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Investigating a Coupled System with Riemann-Liouville Fractional Derivative by Modified Monotone Iterative Technique

Abid Hussain, Kamal Shah*

Department of Mathematics, University of Malakand, Dir(L), Khyber Pakhtunkhwa, Pakistan

Abstract

In recent time, the area of arbitrary order differential equations (AODEs) has been considered very well. Different aspects have been investigated for the said area. One of the important and most warm area is devoted to study multiplicity results along with existence and uniqueness of solutions for the said equations. In this regard various techniques have been utilized to investigate the said area. Monotone iterative technique (MIT) coupled with the method of extremal solutions has been used recently to investigate multiplicity of solutions to some AODEs. In this research work, we deal a coupled system of nonlinear AODEs under boundary conditions (BCs) involving Riemann-Liouville fractional derivative by using fixed point theorems due to Perov's and Schuader's to study existence and uniqueness results. Using Perove's fixed point theorem ensures uniqueness of solution to systems of equations, while existence of at least one solution is achieved by Schauder's fixed point theorem. Then we come across the multiplicity of solutions and establish some criteria for the iterative solutions via using updated type MIT together with the method of upper and lower solutions for the considered system of AODEs. Corresponding to multiplicity results of solutions, we first establish two sequences of extremal solutions. One of the sequence is monotonically decreasing and converging to lower solution. On the other hand, the other sequence is monotonically increasing and converging to the upper solution. In last we give suitable examples to illustrate the main results.

Keywords: Arbitrary order differential equations, Monotone iterative technique, Riemann-Liouville fractional derivative, Perov's and Schuader's fixed point theorems.
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1. Introduction

Coupled systems of AODEs have been considered by many authors in literatures. Since, such systems are increasingly arisen in the mathematical modeling of many real world problems. Therefore, in last few

*Corresponding author
Email addresses: abid3877@gmail.com (Abid Hussain), kamalshah408@gmail.com (Kamal Shah)

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decades coupled system of AODEs have been investigated very well, for detail, we refer [1, 2, 3, 4] and the references theorem. To establish existence and uniqueness of solutions to coupled systems of AODEs, the authors have been used classical fixed point theorems of Banach, Schauder's, Sheafers and Krassnoselskii's type very well [5, 6]. Some authors have also applied Perov's fixed point theorem [7]. In the last few years some authors also considered AODEs for monotone iterative solution, in this regard large numbers of articles have been written, for detail, we refer [8, 9, 10, 11]. The obtained research was performed by using classical monotone technique combine with the method of upper and lower solutions. For this reasons authors have been developed many comparison results which are valuable for certain problems but not general. In 2012, some updated MIT were developed to generalize the concerned technique. Al-Refi first time obtain the generalized comparison results which have not properly used for AODEs under BCs. We are going to use the MIT for more general coupled system of AODEs with BCs of Dirichlet type as

$$\begin{cases} {}_0^R D_t^{p_1} z_1(t) = \theta_1(t, z_1(t), z_2(t)), & 1 < p_1 \leq 2,\ t \in (0,1), \\ {}_0^R D_t^{p_2} z_2(t) = \theta_2(t, z_2(t), z_1(t)), & 1 < p_2 \leq 2,\ t \in (0,1), \\ z_1(0) = 0, \quad z_1(1) = 0, \\ z_2(0) = 0, \quad z_2(1) = 0. \end{cases} \tag{1.1}$$

Where $\theta_i \in (I \times R^2, R)$, for $i = 1, 2$ are nonlinear continuous function. We also develop existence theory of solution via fixed point theorem and also we construct existence criteria and iterative solutions for a given coupled system (1.1). The considered problem under the mentioned conditions have a lots of applications in mechanical and civil engineering (beam theory), electrostatic and thermodynamics where a surface is kept on a fixed temperature. Further, the said kind of problem with mentioned boundary conditions have many applications in fluid mechanics also. Keeping in mind that we used Riemann-Liouville fractional derivative instead of Caputo fractional derivative because under Riemann-Liouville fractional derivative the updated MIT has not yet applied for the considered system (1.1). The comparison results are completely modified to fix the iterative solutions for the system (1.1). We have to develop two sequences in which one is monotonically increasing that converges to maximal solution, while the other sequence is monotonically decreasing that converges to minimal solution and the obtain results are displayed through plots to see the behavior of the corresponding extremal solutions respectively. Further, to justify our analysis we provide some interesting examples.

2. Preliminaries

Here, we review some fundamental results of fractional calculus.

Definition 2.1. [12] **Fractional order integration:** The fractional integral of order $p > 0$ of a function $z : (0, \infty) \to R$ is given by

$${}_0I_t^p z(t) = \frac{1}{\Gamma(p)} \int_0^t \frac{z(s)ds}{(t-s)^{1-p}},$$

provided that the integral exists on the right side.

Example 2.2. Consider $z(t) = t^3 + \exp(4t)$, then fractional order integration of z is given by

$${}_0I_t^p [t^3 + \exp(4t)] = \frac{6t^{p+3}}{\Gamma(4+p)} + t^p E_p(4t).$$

Definition 2.3. [13] **Fractional order derivative:** The fractional derivative of order $p > 0$ of a continues function $z : (0, \infty) \to R$ is given by

$${}_0^R D_t^p z(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{z(s)ds}{(t-s)^{p-n+1}},$$

$n - 1 < p \leq n$, where $n = [p] + 1$.

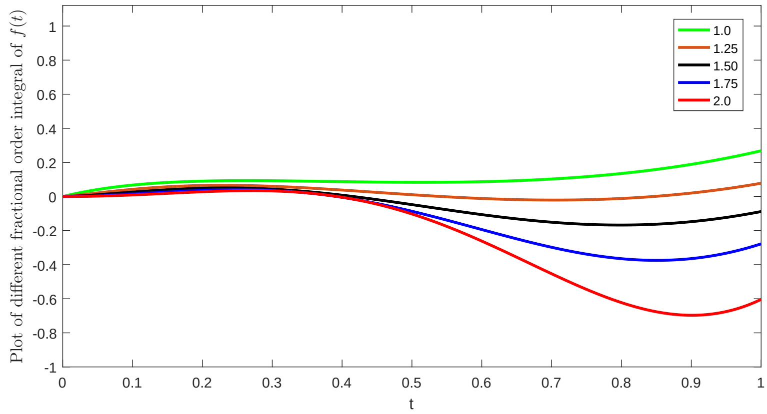

Figure 1: Plot of fractional order integration at different values of p of $z(t)$.

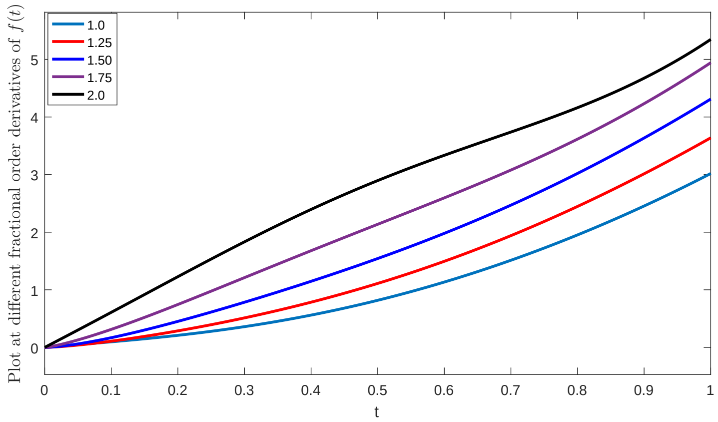

Figure 2: Plot of fractional order derivatives at different values of p of $z(t)$.

