of (1.2) to a fixed point of T was established.

A modification of (1.2) which is being called monotone hybrid method for a nonexpansive mapping T in a Hilbert space H is given as

$$\begin{cases} x_1 = x \in K, K_0 = Q_0 = K, \\ u_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ K_n = \{y \in K_{n-1}\cap Q_{n-1} : \|y-u_n\| \le \|y-x_n\|\} \\ Q_n = \{y \in K_{n-1}\cap Q_{n-1} : \langle x_n - y, x - x_n\rangle \ge 0\} \\ x_{n+1} = P_{K_n \cap Q_n} x. \end{cases} \tag{1.3}$$

A strong convergence theorem was established for (1.3) under appropriate control conditions (see, e.g., [6]). Moreover, there are studies on using monotone hybrid method for a family of generalized nonexpansive mappings in a Banach space E, which is given as

$$\begin{cases} x_1 = x \in K, K_0 = Q_0 = K, \\ u_n = \beta_n x_n + (1-\beta_n)S_n x_n, \\ K_n = \{y \in K_{n-1}\cap Q_{n-1} : \varphi(y,u_n) \le \varphi(y,x_n)\} \\ Q_n = \{y \in K_{n-1}\cap Q_{n-1} : \langle x_n - y, Jx - Jx_n\rangle \ge 0\} \\ x_{n+1} = R_{K_n \cap Q_n} x, \end{cases} \tag{1.4}$$

where J is the duality mapping on E, $R_{K_n\cap Q_n}$ is the sunny nonexpansive retraction from K onto $K_n \cap Q_n$ and $\{S_n\}$ is a countable family of generalized nonexpansive mappings which is defined from a generalized nonexpansive mapping $T : K \to E$ by

$$S_n x = \alpha_n x + (1-\alpha_n)Tx,$$

for all $x \in K$, and $\{\alpha_n\} \subset (0,1)$. Under the conditions that $\{\alpha_n\}$ and $\{\beta_n\}$ respectively satisfy $\liminf_{n\to\infty} \alpha_n(1-\alpha_{nn}) > 0$ $\liminf_{n\to\infty}(1-\beta_n) > 0$ and the family $\{T_n\}$ satisfies NST -condition, the strong convergence of the sequence $\{x_n\}$ generated by (1.4) was established (see, e.g., [7, 8]). For more information on the class of generalized nonexpansive mappings in Banach spaces, interested readers are referred to [9, 10, 11, 12].

Consider a problem of finding a solution of the equation $Au = 0$, where A is a maximal monotone operator. Such a problem is associated with convex minimization problems. Indeed, for a proper lower semi continuous convex function $f : E \to (-\infty, +\infty]$, the subdifferential mapping $\partial f : E \to 2^{E^*}$, is defined at $x \in E$ by

$$\partial f(x) = \{x^* \in E^* : fy - fx \geq \langle y - x, x^* \rangle \,\forall\;\; y \in E\},$$

and it known to be maximal monotone (See, e.g., [14]). Solving the equation $Au = 0$ is equivalent to finding $f(u) = \min_{x \in E} f(x)$ by setting $\partial f \equiv A$.

Inspired by the previous results in this inclination, the goal of this study is to introduce a new monotone hybrid algorithm which is suitable for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a family of generalized nonexpansive mapping in a Banach space. This study establishes the conditions which guarantee the strong convergence of the generated sequence of iteration.

2. Preliminaries

Let E be a real Banach space and $S := \{x \in E : \|x\| = 1\}$. E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S$. E is said to be uniformly smooth if it is smooth and the limit (2.1) is attained uniformly for each $x, y \in S$. The modulus of convexity of a Banach space E, $\delta_E : (0, 2] \to [0, 1]$ is defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| > \epsilon \right\}.$$

E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. A Banach space E is said to be strictly convex if

$$\|x\| = \|y\| = 1, x \neq y \Rightarrow \frac{\|x + y\|}{2} < 1.$$

It is well known that a space E is uniformly smooth if and only if E^* is uniformly convex, where E^* is its dual. The sets of all positive integers and real numbers will be denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. The normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \|x^*\|, \|x^*\| = \|x\|\} \;\forall\; x \in E.$$

J is known to be uniformly norm-to-norm continuous on bounded sets of E if E is uniformly smooth. Let $A \subset E \times E^*$ be a multi-valued operator. A is said to be monotone if for all $(x, x^*), (y, y^*) \in A$,

$$\langle x - y, x^* - y^* \rangle \geq 0,$$

and it is said to be maximal monotone if it is monotone and its graph is not properly contained in the graph of any other monotone mapping. For a maximal monotone operator A, the set $A^{-1}(0) := \{x \in E : Ax = 0\}$ is closed and convex. According to a result of Rockafellar [13], A is said to be maximum monotone if it is monotone and the range of $(J + rA)$ is all of E^* for some $r > 0$.