Example 2.4. Consider $z(t) = t^3 + \exp(4t)$, then fractional order derivative of z is given by

$$ {}_0^R D_t^p [t^3 + \exp(4t)] = \frac{6t^{3-p}}{\Gamma(t^3 - p)} + t^{-p} E_p(4t). $$

For various fractional order in Caputo sense we provide derivative plot in Figure 2.

Definition 2.5. [13] **Caputo's fractional derivative:** The fractional derivative of order $p > 0$, of a continues function $z : (0, \infty) \to R$ is given by

$$ {}_0^c D_t^p z(t) = \frac{1}{\Gamma(n-p)} \int_0^t \frac{z^n(s)ds}{(t-s)^{p+1-n}}, $$

$n - 1 < 1 \leq n$, where $n \in N,\ n = [p] + 1$.

Definition 2.6. [13] **Riemann-Liouville fractional derivative:** The fractional derivative of order $p > 0$ of a continues function $z : (0, \infty) \to R$ is given by

$$ {}_0^R D_t^p z(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{z(s)ds}{(t-s)^{p+1-n}}, $$

$n - 1 < 1 \leq n$, where $n \in N,\ n = [p] + 1$.

Definition 2.7. [14]. **Upper and lower solutions:** We define upper and lower solutions for considered problem (1.1). The pair of functions $(\underline{z}_1, \underline{z}_2) \in C[0,1] \times C[0,1]$ is called lower solutions of system (3.9), if the following condition holds,

$$\begin{cases} {}_0^R D_t^{p_1} \underline{z}_1(t) + \theta_1(t, \underline{z}_1(t), \underline{z}_2(t)) \geq 0, & 1 < p_1 \leq 2, \quad t \in (0,1), \\ {}_0^R D_t^{p_2} \underline{z}_2(t) + \theta_2(t, \underline{z}_2(t), \underline{z}_1(t)) \geq 0, & 1 < p_2 \leq 2, \quad t \in (0,1), \\ \underline{z}_1(0) \leq 0, \quad \underline{z}_1(1) \leq 0, \\ \underline{z}_2(0) \leq 0, \quad \underline{z}_2(1) \leq 0. \end{cases}$$

Similarly $(z_1, z_2) \in C[0,1] \times C[0,1]$ is called upper solutions of system (1.1), if the following condition holds,

$$\begin{cases} {}_0^R D_t^{p_1} \overline{z}_1(t) + \theta_1(t, \overline{z}_1(t), \overline{z}_2(t)) \leq 0, & 1 < p_1 \leq 2, \quad t \in (0,1), \\ {}_0^R D_t^{p_2} \overline{z}_2(t) + \theta_2(t, \overline{z}_2(t), \overline{z}_1(t)) \leq 0, & 1 < p_2 \leq 2, \quad t \in (0,1), \\ \overline{z}_1(0) \geq 0, \quad \overline{z}_1(1) \geq 0, \\ \overline{z}_2(0) \geq 0, \quad \overline{z}_2(1) \geq 0. \end{cases}$$

Lemma 2.8. *[15] Let $z_1 \in C[0,1]$, $1 < p_1 \leq 2$, attain its minimum at $t_0 \in (0,1)$, such that*

$${}_0^R D_{t_0}^{p_1} z_1(t) \geq \frac{1}{\Gamma(2-p_1)} \left[(p_1 - 1) t_0^{-p_1} (z_1(0) - z_1(t_0)) - t_0^{1-p_1} \acute{z}_1(0) \right], \text{ for all } 1 < p_1 \leq 2.$$

Then

$${}_0^R D_{t_0}^{p_1} z_1(t) \geq 0, \text{ for all } 1 < p_1 \leq 2,$$

the above result is called comparison result.

Lemma 2.9. *[15] Assume that $z_1 \in C[0,1]$, $1 < p_1 \leq 2$, attains its minimum at $t_0 \in (0,1)$ and if $\acute{z}_1(0) \leq 0$. Then ${}_0^R D_{t_0}^{p_1} z_1(t) \geq 0$, for all $1 < p_1 \leq 2$.*

Theorem 2.10. *[15]. Assume that $z_1, z_2 \in C[0,1]$, $\theta_1(t, z_1, z_2), \theta_2(t, z_2, z_1) \in C([0,1] \times R^2)$, such that $\theta_1(t, z_1, z_2) < 0, \theta_2(t, z_2, z_1) < 0$, for all $t \in (0,1)$. If $z_1(t), z_2(t)$ satisfy the following inequalities,*

$$\begin{cases} {}_0^R D_t^{p_1} z_1(t) + \theta_1(t, z_1(t), z_2(t)) \leq 0, & 1 < p_1 \leq 2, \quad t \in (0,1), \\ {}_0^R D_t^{p_2} z_2(t) + \theta_2(t, z_2(t), z_1(t)) \leq 0, & 1 < p_2 \leq 2, \quad t \in (0,1), \\ z_1(0) \geq 0, \quad z_1(1) \geq 0, \\ z_2(0) \geq 0, \quad z_2(1) \geq 0, \end{cases}$$

then $z_1(t) \geq 0, z_2(t) \geq 0$, for all $t \in [0,1]$.

Proof. Assume that the conclusion is not true, then $z_1(t)$ and $z_2(t)$ have absolute minimum at some t_0 with $z_1(t_0) < 0$ and $z_2(t_0) < 0$. If $0 < t_0 < 1$, then $\acute{z}_1(t_0) = 0$, $\acute{z}_2(t_0) = 0$.
Therefore, we have to prove that ${}_0^R D_{t_0}^{p_1} z_1(t_0) \geq 0$ and ${}_0^R D_{t_0}^{p_2} z_2(t_0) \geq 0$.

$$\begin{aligned} {}_0^R D_{t_0}^{p_1} z_1(t) &\geq \frac{1}{\Gamma(2-p_1)} \left[(p_1 - 1) t_0^{-p_1} (z_1(0) - z_1(t_0)) - t_0^{1-p_1} \acute{z}_1(0) \right], \text{ for all } 1 < p_1 \leq 2, \\ {}_0^R D_{t_0}^{p_2} z_2(t) &\geq \frac{1}{\Gamma(2-p_2)} \left[(p_2 - 1) t_0^{-p_2} (z_2(0) - z_2(t_0)) - t_0^{1-p_2} \acute{z}_2(0) \right], \text{ for all } 1 < p_2 \leq 2. \end{aligned} \tag{2.1}$$