Definition 2.1. Let E be a smooth Banach space. The function $\varphi : E \times E \to \mathbb{R}$ is defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2,$$

for all $x, y \in E$. In a Hilbert space, it is expressed as $\varphi(x, y) = \|x - y\|^2 \geq 0$. The following identities hold for all $x, y, z \in E$:

(i) $(\|x\| - \|y\|)^2 \leq \varphi(x, y) \leq (\|x\| + \|y\|)^2$,

(ii) $\varphi(x,y) = \varphi(x,z) + \varphi(z,y) + 2\langle x-z, Jz-Jy\rangle$,

(iii) $\varphi(x,y) = \langle x, Jx-Jy\rangle + \langle x-y, Jy\rangle \leq \|x\|\|Jx-Jy\| + \|x-y\|\|y\|$.

Definition 2.2. Resolvent: Let E be a strictly convex, smooth, and reflexive Banach space and $A \subset E\times E^*$ a maximal monotone mapping. Given $r > 0$ and $x \in E$, then there exists a unique $x_r \in D(A)$ such that $Jx \in Jx_r + rAx_r$. Thus one can define a single-valued mapping $J_r : E \to D(A)$ by

$$J_r x = \{z \in D(A) : Jx \in Jz + rAz\},$$

which is being called the resolvent of A. $J_r x$ consists of one point and for all $r > 0$, $A^{-1}(0) = F(J_r)$, where $F(J_r)$ is the set of fixed points of J_r. Also, for all $r > 0$ and $x \in E$, the Yosida approximation $A_r : C \to E^*$ is defined by

$$A_r x = \frac{1}{r}(J - JJ_r)x.$$

For all $r > 0$ and $x \in E$, the following hold (See e.g, [17, 18])

(i) $\varphi(p, J_r x) + \varphi(J_r x, x) \leq \varphi(p, x)$ for all $p \in A^{-1}(0)$.

(ii) $(J_r x, A_r x) \in A$.

Definition 2.3. Metric projection: Let K be a nonempty closed convex subset of a Hilbert space H. A mapping $P_K : H \to K$ of H onto K satisfying

$$\|x - P_K x\| = \min_{y\in K} \|x-y\|,$$

is called the metric projection. This set is known to be singleton. The metric projection has the important property that; for $x \in H$ and $x_0 \in K, x_0 = P_K x$ if and only if

$$\langle x - x_0, x_0 - y\rangle \geq 0 \ \forall\ y \in K.$$

Definition 2.4. Retraction: Let K be nonempty subset of a Banach space E. A mapping $R : E \to K$ is called sunny if

$$R(Rx + \alpha(x - Rx)) = Rx,$$

for all $x \in E$ and all $\alpha \geq 0$. If $Rx = x$ for all $x \in K$, it is also called a retraction. A retraction which is also sunny and nonexpansive is called a sunny nonexpansive retraction. If E is a smooth Banach space, the sunny nonexpansive retraction of E onto K is denoted by R_K. K is said to be a sunny generalized nonexpansive retract of E provided that there exists a sunny generalized nonexpansive retraction R from E onto K.

The following results on sunny generalized nonexpansive retraction will be needed and for their proof, interested readers are referred to see [18, 19].

Lemma 2.5. *Let K be a nonempty closed subset of a smooth and strictly convex Banach space E. Let R_K be a retraction of E onto K. Then R_K is sunny and generalized nonexpansive if and only if*

$$\langle x - R_K x, JR_K x - Jy\rangle \geq 0$$

for each $x \in E$ and $y \in K$.

Lemma 2.6. *Let K be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto K and let $(x,z) \in E \times K$. Then the following hold:*

(i) $z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in K$;

(ii) $\varphi(x, R_K y) + \varphi(R_K y, y) \le \varphi(x, y)$.

Lemma 2.7. *Let E be a smooth, strictly convex and reflexive Banach space and let K be a nonempty closed subset of E. Then the following are equivalent:*

(i) *K is a sunny generalized nonexpansive retract of E;*

(ii) *K is a generalized nonexpansive retract of E;*

(iii) *JK is closed and convex.*

The following results are well known results and will be applied to establish the main results.

Lemma 2.8. *Let E be a uniformly convex and smooth Banach space and let $\{u_n\}$ and $\{v_n\}$ be two sequences in E such that either $\{u_n\}$ or $\{v_n\}$ is bounded. If $\lim_{n\to\infty} \varphi(u_n, v_n) = 0$, then $\lim_{n\to\infty} \|u_n - v_n\| = 0$ (See [15]).*

Lemma 2.9. *Let E be a uniformly convex and smooth Banach space and let $d > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2d] \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \le \varphi(x, y)$$

for all $x, y \in B_d(0)$, where $B_d(0) = \{z \in E : \|z\| \le d\}$ (See e.g, [15]).