Since $z_1(t_0) \leq z_1(0),\ z_2(t_0) \leq z_2(0), t_0 > 0$ and $z_1'(0) \leq 0, z_2'(0) \leq 0$, by using Lemma 2.8, from the first inequality of Theorem 2.1 and boundary condition $z_1(t) \geq 0$, we have

$$\begin{aligned}
{}_0^R D_{t_0}^{p_1} z_1(t) &\geq \frac{1}{\Gamma(2-p_1)}\left[(p_1-1)t_0^{-p_1}(z_1(0)-z_1(t_0)) - t_0^{1-p_1} z_1'(0)\right], \quad \text{for all } 1 < p_1 \leq 2,\\
&\geq \frac{t_0^{p_1}}{\Gamma(2-p_1)}\left[(p_1-1)t_0^{-p_2}(z_1(0)-z_1(t_0)) - t_0 z_1'(0)\right], \ \text{for all } 1 < p_1 \leq 2,\\
&\geq \frac{t_0^{p_1}}{\Gamma(2-p_1)}\left[-t_0 z_1'(0)\right] \geq 0, \ \text{as } z_1(t_0) \leq 0, \ \text{for all } 1 < p_1 \leq 2,\\
{}_0^R D_{t_0}^{p_1} z_1(t) &\geq 0.
\end{aligned}$$

Similarly, we can prove that ${}_0^R D_{t_0}^{p_2} z_2(t) \geq 0$. If $z_1'(t) > 0$ and $z_2'(t) > 0$, for all $t \in [0,1]$, we can obtain the same result by using Lemma 2.9 and both cases the conclusion is that $z_1(t) \geq 0$ and $z_2(t) \geq 0$, for all $t \in [0,1]$. □

Lemma 2.11. *[16]. The unique solution of AODEs involving Riemann-Liouville fractional derivative as* ${}_0^R D_t^p z(t) = 0$, *for* $z \in C(0,1) \cap L(0,1)$ *is provided by*

$${}_0 I_t^p [{}_0^R D_t^p z(t)] = z(t) + \sum_{k=0}^{n} c_k t^{p-k},$$

for arbitrary $c_k \in R$, *for* $k = 0, 1, 2, ..., n-1$.

3. Main Results

This portion of the research is devoted to the main results.

3.1. Derivative of Solution in Terms of Green Functions

In this section, we convert our considered problem (1.1) to corresponding coupled system of fractional integral equations. In this regard, we present the following results.

Lemma 3.1. *Consider* $h \in L[0,1]$, *then the following boundary value problem of AODEs*

$$\begin{cases} {}_0^R D_t^{p_1} z_1(t) = -h(t), \ 1 < p_1 \leq 2, \ t \in [0,1], \\ z_1(0) = z_1(1) = 0, \end{cases} \tag{3.1}$$

is equivalent to the given integral equation as

$$z_1(t) = \int_0^1 G_1(t,s) h(s) ds,$$

where $G_1(t,s)$ *is the Green's function defined as*

$$G_1(t,s) = \frac{1}{\Gamma(p_1)} \begin{cases} (t-s)^{p_1-1} - [t(1-s)]^{p_1-1}, \ 0 \leq s \leq t \leq 1, \\ -[t(1-s)]^{p_1-1}, \ 0 \leq t \leq s \leq 1. \end{cases} \tag{3.2}$$

Lemma 3.2. *Consider* $h \in L[0,1]$, *then the following boundary value problem of AODEs*

$$\begin{cases} {}_0^R D_t^{p_1} z_1(t) = -h(t), \ 1 < p_1 \leq 2, \ t \in [0,1], \\ z_1(0) = z_1(1) = 0, \end{cases} \tag{3.3}$$

is equivalent to the given integral equation as

$$z_1(t) = \int_0^1 G_1(t,s)h(s)ds,$$

where $G_1(t,s)$ is the Green's function defined as

$$G_1(t,s) = \frac{1}{\Gamma(p_1)} \begin{cases} (t-s)^{p_1-1} - [t(1-s)]^{p_1-1}, \ 0 \le s \le t \le 1, \\ -[t(1-s)]^{p_1-1}, \ 0 \le t \le s \le 1. \end{cases} \tag{3.4}$$

Proof. In view of Lemma 2.11, the given problem (3.3) yields

$$z_1(t) = C_1 t^{p_1-1} + C_2 t^{p_2-2} -_0 I_t^{p_1} h(t), \tag{3.5}$$

using initial condition $z_1(0) = 0$, boundary condition $z_1(1) = 0$ and taking $\lim_{t\to 0}$, on omitting singularity, we get $C_2 = 0$ and $C_1 =_0 I_t^{p_1} h(1)$. Putting C_1, C_2 in (3.5), we have

$$z_1(t) = t^{p_1-1}{}_0I_1^{p_1}h(1) - {}_0I_t^{p_1}h(t). \tag{3.6}$$

Then (3.6) can be written as

$$\begin{aligned} z_1(t) &= \frac{1}{\Gamma(p_1)}\int_0^1 t^{p_1-1}(1-s)^{p_1-1}h(s)ds - \frac{1}{\Gamma(p_1)}\int_0^t (t-s)^{p_1-1}h(s)ds, \\ &= \int_0^1 G_1(t,s)h(s)ds, \end{aligned}$$

where $G_1(t,s)$ is the Green's function given in (3.4). □

Corollary 3.3. *In view of Lemma 3.2, our considered coupled system under BCs (1.1) of AODEs is equivalent to the following coupled system of fractional integral equations as*

$$\begin{cases} z_1(t) = \int_0^1 G_1(t,s)\theta_1(s, z_1(s), z_2(s))ds, \\ z_2(t) = \int_0^1 G_2(t,s)\theta_2(s, z_2(s), z_1(s))ds, \end{cases} \tag{3.7}$$

where $G_2(t,s)$ is the Green's function of second equation of the considered problem (1.1) and given by

$$G_2(t,s) = \frac{1}{\Gamma(p_2)} \begin{cases} (t-s)^{p_2-1} - [t(1-s)]^{p_2-1}, \ 0 \le s \le t \le 1, \\ -[t(1-s)]^{p_2-1}, \ 0 \le t \le s \le 1. \end{cases} \tag{3.8}$$

Now, we define mapping $T : Z_1 \times Z_2 \to Z_1 \times Z_2$ by

$$T(z_1, z_2)(t) = \begin{pmatrix} T_1(z_1, z_2) \\ T_2(z_1, z_2) \end{pmatrix}(t), \tag{3.9}$$

where,

$$\begin{cases} T_1(z_1, z_2)(t) = \int_0^1 G_1(t,s)\theta_1(s, z_1(s), z_2(s))ds, \\ T_2(z_1, z_2)(t) = \int_0^1 G_2(t,s)\theta_2(s, z_2(s), z_1(s))ds. \end{cases} \tag{3.10}$$

3.2. Existence and Uniqueness Criteria for Solutions of Considered Problems

Now, we study existence criteria of solutions for the system (3.10), which lead us to the existence of solution to the system (1.1).