Lemma 2.10. *Let E be a uniformly convex Banach space and let $d > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\alpha x + (1-\alpha)y\|^2 \le \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x - y\|)$$

for all $x, y \in B_d(0)$ and $\alpha \in [0, 1]$, where $B_d(0) = \{w \in E : \|w\| \le d\}$ (See e.g, [16]).

3. Main Results

Lemma 3.1. *Let E be a strictly convex, smooth, and reflexive Banach space and let $A \subset E \times E^*$ be a maximal monotone mapping with $A^{-1}(0) \neq \emptyset$. For each $r > 0$, let $J_r : E \to E$ be the resolvent of A for r. Then J_r is a generalized nonexpansive mapping.*

Proof. Let $x \in E, y \in F(J_r)$ and $r > 0$. Since A is maximal monotone, recall that $A^{-1}(0) = F(J_r)$. Apply Definition 2.2(i) to have

$$\varphi(y, J_r x) + \varphi(J_r x, x) \le \varphi(y, x) \text{ for all } y \in A^{-1}(0).$$

By Definition 2.1(i), $\varphi(J_r x, x) \ge 0$. Consequently

$$\varphi(y, J_r x) \le \varphi(y, x).$$

□

Theorem 3.2. *Let E be a uniformly convex and uniformly smooth Banach space, K be a nonempty closed convex subset of E and $R_K : E \to K$ be a sunny and generalized nonexpansive retraction from E onto K. For all $r > 0$, let $J_r : E \to E$ denote the resolvent which is associated with a maximal monotone mapping $A \subset E \times E^*$. Let $T : K \to E$ be a closed generalized nonexpansive mapping such that $F(T) \cap A^{-1}(0) \neq \emptyset$ and for each $n \in N$, define the sequence $\{x_n\}$ by*

$$\begin{cases} x_1 = x \in K, \ \ K_0 = Q_0 = K, \\ u_n = J^{-1}\left(\beta_n J x_n + (1-\beta_n) J T_n R_K\left(J_{r_n} x_n\right)\right), \\ K_n = \{y \in K_{n-1} \cap Q_{n-1} : \varphi(y, u_n) \le \varphi(y, x_n)\} \\ Q_n = \{y \in K_{n-1} \cap Q_{n-1} : \langle x_n - y, Jx - Jx_n \rangle \ge 0\} \\ x_{n+1} = R_{K_n \cap Q_n} x, \end{cases}$$

where J is the duality mapping on E and $\{T_n\}$ is a countable family of generalized nonexpansive mappings such that the mapping T_n from K into E is given by

$$T_n x = J^{-1}\left(\alpha_n Jx + (1-\alpha_n) JTx\right), \tag{3.1}$$

for all $x \in K$. The real sequence $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset [a,\infty)$ for some $a > 0$, are such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty}(1-\beta_n) > 0$. Then the sequence $\{x_n\}$ converges strongly to $R_{F(T)\cap A^{-1}(0)}x$, where $R_{F(T)\cap A^{-1}(0)}$ is the sunny nonexpansive retraction from K onto $F(T)\cap A^{-1}(0)$.

Proof. **Step 1**: T_n is a generalized nonexpansive mapping for each $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} F(T_n) = F(T)$. Indeed, for $p \in \bigcap_{n=1}^{\infty} F(T_n)$ and $x \in K$,

$$\begin{aligned}
\varphi(p, T_n x) &= \varphi\left(p, J^{-1}\left(\alpha_n Jx + (1-\alpha_n) JTx\right)\right) \\
&= \|p\|^2 - 2\left\langle p, \alpha_n Jx + (1-\alpha_n) JTx\right\rangle + \|\alpha_n Jx + (1-\alpha_n) JTx\|^2 \\
&\leq \|p\|^2 - 2\alpha_n \langle p, Jx\rangle - 2(1-\alpha_n)\langle p, JTx\rangle + \alpha_n\|x\|^2 + (1-\alpha_n)\|Tx\|^2 \\
&= \alpha_n\varphi(p,x) + (1-\alpha_n)\varphi(p,Tx) \\
&\leq \alpha_n\varphi(p,x) + (1-\alpha_n)\varphi(p,x) \\
&= \varphi(p,x).
\end{aligned}$$

Thus, T_n is generalized nonexpansive. Furthermore, suppose $p \in \bigcap_{n=1}^{\infty} F(T_n)$, then $p \in F(T_n)$ for each $n \in \mathbb{N}$. Therefore, by (3.1),

$$p = Tp,$$

which shows that $p \in F(T)$ and consequently $\bigcap_{n=1}^{\infty} F(T_n) \subset F(T)$. Also, suppose $p \in F(T)$, then by (3.1),

$$T_n p = p$$

which shows that $p \in F(T_n)$ for each $n \in \mathbb{N}$ and thus, $p \in \bigcap_{n=1}^{\infty} F(T_n)$, which justifies that $F(T) \subset \bigcap_{n=1}^{\infty} F(T_n)$. Hence $\bigcap_{n=1}^{\infty} F(T_n) = F(T)$.

Step 2: To show that K_n and Q_n are closed and convex for all $n \in \mathbb{N}$. By definition, K_n is closed and Q_n is closed and convex for each $n \in \mathbb{N}$. How to establish that K_n is convex is the burden. Notice that

$$\varphi(y, u_n) \leq \varphi(y, x_n)$$

implies that for all $y \in K_n$,

$$\|x_n\|^2 - \|u_n\|^2 - 2\langle y, Jx_n - Ju_n\rangle \geq 0,$$

which is affine in y, and thus K_n is convex. So for all $n \in \mathbb{N}$, $K_n \cap Q_n$ is a closed and convex subset of E.

Step 3: To show that $F(T) \cap A^{-1}(0) \subset K_n \cap Q_n$. Setting $v_n = R_K(J_{r_n}x_n)$ and for $p \in F(T) \cap A^{-1}(0)$,

$$\begin{aligned}
\varphi(p, u_n) &= \varphi\left(p, J^{-1}(\beta_n Jx_n + (1-\beta_n)JT_nR_KJ_{r_n}x_n)\right) \\
&= \|p\|^2 - 2\langle p, \beta_n Jx_n + (1-\beta_n)JT_nv_n\rangle + \|\beta_n Jx_n + (1-\beta_n)JT_nv_n\|^2 \\
&\leq \|p\|^2 - 2\beta_n\langle p, Jx_n\rangle - 2(1-\beta_n)\langle p, JT_nv_n\rangle + \beta_n\|x_n\|^2 + (1-\beta_n)\|T_nv_n\|^2 \\
&= \beta_n\varphi(p, x_n) + (1-\beta_n)\varphi(p, T_nv_n) \\
&\leq \beta_n\varphi(p, x_n) + (1-\beta_n)\varphi(p, v_n) \text{ (By generalized nonexpansive property of } T_n) \qquad (3.2) \\
&= \beta_n\varphi(p, x_n) + (1-\beta_n)\varphi(p, R_K(J_{r_n}x_n)) \\
&\leq \beta_n\varphi(p, x_n) + (1-\beta_n)\varphi(p, J_{r_n}x_n) \text{ (By the property of } R_K) \\
&\leq \beta_n\varphi(p, x_n) + (1-\beta_n)\varphi(p, x_n) \text{ (By generalized nonexpansive property of } J_{r_n}) \\
&= \varphi(p, x_n).
\end{aligned}$$

This shows that $p \in K_n$ for all $n \in \mathbb{N}$, wherefore $F(T) \cap A^{-1}(0) \subset K_n$. Induction method will be applied to show that $F(T) \cap A^{-1}(0) \subset Q_n$ for all $n \in \mathbb{N}$. Recall that J is one-to-one in a strictly convex Banach space. Wherefore, $J(K_n \cap Q_n) = JK_n \cap JQ_n$ is closed convex for each $n \in \mathbb{N}$. Also, it is known that $K_n \cap Q_n$ is a sunny generalized nonexpansive retract of E (By Lemma 2.7). Observe that by definition, for $n = 1, F(T) \cap A^{-1}(0) \subset K = K_0 \cap Q_0$. For some $j \in \mathbb{N}$, assume that $F(T) \cap A^{-1}(0) \subset K_{j-1} \cap Q_{j-1}$. Since $x_j = R_{K_{j-1}\cap Q_{j-1}}y$, Lemma 2.5 gives that

$$\langle x - x_j, Jx_j - Jy\rangle \geq 0,$$

for all $y \in K_{j-1} \cap Q_{j-1}$. Therefore, it can be deduced that

$$\langle x - x_j, Jx_j - Jy\rangle \geq 0, \ \forall\, y \in F(T) \cap A^{-1}(0) \qquad (3.3)$$

since $F(T)\cap A^{-1}(0) \subset K_{j-1}\cap Q_{j-1}$. The inequality (3.3) and the definition of Q_n gives that $F(T)\cap A^{-1}(0) \subset Q_k$ and thus $F(T) \cap A^{-1}(0) \subset Q_n$ for all $n \in \mathbb{N}$. Hence, $F(T) \cap A^{-1}(0) \subset K_n \cap Q_n$ for all $n \in \mathbb{N}$, which confirms that $\{x_n\}$ is well defined.