Lemma 3.4. *[17] Suppose that $\theta_1, \theta_2 : I \times R \times R \to R$ are continuous functions. Then $(z_1, z_2) \in Z_1 \times Z_2$ is a solution of system (1.1) if and only if $(z_1, z_2) \in Z_1 \times Z_2$ is a solution of system (3.7).*

Proof. Let $(z_1, z_2) \in Z_1 \times Z_2$ ba solution of system (1.1), then we have already proved Lemma 3.2, that (z_1, z_2) is a solution of system (1.1). Conversely, let $(z_1, z_2) \in Z_1 \times Z_2$ be solution of system (3.7). We denote the right hand side of the first equation in (3.7) by

$$\begin{aligned} z_1(t) &= {}_0I_t^{p_1}\theta_1(t, z_1(t), z_2(t)) - t^{p_1-1}\ {}_0I_t^{p_1}\theta_1(1, z_1(1), z_2(1)), \\ {}_0^RD_t^{p_1} z_1(t) &= {}_0^RD_t^{p_1} z_1(t)[{}_0I_t^{p_1}\theta_1(t, z_1(t), z_2(t)) - t^{p_1-1}\ {}_0I_t^{p_1}\theta_1(1, z_1(1), z_2(1))], \end{aligned}$$

by using the remarks that ${}_0^RD_t^{p_1}(t)t^{\alpha} = 0$, if $p_1 < \alpha$,

$$\begin{aligned} {}_0^RD_t^{p_1} z_1(t) &= {}_0^RD_t^{p_1}[{}_0I_t^{p_1}\theta_1(t, z_1(t), z_2(t)) - t^{p_1-1}\ {}_0I_t^{p_1}\theta_1(1, z_1(1), z_2(1)), \\ &= \theta_1(t, z_1(t), z_2(t)), \quad z_1(0) = 0,\ z_2(1) = 0, \end{aligned}$$

are satisfied. Similarly, for z_2

$$z_2(t) = {}_0I_t^{p_2}\theta_2(t, z_2(t), z_1(t)) - t^{p_2-1}\ {}_0I_t^{p_2}\theta_2(1, z_2(1), z_1(1)),$$

using the remarks that ${}_0^RD_t^{p_2}(t)t^{\beta} = 0$, if $p_2 < \beta$,

$$\begin{aligned} {}_0^RD_t^{p_2} z_2(t) &= {}_0^RD_t^{p_2}[{}_0I_t^{p_2}\theta_2(t, z_1(t), z_2(t)) - t^{p_2-1}\ {}_0I_t^{p_2}\theta_2(1, z_1(1), z_2(1)), \\ &= \theta_2(t, z_2(t), z_1(t)),\ z_1(0) = 0,\ z_2(1) = 0, \end{aligned}$$

are hold. Therefore (z_1, z_2) is a solution of (1.1). □

By Lemma 3.4, the fixed point of operator T coincides with the solution of system (1.1). For our main work, we use the following notation,

$$A = \frac{1}{\Gamma(p_1 + 1)}, \quad B = \frac{1}{\Gamma(p_2 + 1)}.$$

For the solution of considered system (1.1), we have to prove the following assumptions,

(H_1) There exist two non-negative functions $a(t), b(t) \in L[0, 1]$, such that

$$\begin{aligned} &\|\theta_1(t, z_1, z_2)\| < a(t) + c_1\|z_1\|^{\rho_1} + c_2\|z_2\|^{\rho_2}, \\ &\|\theta_2(t, z_1, z_2)\| < b(t) + d_1\|z_1\|^{\theta_1} + d_2\|z_2\|^{\theta_2}, \\ &\text{where } c_i, d_1 \geq 0,\ 0 < \rho_i,\ \theta_i < 1,\ \text{for } i = 1, 2. \end{aligned}$$

(H_2)

$$\begin{aligned} &\|\theta_1(t, z_1, z_2)\| < c_1\|z_1\|^{\rho_1} + c_2\|z_2\|^{\rho_2}, \\ &\|\theta_2(t, z_1, z_2)\| < d_1\|z_1\|^{\theta_1} + d_2\|z_2\|^{\theta_2}, \\ &\text{where } c_i, d_1 > 0,\ \rho_i, \theta_i \geq 1,\ \text{for } i = 1, 2. \end{aligned}$$

Theorem 3.5. *Let $\theta_1, \theta_2 : I \times R \times R \to R$ be continuous functions, then under the assumptions $(H_1), (H_2)$, the considered system (1.1) has at least one solution.*

Proof. We shall prove the result in view of Schauder's fixed point Theorem **??** under assumption (H_1). Let us define a set

$$S = \{(z_1(t), z_2(t))|(z_1(t), z_2(t)) \in Z_1 \times Z_2.\|(z_1(t), z_2(t))\|_{Z_1\times Z_2} \leq \mathcal{R},\ t \in [0,1]\},$$

where

$$\mathcal{R} \geq \max\left[(3Ac_1)^{\frac{1}{1-\rho_1}}, (3Ac_2)^{\frac{1}{1-\rho_2}}, (3Bd_1)^{\frac{1}{1-\theta_1}}, (3Bd_2)^{\frac{1}{1-\theta_2}}, 3k, 3l\right],$$

such that,

$$\begin{aligned} k &= \max_{t\in[0,1]} \int_0^1 |G_1(t,s)a(s)|ds, \\ l &= \max_{t\in[0,1]} \int_0^1 |G_2(t,s)b(s)|ds. \end{aligned}$$

Now for any $(z_1, z_2) \in Z_1$, for operator (1.1), we have

$$\begin{aligned} |T_1(z_1,z_2)(t)| &= \left|\int_0^1 G_1(t,s)\theta_1(s, z_1(s), z_2(s)ds)\right| \\ &\leq \int_0^1 |G_1(t,s)a(s)|ds + (c_1\mathcal{R}^{\rho_1} + c_2\mathcal{R}^{\rho_2})\int_0^1 |G_1(t,s)|ds \\ &= \int_0^1 |G_1(t,s)a(s)|ds + (c_1\mathcal{R}^{\rho_1} + c_2\mathcal{R}^{\rho_2})\left(\int_0^1 \frac{-(t-s)^{p_1-1}}{\Gamma(p_1)} + \int_0^1 \frac{t(1-s)^{p_1-1}}{\Gamma(p_1)}\right) \\ &= \int_0^1 |G_1(t,s)a(s)|ds + (c_1\mathcal{R}^{\rho_1} + c_2\mathcal{R}^{\rho_2})\left(\frac{t^{p_1-1}}{\Gamma(p_1+1)} - \frac{t^{p_1}}{\Gamma(p_1+1)}\right) \\ &\leq \int_0^1 |G_1(t,s)a(s)|ds + (c_1\mathcal{R}^{\rho_1} + c_2\mathcal{R}^{\rho_2})\left(\frac{1}{\Gamma(p_1+1)}\right),\ \text{where } A = \frac{1}{\Gamma(p_1+1)}. \end{aligned}$$

Taking maximum of both sides, we have

$$\begin{aligned} \|T_1(z_1,z_2)\| &\leq k + (c_1\mathcal{R}^{\rho_1} + c_2\mathcal{R}^{\rho_2})A \\ &\leq \frac{\mathcal{R}}{3} + \frac{\mathcal{R}}{3} + \frac{\mathcal{R}}{3} \\ &\leq \mathcal{R}. \end{aligned}$$