Step 4: The climax is to show that $x_n \to R_{F(T)\cap A^{-1}(0)}x$ as $n \to \infty$. By the definition of $Q_n, x_n = R_{Q_n}x$. Using Lemma 2.6(ii) gives

$$\varphi(x, x_n) = \varphi(x, R_{Q_n}x) \leq \varphi(x, u) - \varphi(R_{Q_n}x, u) \leq \varphi(x, u),$$

for all $F(T) \cap A^{-1}(0) \subset Q_n$, so, $\{\varphi(x, x_n)\}$ is bounded. Moreover, it can be deduced from the definition of φ that $\{x_n\}, \{u_n\}$ and $\{v_n\}$ are bounded. Thus, the limit of $\{\varphi(x, x_n)\}$ exists. Given a positive integer k, from $x_n = R_{Q_n}x$ for each $n \in N$, one can have that

$$\varphi(x_n, x_{n+k}) = \varphi(R_{Q_n}x, x_{n+k}) \leq \varphi(x, x_{n+k}) - \varphi(x, R_{Q_n}x) \leq \varphi(x, x_{n+k}) - \varphi(x, x_n),$$

which leads to

$$\lim_{n\to\infty} \varphi(x_n, x_{n+k}) = 0. \qquad (3.4)$$

Using Lemma 2.9, a strictly increasing, convex and continuous function $g : [0, 2r] \to [0, \infty)$ exists such that for $m, n \in \mathbb{N}$ with $m > n$,

$$g(\|x_m - x_n\|) \leq \varphi(x_m, x_n) \leq \varphi(x_m, x_0) - \varphi(x_n, x_0).$$

It is obvious by the the property of g that $\{x_n\}$ is Cauchy. Thus there exists $z \in K$ so that $x_n \to z$. Considering $x_{n+1} = R_{K_n\cap Q_n}x \in K_n$ and by the definition of K_n, one can have that

$$\varphi(x_{n+1}, x_n) - \varphi(x_{n+1}, u_n) \geq 0, \ \forall\, n \in \mathbb{N}. \qquad (3.5)$$

By (3.4) and (3.5), it can be deduced that $\lim_{n\to\infty}\varphi(x_{n+1},x_n)=\lim_{n\to\infty}\varphi(x_{n+1},u_n)=0$. Due to uniform convexity and smoothness of E, using Lemma 2.8 gives

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\|x_{n+1}-u_n\|=0,$$

therefore

$$\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{3.6}$$

Since the duality mapping J is norm-to-norm uniform continuous on bounded sets, it can be obtained that

$$\lim_{n\to\infty}\|Jx_{n+1}-Jx_n\|=\lim_{n\to\infty}\|Jx_{n+1}-Ju_n\|=\|Jx_n-Ju_n\|=0. \tag{3.7}$$

It can be observed from (3.2) that

$$\varphi(p,v_n)\geq\frac{1}{(1-\beta_n)}\left(\varphi(p,u_n)-\beta_n\varphi(p,x_n)\right).$$

Since $v_n:=R_K\left(J_{r_n}x_n\right)$, wherefore,

$$\begin{aligned}
\varphi(v_n,x_n) &= \varphi\left(R_K\left(J_{r_n}x_n\right),x_n\right)\leq\varphi(p,x_n)-\varphi(p,v_n)\ \text{(by Lemma 2.6 (ii),)}\\
&\leq \varphi(p,x_n)-\frac{1}{(1-\beta_n)}\left(\varphi(p,u_n)-\beta_n\varphi(p,x_n)\right)\\
&= \frac{1}{(1-\beta_n)}\left(\varphi(p,x_n)-\varphi(p,u_n)\right)\\
&= \frac{1}{(1-\beta_n)}\left(\|x_n\|^2-\|u_n\|^2-2\langle p,Jx_n-Ju_n\rangle\right)\\
&\leq \frac{1}{(1-\beta_n)}\left(\left|\|x_n\|^2-\|u_n\|^2\right|+2\left|\langle p,Jx_n-Ju_n\rangle\right|\right)\\
&\leq \frac{1}{(1-\beta_n)}\left(\left|\|x_n\|-\|u_n\|\right|\left(\|x_n\|+\|u_n\|\right)+2\|p\|\|Jx_n-Ju_n\|\right)\\
&\leq \frac{1}{(1-\beta_n)}\left(\|x_n-u_n\|\left(\|x_n\|+\|u_n\|\right)+2\|p\|\|Jx_n-Ju_n\|\right).
\end{aligned}$$