Similarly, we can prove that $\|T_2(z_1, z_2)\| \leq \mathcal{R}$.
Therefore, we get

$$\|T(z_1, z_2)\| \leq \mathcal{R},$$

Hence, T is bounded.
Also, under assumption (H_2), we have

$$\|T(z_1, z_2)\| \leq \mathcal{R}.$$

In this case, the operator T is also bounded. Now we have to show that T is completely continuous operator. We take

$$\begin{aligned} M &= \max_{t\in[0,1]} |(\theta_1, z_1(t), z_2(t))|, \\ N &= \max_{t\in[0,1]} |(\theta_2(t, z_2(t).z_1(t))|. \end{aligned}$$

For any $(z_1, z_2) \in Z_1$. Let $t, \tau \in Z_1$ such that, $t < \tau$. Then, we have

$$\begin{aligned}
|T_1(z_1,z_2)(t) - T_1(z_1,z_2)(\tau)| &= \left| \int_0^1 (G_1(t,s) - G_1(\tau,s))\theta_1(s, z_1(s), z_2(s)ds \right| \\
&\leq \frac{M}{\Gamma(p_1)} \Big[\int_0^t (\tau - s)^{p_1-1} - (t-s)^{p_1-1} \\
&- \tau^{p_1-1}(1-s)^{p_1-1} - t^{p_1-1}(1-s)^{p_1-1} ds \Big] \\
&+ \int_t^\tau \tau^{p_1-1}(1-s)^{p_1-1} - t^{p_1-1}(1-s)^{p_1-1} + (\tau-s)^{p_1-1})ds \\
&+ \int_\tau^1 (\tau^{p_1-1}(1-s)^{p_1-1} - \tau^{p_1-1}(1-s)^{p_1-1} \Big] ds \\
&= \frac{M}{\Gamma(p_1)} \Big[\int_0^1 (1-s)^{p_1-1}(\tau^{p_1-1} - t^{p_1-1})ds \\
&+ \int_0^\tau (\tau - s)^{p_1-1} ds - \int_0^t (t-s)^{p_1-1} ds \Big] \\
&= \frac{M}{\Gamma(p_1+1)} (\tau^{p_1-1} - t^{p_1-1} + \tau^{p_1} - t^{p_1}).
\end{aligned}$$

Hence, we have

$$|T_1(z_1,z_2)(t) - T_1(z_1,z_2)(\tau)| \leq \frac{M}{\Gamma(p_1+1)} (\tau^{p_1-1} - t^{p_1-1} + \tau^{p_1} - t^{p_1}) \longrightarrow 0 \text{ as } t \longrightarrow \tau.$$

Similarly,

$$|T_2(z_1,z_2)(t) - T_2(z_1,z_2)(\tau)| \leq \frac{N}{\Gamma(p_2+1)} (\tau^{p_2-1} - t^{p_2-1} + \tau^{p_2} - t^{p_2}) \longrightarrow 0, \text{ if } t \longrightarrow \tau.$$

Also $(\tau^{p_1-1} - t^{p_1-1} + \tau^{p_1} - t^{p_1})$ and $(\tau^{p_2-1} - t^{p_2-1} + \tau^{p_2} - t^{p_2})$ are uniformly continues therefore,

$$\|T_1(z_1,z_2)\| \longrightarrow 0 \text{ and } \|T_2(z_1,z_2)\| \longrightarrow 0 \text{ as } t \longrightarrow \tau.$$

So, we have

$$\|T_1(z_1,z_2)\| + \|T_2(z_1,z_2)\| \longrightarrow 0 \text{ as } t \longrightarrow \tau.$$

Hence, T is uniformly continuous. Since, the operator T is completely continuous and compact. So by Arzelá-Ascoli theorem, the operator T has at least one fixed point which means that the considered system (1.1) has at least one solutions.
Under condition (H_2), we consider

$$0 < \mathcal{R} \leq \min \left[(\frac{1}{2Ac_1})^{\frac{1}{\rho_1-1}}, (\frac{1}{2Ac_2})^{\frac{1}{\rho_2-1}}, (\frac{1}{2Bd_1})^{\frac{1}{\theta_1-1}}, (\frac{1}{2Bd_2})^{\frac{1}{\theta_2-1}} \right].$$

Proceeding on same line, we can prove that the operator $T : Z_1 \times Z_2 \longrightarrow Z_1 \times Z_2$ is completely continuous and uniformly bounded which yields that the considered system under BCs (1.1) has at least one solution due to Arzelá-Ascoli theorem. □

Next, to prove uniqueness of the solutions, we define the following assumption;

(H_3) There exist a functions a_i and b_i, $a_i, b_i : (0,1) \longrightarrow [0,\infty)$, for $i = 1,2$, such that for any $z_1, z_2, \overline{z}_1, \overline{z}_2 \in R$, we have

$$\begin{aligned}
|\theta_1(t,z_1,z_2) - \theta_1(t,\overline{z}_1,\overline{z}_2)| &\leq a_1(t)|z_1 - \overline{z}_1| + b_1|z_2 - \overline{z}_2| \\
|\theta_2(t,z_1,z_2) - \theta_2(t,\overline{z}_1,\overline{z}_2)| &\leq a_2(t)|z_1 - \overline{z}_1| + b_2|z_2 - \overline{z}_2|,
\end{aligned}$$

Theorem 3.6. *Under the assumption (H_3), the considered system under BCs (1.1) has unique solution if and only if there exists matrix given by*

$$B = \begin{bmatrix} \int_0^1 G_1(t,s)a_1(s)ds & \int_0^1 G_1(t,s)b_1(s)ds \\ \int_0^1 G_2(t,s)a_2(s)ds & \int_0^1 G_2(t,s)b_2(s)ds \end{bmatrix}, \tag{3.11}$$

such that $\rho\ (B) < 1$, where $\rho(B)$ is the spectral radius of matrix B.

Proof. Consider for $z_1, z_2, \overline{z}_1, \overline{z}_2 \in R$, we have

$$\begin{aligned}\|T_1(z_1,z_2) - T_1(\overline{z}_1,\overline{z}_2)\| &= \max_{t\in[0,1]} \left| \int_0^1 G_1(t,s)[\theta_1(s,z_1(s),z_2(s)) - \theta_1(s,\overline{z}_1(s),\overline{z}_z(s)]ds \right| \\ &\leq \max_{t\in[0,1]} \int_0^1 |G_1(t,s)||\theta_1(s,z_1(s),z_2(s)) - \theta_1(s,\overline{z}_1(s),\overline{z}_z(s)|ds, \\ &\leq \int_0^1 |G_1(1,s)[a_1(s)\|z_1 - \overline{z}_1\| + b_1(s)\|z_2 - \overline{z}_2\|]ds, \\ &= \|z_1 - \overline{z}_1\| \int_0^1 |G_1(1,s)a_1(s)ds + \|z_2 - \overline{z}_2\| \int_0^1 G_1(1,s)b_1(s)ds, \\ \|T_1(z_1,z_2) - T_1(\overline{z}_1,\overline{z}_2)\| &\leq \|z_1 - \overline{z}_1\| \int_0^1 G_1(1,s)a_1(s)ds + \|z_2 - \overline{z}_2\| \int_0^1 G_1(1,s)b_1(s)ds,\end{aligned}$$

Similarly, we get

$$\|T_2(z_1,z_2) - T_2(\overline{z}_1,\overline{z}_2)\| \leq \|z_1 - \overline{z}_1\| \int_0^1 G_2(1,s)a_2(s)ds + \|z_2 - \overline{z}_2\| \int_0^1 G_2(1,s)b_2(s)ds,$$

upon re-arrangement and simplifications, we have

$$\|T_1(z_1,z_2) - T_1(\overline{z}_1,\overline{z}_2)\| \leq \begin{bmatrix} \int_0^1 G_1(t,s)a_1(s)ds & \int_0^1 G_1(t,s)b_1(s)ds \\ \int_0^1 G_2(t,s)a_2(s)ds & \int_0^1 G_2(t,s)b_2(s)ds \end{bmatrix} \begin{bmatrix} \|z_1 - \overline{z_1}\| \\ \|z_2 - \overline{z_2}\| \end{bmatrix},$$

which can be written as

$$\|T_1(z_1,z_2) - T_1(\overline{z}_1,\overline{z}_2)\| \leq B \begin{bmatrix} \|z_1 - \overline{z_1}\| \\ \|z_2 - \overline{z_2}\| \end{bmatrix},$$

where

$$B = \begin{bmatrix} \int_0^1 G_1(t,s)a_1(s)ds & \int_0^1 G_1(t,s)b_1(s)ds \\ \int_0^1 G_2(t,s)a_2(s)ds & \int_0^1 G_2(t,s)b_2(s)ds \end{bmatrix},$$

where $\rho(B) < 1$. Which show that the system (1.1) has unique solution. □

3.3. Examples to demonstrate existence results

Example 3.7. Consider we have a coupled system under BCs as

$$\begin{cases} {}^R_0D_t^{\frac{3}{2}} z_1(t) = \frac{t}{4} + (t - \frac{1}{2})^3[\sqrt{z_1(t)} + \sqrt{z_2(t)}], \\ {}^R_0D_t^{\frac{3}{2}} z_2(t) = \frac{t^2}{2} + (t - \frac{1}{2})^3[\sqrt[3]{(z_1(t)} + \sqrt[3]{z_2(t)}], \\ z_1(0) = 0, \quad z_1(1) = 0, \\ z_2(0) = 0, \quad z_2(1) = 0. \end{cases} \tag{3.12}$$

Where $\rho_i = \frac{1}{2}$, $\theta_i = \frac{1}{3} < 1$, we see that $a(t) = \frac{t}{4}$, $b(t) = \frac{t^2}{2}$ and $c_i, d_i = \frac{1}{2}$, for $i = 1, 2$. Also $k = \frac{\sqrt{\pi}}{8}$, $l = \frac{15\sqrt{\pi}}{128}$. Then for $0 < \rho_i,\ \theta_i < 1$, we see that

$$\begin{aligned}\mathcal{R} &\geq \max\left[(3Ac_1)^{\frac{1}{1-\rho_1}}, (3Ac_2)^{\frac{1}{1-\rho_2}}, (3Bd_1)^{\frac{1}{1-\theta_1}}, (3Bd_2)^{\frac{1}{1-\theta_2}}, 3k, 3l\right],\\ &= \max\{2.5827704, 2.45990, 0.6646701, 0.207709\} = 2.5827704.\end{aligned}$$

Thus by using Theorem 3.5, the existence of at least one solution is obvious with $\mathcal{R} \geq 2.5827704$ under assumption (H_1).

In same line to verify the aforesaid theorem under assumption (H_2), we take the following example.

Example 3.8. Consider we have a coupled system under BCs as

$$\begin{cases} {}_0^R D_t^{\frac{3}{2}} z_1(t) = \dfrac{t}{40}[z_1^2(t) + z_2^2(t)],\\ {}_0^R D_t^{\frac{3}{2}} z_2(t) = \dfrac{t^2}{50}[z_1^3(t) + z_2^3(t)],\\ z_1(0) = 0, \quad z_1(1) = 0,\\ z_2(0) = 0, \quad z_2(1) = 0. \end{cases} \tag{3.13}$$

From the given system (3.13), we see that $c_i = \frac{1}{40}$, $d_i = \frac{1}{50}$, for $i = 1, 2$ and $\rho_i = 2$, $\theta_i = 3 > 1$. Using assumption (H_2) of Theorem 3.5, we can see that

$$\mathcal{R} \leq \min\{15.04505, 10.2336\} = 10.2336,$$

hence the given system under BCs has at least one solution.

Example 3.9. Consider we have a coupled system under BCs as

$$\begin{cases} {}_0^R D_t^{\frac{3}{2}} z_1(t) = \dfrac{t}{40}\sqrt{z_1(t)} + \dfrac{t^2}{50}\sqrt{z_2(t)},\\ {}_0^R D_t^{\frac{3}{2}} z_2(t) = \dfrac{t^2}{50}\sqrt{z_1(t)} + \dfrac{t}{40}\sqrt{z_2(t)},\\ z_1(0) = 0, \quad z_1(1) = 0,\\ z_2(0) = 0, \quad z_2(1) = 0. \end{cases} \tag{3.14}$$

From the system (3.14), we have $a_1(t) = b_2(t) = \frac{t}{40}$, $b_1(t) = a_2(t) = \frac{t^2}{50}$, we see in view of Theorem 3.6 that

$$B = \begin{bmatrix} \int_0^1 G_1(s,s)a_1(s)ds & \int_0^1 G_1(s,s)b_1(s)ds \\ \int_0^1 G_2(s,s)a_2(s)ds & \int_0^1 G_2(s,s)b_2(s)ds \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{\pi}}{320} & \frac{\sqrt{\pi}}{640} \\ \frac{\sqrt{\pi}}{640} & \frac{\sqrt{\pi}}{320} \end{bmatrix}.$$

On calculation we get $\rho(B) = 4.8 \times 10^{-6}$. Clearly $\rho(B) < 1$ and hence thank to Perove's Theorem the given system under BCs has unique solution.

3.4. Investigation of Monotone Iterative Solutions

In this section, we establish two monotone sequences in which one is decreasing and other is increasing first we prove some assumptions as;

(H_4) The nonlinear function $\theta_1(t, z_1(t), z_2(t))$ is strictly decreasing in z_1,

(H_5) The nonlinear function $\theta_2(t, z_2(t), z_1(t))$ is strictly decreasing in z_2.