By (3.6) and (3.7), $\lim_{n\to\infty}\varphi(v_n,x_n)=0$. Apply Lemma 2.8 to have

$$\lim_{n\to\infty}\|v_n-x_n\| = 0. \tag{3.8}$$

Furthermore, it can be obtained that

$$\begin{aligned}
\|Jx_{n+1}-Ju_n\| &= \|Jx_{n+1}-\beta_nJx_n-(1-\beta_n)JT_nv_n\|\\
&= \|(1-\beta_n)\left(Jx_{n+1}-JT_nv_n\right)-\beta_n\left(Jx_n-Jx_{n+1}\right)\|\\
&\geq (1-\beta_n)\|Jx_{n+1}-JT_nv_n\|-\beta_n\|Jx_n-Jx_{n+1}\|.
\end{aligned}$$

Then

$$\|Jx_{n+1}-JT_nv_n\|\leq\frac{1}{(1-\beta_n)}\left(\|Jx_{n+1}-Ju_n\|+\beta_n\|Jx_n-Jx_{n+1}\|\right).$$

Since $\liminf_{n\to\infty}(1-\beta_n)>0$ and by considering (3.6), it leads to

$$\lim_{n\to\infty}\|Jx_{n+1}-JT_nv_n\|=0.$$

By the property of J^{-1} which is norm-to-norm uniformly continuous on bounded sets, it follows that

$$\lim_{n\to\infty}\|x_{n+1}-T_nv_n\|=0.$$

It should be noted that

$$\|x_n - T_n v_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_n v_n\|,$$

which results in

$$\lim_{n\to\infty} \|x_n - T_n v_n\| = 0. \tag{3.9}$$

Notice that

$$\|v_n - T_n v_n\| \leq \|v_n - x_n\| + \|x_n - T_n v_n\|.$$

Wherefore by (3.8) and (3.9),

$$\lim_{n\to\infty} \|v_n - T_n v_n\| = 0. \tag{3.10}$$

Since E is uniformly smooth and by (3.10), one can have that

$$\lim_{n\to\infty} \|Jv_n - JT_n v_n\| = 0.$$

It is known that $\{Tv_n\}$ is bounded since $\{v_n\}$ is bounded. Let $d = \max\{\sup_n \|v_n\|, \sup_n \|Tv_n\|\}$. Therefore, there exists $d > 0$ such that $B_d(0) = \{w \in E : \|w\| \leq d\}$ and $\{v_n\}, \{Tv_n\} \subset B_d(0)$. Using Lemma 2.10, for $p \in \bigcap_{n=1}^{\infty} F(T_n)$,

$$\begin{aligned}
\varphi(p, T_n v_n) &= \varphi\left(p, J^{-1}(\alpha_n J v_n + (1-\alpha_n) JTv_n)\right) \\
&= \|p\|^2 - 2\langle p, \alpha_n J v_n + (1-\alpha_n) JTv_n\rangle + \|\alpha_n J v_n + (1-\alpha_n) JTv_n\|^2 \\
&\leq \|p\|^2 - 2\alpha_n \langle p, Jv_n\rangle - 2(1-\alpha_n)\langle p, JTv_n\rangle + \alpha_n \|v_n\|^2 \\
&\quad +(1-\alpha_n)\|Tv_n\|^2 - \alpha_n(1-\alpha_n) g(\|v_n - Tv_n\|) \\
&= \alpha_n \varphi(p, v_n) + (1-\alpha_n)\varphi(p, Tv_n) - \alpha_n(1-\alpha_n) g(\|v_n - Tv_n\|) \\
&\leq \alpha_n \varphi(p, v_n) + (1-\alpha_n)\varphi(p, v_n) - \alpha_n(1-\alpha_n) g(\|v_n - Tv_n\|) \\
&= \varphi(p, v_n) - \alpha_n(1-\alpha_n) g(\|v_n - Tv_n\|).
\end{aligned}$$

For this reason,

$$\alpha_n(1-\alpha_n) g(\|v_n - Tv_n\|) \leq \varphi(p, v_n) - \varphi(p, T_n v_n). \tag{3.11}$$

Let $\{\|v_{n_j} - Tv_{n_j}\|\}$ be any subsequence of $\{\|v_n - Tv_n\|\}$. It is known that $\{v_{n_j}\}$ is bounded. Therefore, there exists a subsequence $\left\{v_{n'_k}\right\}$ of $\{v_{n_j}\}$ such that

$$\lim_{k\to\infty} \varphi\left(p, v_{n'_k}\right) = \limsup_{j\to\infty} \varphi(p, v_{n_j}) = 0.$$