Lemma 3.10. *[17] Let $(\underline{z}_1(t), \underline{z}_2(t))$ and $(\overline{z}_1(t), \overline{z}_2(t))$ be the lower and upper solution such that $\theta_1(t, z_1(t), z_2(t))$ is strictly decreasing with respect to z_1 and $\theta_2(t, z_2(t), z_1(t))$ is strictly decreasing with respect to z_2. Then $(\underline{z}_1, \underline{z}_2) \leq (\overline{z}_1, \overline{z}_2)$, for $t \in [0, 1]$.*

Proof. According to definition of the lower and upper solutions, we have

$$\begin{cases} {}_0^R D_t^{p_1} \underline{z}_1(t) + \theta_1(t, \underline{z}_1(t), z_2(t)) \geq 0, \ 1 < p_1 \leq 2, \quad t \in (0,1), \\ {}_0^R D_t^{p_2} \underline{z}_2(t) + \theta_2(t, \underline{z}_1(t), \underline{z}_2(t)) \geq 0, \ 1 < p_2 \leq 2, \quad t \in (0,1), \\ z_1(0) \leq 0, \ z_1(1) \leq 0, \\ z_2(0) \leq 0, \ z_2(1) \leq 0, \end{cases} \tag{3.15}$$

and

$$\begin{cases} {}_0^R D_t^{p_1} \overline{z}_1(t) + \theta_1(t, \overline{z}_1(t), \overline{z}_2(t)) \leq 0, \ 1 < p_1 \leq 2, \ t \in (0,1), \\ {}_0^R D_t^{p_2} \overline{z}_2(t) + \theta_2(t, \overline{z}_1(t), \overline{z}_2(t)) \leq 0, \ 1 < p_2 \leq 2, \ t \in (0,1), \\ z_1(0) \geq 0, \ z_1(1) \geq 0, \\ z_2(0) \geq 0, \ z_2(1) \geq 0, \end{cases} \tag{3.16}$$

from (3.15) and (3.16), we have

$$\begin{aligned} &{}_0^R D_t^{p_1} (\overline{z}_1 - \underline{z}_1) + \theta_1(t, \overline{z}_1(t), \overline{z}_2(t)) - \theta_1(t, \underline{z}_1(t), \underline{z}_2(t)) \leq 0, \\ &{}_0^R D_t^{p_2} (\overline{z}_2 - \underline{z}_2) + \theta_2(t, \overline{z}_1(t), \overline{z}_2(t)) - \theta_2(t, \underline{z}_1(t), \underline{z}_2(t)) \leq 0. \end{aligned}$$

By using Mean value theorem and taking $x = (\overline{z}_1 - \underline{z}_1), \ y = (\overline{z}_2 - \underline{z}_2)$, we have

$$\begin{aligned} &{}_0^R D_t^{p_1} x + \frac{\partial \theta_1}{\partial z_1}(m) \leq 0, \ m = \partial \overline{z}_1 + (1 - \partial) z_1, \ m \in [0,1], \\ &{}_0^R D_t^{p_2} y + \frac{\partial \theta_2}{\partial z_2}(n) \leq 0, \ n = \partial \overline{z}_2 + (1 - \partial) z_2, \quad n \in [0,1], \\ &z_1(0) = 0, \ z_1(1) = 0, \ z_2(0) = 0, \ z_2(1) = 0. \end{aligned}$$

Since θ_1 and θ_2 are strictly decreasing with respect z_1, z_2 respectively,

$$\begin{aligned} &\frac{\partial \theta_1}{\partial z_1}(m) < 0, \\ &\frac{\partial \theta_2}{\partial z_2}(n) < 0. \end{aligned}$$

We have $z_1 \geq 0, z_2 \geq 0$. Therefore, $\overline{z}_1 \geq \underline{z}_1$ and $\overline{z}_2 \geq \underline{z}_2$.

$$\Rightarrow (\underline{z}_1, \underline{z}_2) \leq (\overline{z}_1, \overline{z}_2).$$

□

In the following theorem, we construct monotone iterative sequences that describe lower and upper solutions to system of boundary value problem (3.9).

Theorem 3.11. *[17] Assume that the hypotheses (H_4) and (H_5) together with initial approximation $(\underline{z}_1^{(0)}, \underline{z}_2^{(0)})$ and $(\overline{z}_{1(0)}, \overline{z}_{2(0)})$ of the ordered lower and upper system of solutions for the coupled system (1.1), respectively in*

$$S = [\underline{z}_1, \overline{z}_1] \times [\underline{z}_2, \overline{z}_2] = \{(z_1, z_2) \in C[0,1] \times C[0,1] : (\underline{z}_1, \underline{z}_2) \leq (z_1, z_2) \leq (\overline{z}_1, \overline{z}_2)\}$$

be the solution of

$$\begin{cases} -{}_0^R D_t^{p_1} \underline{z}_1^{(n)}(t) + k\underline{z}_1^{(n)} = k\underline{z}_1^{(n-1)} + \theta_1(t, \underline{z}_1^{(n-1)}(t), \underline{z}_2^{(n-1)}(t)), \quad t \in [0,1],\ 1 < p_1 \leq 2, \\ -{}_0^R D_t^{p_2} \underline{z}_2^{(n)}(t) + l\underline{z}_2^{(n)} = l\underline{z}_1^{(n-1)}(t) + \theta_2(t, \underline{z}_1^{(n-1)}(t), \underline{z}_2^{(n-1)}(t)),\ t \in [0,1],\ 1 < p_2 \leq 2, \\ \underline{z}_1^{(n)}(0) = \underline{z}_{1(0)}^{(n)} \geq \underline{z}_1^{(n-1)}(0), \quad \underline{z}_{1(1)}^{(n)}(1) = \underline{z}_1^{(n)}(1) \geq \underline{z}_1^{(n-1)}(1), \\ \underline{z}_2^{(n)}(0) = \underline{z}_{2(0)}^{(n)} \geq \underline{z}_2^{(n-1)}(0), \quad \underline{z}_{2(1)}^{(n)}(1) = \underline{z}_{2(1)}^{(n)}(1) \geq \underline{z}_2^{(n-1)}(1), \end{cases} \tag{3.17}$$

and

$$\begin{cases} -{}_0^R D_t^{p_1} \overline{z}_{1(n)} + l\overline{z}_{1(n)} = l\overline{z}_{1(n-1)} + \theta_1(t, \overline{z}_{1(n-1)}(t), \overline{z}_{2(n-1)}, (t)),\ t \in [0,1],\ 1 < p_1 \leq 2, \\ -{}_0^R D_t^{p_2} \overline{z}_{2(n)} + l\overline{z}_{2(n)} = l\overline{z}_{2(n-1)} + \theta_2(t, \overline{z}_{1(n-1)}(t), \overline{z}_{2(n-1)})(t)),\ t \in [0,1],\ 1 < p_2 \leq 2, \\ \overline{z}_{1(n)}(0) = \overline{z}_{1(n)}^{n}(0) \geq \overline{z}_{1(n-1)}(0), \quad \overline{z}_{1(n)}(1) = \overline{z}_{1(n)}^{(1)}(1) \geq \overline{z}_{1(n-1)}(1), \\ \overline{z}_{2(n)}(0) = \overline{z}_{2(0)}^{(0)}(0) \geq \overline{z}_{2(n-1)}(0), \quad \overline{z}_{2(n)}(1) = \overline{z}_{2(n)}^{(1)}(1) \geq \overline{z}_{2(n-1)}(1). \end{cases} \tag{3.18}$$

Then, we have

(H_6) *The sequence* $(\underline{z}_1^{(n)}, \underline{z}_2^{(n)})$, $n \geq 1$ *is an increasing sequence of lower solution of BVP(1.1).*

(H_7) *The sequence* $(\overline{z}_1^{(n)}, \overline{z}_2^{(n)})$, $n \geq 1$ *is decreasing sequence of upper solution under BCs (1.1).*

(H_8) $(\underline{z}_1^{(n)}, \underline{z}_2^{(n)}) \leq (\overline{z}_1^{(n)}, \overline{z}_2^{(n)})$, *for all* $n \geq 1$.