Applying Definition 2.1 ((ii) and (iii)) gives

$$\begin{aligned}
\varphi\left(p, v_{n'_k}\right) &= \varphi\left(p, T_{n'_k} v_{n'_k}\right) + \varphi\left(T_{n'_k} v_{n'_k}, v_{n'_k}\right) \\
&\quad + 2\left\langle p - T_{n'_k} v_{n'_k}, JT_{n'_k} v_{n'_k} - Jv_{n'_k}\right\rangle \\
&\leq \varphi\left(p, T_{n'_k} v_{n'_k}\right) + \|T_{n'_k} v_{n'_k}\| \|JT_{n'_k} v_{n'_k} - Jv_{n'_k}\| \\
&\quad + \|T_{n'_k} v_{n'_k} - v_{n'_k}\| \|v_{n'_k}\| + 2\|p - T_{n'_k} v_{n'_k}\| \|JT_{n'_k} v_{n'_k} - Jv_{n'_k}\|.
\end{aligned} \tag{3.12}$$

Therefore

$$c = \liminf_{k\to\infty} \varphi(p, v_{n_k}) = \liminf_{k\to\infty} \varphi\left(p, T_{n'_k} v_{n'_k}\right).$$

Alternatively, $\varphi(p, T_n v_n) \leq \varphi(p, v_n)$ leads to

$$\limsup_{k\to\infty} \varphi\left(p, T_{n'_k} v_{n'_k}\right) = \limsup_{k\to\infty} \varphi(p, v_{n_k}) = c,$$

for that reason

$$\lim_{k\to\infty} \varphi(p, v_{n_k}) = \lim_{k\to\infty} \varphi\left(p, T_{n'_k} v_{n'_k}\right) = c.$$

Given that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, it can be obtained from (3.11) that $\lim_{k\to\infty} g\left(\|v_{n'_k} - Tv_{n'_k}\|\right) = 0$. According to the properties of the function g, it can be deduced that $\lim_{k\to\infty} \|v_{n'_k} - Tv_{n'_k}\| = 0$ and consequently

$$\lim_{n\to\infty} \|v_n - Tv_n\| = 0. \tag{3.13}$$

By (3.8) and (3.13),

$$\|x_n - Tv_n\| \leq \|x_n - v_n\| + \|v_n - Tv_n\| \to 0 \text{ as } n \to \infty.$$

Forasmuch as $x_n \to z$ and by (3.8), therefore $v_n \to z$. Given that T is closed and $v_n \to z$, then z is a fixed point of T.

The next thing is to establish that $z \in A^{-1}(0)$. Since E is uniformly smooth and by (3.8),

$$\lim_{n\to\infty} \|Jx_n - Jv_n\| = 0.$$

For $r_n \geq a$, one can have that

$$\lim_{n\to\infty} \frac{1}{r_n} \|Jx_n - Jv_n\| = 0.$$

For that reason

$$\lim_{n\to\infty} \|A_{r_n} x_n\| = \lim_{n\to\infty} \frac{1}{r_n} \|Jx_n - Jv_n\| = 0.$$

For $(q, q^*) \in A$, one can have by the monotonicity of A that

$$\langle q - z_n, q^* - A_{r_n} x_n \rangle \geq 0 \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$ results in

$$\langle q - z, q^* \rangle \geq 0.$$

Since A is maximal monotone, then $z \in A^{-1}(0)$. Next is to prove that $z = R_{F(T)\cap A^{-1}(0)} x$. Using Lemma 2.6 leads to

$$\varphi\left(z, R_{F(T)\cap A^{-1}(0)} x\right) + \varphi\left(R_{F(T)\cap A^{-1}(0)} x, x\right) \leq \varphi(z, x).$$

Since $x_{n+1} = R_{K_n \cap Q_n} x$ and $z \in F(T) \cap A^{-1}(0) \subset K_n \cap Q_n$, using Lemma 2.6 leads to

$$\varphi\left(R_{F(T)\cap A^{-1}(0)} x, x_{n+1}\right) + \varphi(x_{n+1}, x) \leq \varphi\left(R_{F(T)\cap A^{-1}(0)} x, x\right).$$

The definition of φ leads to the deduction that $\varphi(z, x) \leq \varphi\left(R_{F(T)\cap A^{-1}(0)} x, x\right)$ and $\varphi(z, x) \geq \varphi\left(R_{F(T)\cap A^{-1}(0)} x, x\right)$, for that reason, $\varphi(z, x) = \varphi\left(R_{F(T)\cap A^{-1}(0)} x, x\right)$. Thus, since $R_{F(T)\cap A^{-1}(0)} x$ is unique, one can conclude that $z = R_{F(T)\cap A^{-1}(0)} x$. □

The following result can be deduced from Theorem 3.2, which is the main result of this paper.

Corollary 3.3. *Let E be a uniformly convex and uniformly smooth Banach space, K be a nonempty closed convex subset of E and $R_K : E \to K$ be a sunny and generalized nonexpansive retraction from E onto K. For all $r > 0$, let $J_r : E \to E$ denote the resolvent which is associated with a maximal monotone mapping $A \subset E \times E^*$. Let $T : K \to E$ be a closed generalized nonexpansive mapping such that $F(T) \cap A^{-1}(0) \neq \emptyset$ and for each $n \in N$, define the sequence $\{x_n\}$ by*

$$\begin{cases} x_1 = x \in K, \;\; K_0 = Q_0 = K, \\ u_n = J^{-1}\left(\beta_n J x_n + (1-\beta_n) J T R_K \left(J_{r_n} x_n\right)\right), \\ K_n = \{y \in K_{n-1} \cap Q_{n-1} : \varphi(y, u_n) \leq \varphi(y, x_n)\} \\ Q_n = \{y \in K_{n-1} \cap Q_{n-1} : \langle x_n - y, Jx - Jx_n \rangle \geq 0\} \\ x_{n+1} = R_{K_n \cap Q_n} x, \end{cases}$$

where J is the duality mapping on E. The real sequence $\{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$, are such that $\liminf_{n\to\infty}(1-\beta_n) > 0$. Then the sequence $\{x_n\}$ converges strongly to $R_{F(T)\cap A^{-1}(0)}x$, where $R_{F(T)\cap A^{-1}(0)}$ is the sunny nonexpansive retraction from K onto $F(T) \cap A^{-1}(0)$.

Proof. By letting $\alpha_n = 0$ for all $n \in \mathbb{N}$ in Theorem 3.2, it is obvious that $\{T_n\} = \{T\}$. Then the desired result follows. □

In the framework of Hilbert spaces, the main result of this paper is given as below.

Corollary 3.4. *Let H be a Hilbert space, K be a nonempty closed convex subset of H and $P_K : H \to K$ be a metric projection from H onto K. For all $r > 0$, let $J_r : H \to H$ denote the resolvent which is associated with a maximal monotone mapping $A \subset H \times H^*$. Let $T : K \to H$ be a closed generalized nonexpansive mapping such that $F(T) \cap A^{-1}(0) \neq \emptyset$ and for each $n \in N$, define the sequence $\{x_n\}$ by*

$$\begin{cases} x_1 = x \in K, \;\; K_0 = Q_0 = K, \\ u_n = \beta_n x_n + (1-\beta_n) T_n R_K \left(J_{r_n} x_n\right), \\ K_n = \{y \in K_{n-1} \cap Q_{n-1} : \|y - u_n\| \leq \|y - x_n\|\} \\ Q_n = \{y \in K_{n-1} \cap Q_{n-1} : \langle x_n - y, x - x_n \rangle \geq 0\} \\ x_{n+1} = P_{K_n \cap Q_n} x, \end{cases}$$

where $\{T_n\}$ is a countable family of generalized nonexpansive mappings such that the mapping T_n from K into H is given by (3.1). The real sequence $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$, are such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty}(1-\beta_n) > 0$. Then the sequence $\{x_n\}$ converges strongly to $R_{F(T)\cap A^{-1}(0)}x$, where $R_{F(T)\cap A^{-1}(0)}$ is the metric projection from K onto $F(T) \cap A^{-1}(0)$.

Proof. Recall that in a Hilbert space, $\varphi(x, y) = \|x - y\|^2$ for all $x, y \in H$ and J is the identity mapping. Therefore, the desired result readily follows from Theorem 3.2. □

4. Conclusion

The class of nonexpansive mappings constitutes an important part of nonlinear operators and the class of maximal monotone operator is indispensable as it is closely associated with convex minimization problems. Results on the algorithms for the class of nonexpansive mappings have several applications such as in signal processing and image restoration. This study establishes the strong convergence of a proposed monotone hybrid algorithm for finding a common element of the set of zeros of a maximum monotone operator and the fixed point set of a family of generalized nonexpansive mappings. The stated conditions for the parameters in the main theorem are readily satisfied by $\{\alpha_n\} = \left\{\frac{2}{3} - \frac{1}{2n}\right\}$ and $\{\beta_n\} = \left\{\frac{1}{2} + \frac{1}{5n}\right\}$.

Abbreviation

lsc: lower semicontinuous. conditions stated in the main theorem are $\{\lambda_n\} = \left\{\frac{1}{8} + \frac{1}{5n}\right\}$ and $\{\beta_n\} = \left\{1 - \frac{1}{2n}\right\}$.

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