Proof. To prove assumption (H_6), we need to show that

(H_9) $(\underline{z}_1^{(n)} - \underline{z}_1^{(n-1)}) \geq 0$ and $(\overline{z}_2^{(n)} - \overline{z}_2^{(n-1)}) \geq 0$, for each $n \geq 1$,

(H_{10}) $(\underline{z}_1^{(n)},\ \underline{z}_2^{(n)})$ is lower solution for each $n \geq 1$.

By using inductive method taking $n = 1$, then from (3.17), we have

$$\begin{cases} -{}_0^R D_t^{p_1} \underline{z}_1^{(1)}(t) + k\underline{z}_1^{(1)} = k\underline{z}_1^{(0)} + \theta_1(t, \underline{z}_1^{(0)}(t), \underline{z}_2^{(0)}(t)),\ t \in [01],\ 1 < p_1 \leq 2, \\ -{}_0^R D_t^{p_2} \underline{z}_2^{(1)}(t) + l\underline{z}_2^{(1)} = l\underline{z}_2^{(0)}(t) + \theta_2(t, \underline{z}_1^{(0)}(t), \underline{z}_2^{(0)}(t)),\ t \in [01],\ 1 < p_2 \leq 2, \\ \underline{z}_1^{(1)}(0) = \underline{z}_1^{(1)}(0) \geq \underline{z}_1^{(0)}(0), \quad \underline{z}_1^{(1)}(1) = \underline{z}_1^{(1)}(1) \geq \underline{z}_1^{(0)}(1), \\ \underline{z}_2^{(1)}(0) = \underline{z}_{2(0)}^{(1)} \geq \underline{z}_2^{(0)}(0), \quad \underline{z}_2^{(1)}(1) = \underline{z}_{2(0)}^{(1)}(1) \geq \underline{z}_2^{(0)}(1). \end{cases} \tag{3.19}$$

Since $(\underline{z}_1^{(0)}, \underline{z}_2^{(0)})$ is a lower solution

$$\begin{cases} {}_0^R D_t^{p_1} \underline{z}_1^{(0)}(t) + \theta_1(t, \underline{z}_1^{(0)}(t), \underline{z}_2^{(0)}(t)) \geq 0, \\ {}_0^R D_t^{p_2} \underline{z}_2^{(0)}(t) + \theta_2(t, \underline{z}_1^{(0)}(t), \underline{z}_2^{(0)}(t)) \geq 0. \end{cases} \tag{3.20}$$

Adding the corresponding equations of the system under BCss (3.19) and (3.20), we get the following system;

$$\begin{cases} {}_0^R D_t^{p_1} (\underline{z}_1^{(1)} - \underline{z}_1^{(0)}) - k(\underline{z}_1^{(1)} - \underline{z}_1^{(0)}) \leq 0, \\ {}_0^R D_t^{p_2} (\underline{z}_2^{(1)} - \underline{z}_2^{(0)}) - l(\underline{z}_2^{(1)} - \underline{z}_2^{(0)}) \leq 0. \end{cases} \tag{3.21}$$

By using $x = \underline{z}_1^{(1)} - \underline{z}_1^{(0)}$, $y = \underline{z}_1^{(1)} - \underline{z}_1^{(0)}$, then (x, y) satisfies

$$\begin{cases} {}_0^R D_t^{p_1} x - kx \leq 0, \quad {}_0^R D_t^{p_2} y - ly \leq 0, \\ x(0) \geq 0,\ y(0) \geq 0,\ x(1) \geq 0,\ y(1) \geq 0. \end{cases} \tag{3.22}$$

Since, $k<0, l<0$ and using Theorem 2.10, we have $x \geq 0, y \geq 0$. Therefore, we have $(\underline{z}_1^{(0)}, \underline{z}_2^{(0)}) \leq (\underline{z}_1^{(1)}, \underline{z}_2^{(1)})$. The result is true for $n=1$. Let the result be true for $m \leq n$ and we will derive the result for $m=n+1$. Now from equation (3.17), we have

$$\begin{aligned}
&-{}_0^R D_t^{p_1}(\underline{z}_1^{(n+1)} - \underline{z}_1^{(n)}) + k(\underline{z}_1^{(n+1)} - \underline{z}_1^{(n)}) \\
&= k(\underline{z}_1^{(n)} - \underline{z}_1^{(n-1)}) + \theta_1(t, \underline{z}_1^{(n)} - \underline{z}_2^{(n)}) - \theta_1(t, \underline{z}_1^{(n-1)} - \underline{z}_2^{(n-1)}), \\
&-{}_0^R D_t^{p_2}(\underline{z}_2^{(n+1)} - \underline{z}_2^{(n)}) + l(\underline{z}_2^{(n+1)} - \underline{z}_2^{(n)}) \\
&= l(\underline{z}_1^{(n)} - \underline{z}_1^{(n-1)}) + \theta_2(t, \underline{z}_1^{(n)} - \underline{z}_2^{(n)}) - \theta_2(t, \underline{z}_1^{(n-1)} - \underline{z}_2^{(n-1)}).
\end{aligned}$$

We using $x = (\underline{z}_1^{(n+1)} - \underline{z}_1^{(n)})$, $y = (\underline{z}_2^{(n+1)} - \underline{z}_2^{(n)})$ and apply the Mean value theorem together with $(\underline{z}_1^{(n-1)}, \underline{z}_1^{(n-1)}) \leq (\underline{z}_1^{(n)}, \underline{z}_1^{(n)})$.
Then, we have

$$\begin{aligned}
{}_0^R D_t^{p_1} x - kx \leq 0, \\
{}_0^R D_t^{p_2} y - ky \leq 0,
\end{aligned}$$

in the view of the Theorem 2.10, $x \geq 0,\ y \geq 0$, which yields

$$(\underline{z}_1^{(n)}, \underline{z}_2^{(n)}) \leq (\underline{z}_1^{(n+1)}, \underline{z}_2^{(n+1)}).$$

The result is true for $n = m+n$, therefore, we have

$$(\underline{z}_1^{(n-1)}, \underline{z}_2^{(n-1)}) \leq (\underline{z}_1^{(n)}, \underline{z}_2^{(n)}),$$

for each $n \geq 1$, which proves (H_9). To derive (H_{10}), we subtract $\theta_1(t, \underline{z}_1^{(n)}(t), \underline{z}_2^{(n)}(t))$ from the first equation of (3.17) and $\theta_2(t, \underline{z}_1^{(n)}(t), \underline{z}_2^{(n)}(t))$ are subtract from the second equation of (3.17) re-arranging and apply Man value theorem, we get

$$\begin{aligned}
{}_0^R D^{p_1} t \underline{z}_1^{(n)}(t) = \theta_1(t, \underline{z}_1^{(n)}(t), \underline{z}_2^{(n)}(t)) \geq 0,\ t \in (0,1), \\
{}_0^R D^{p_2} t \underline{z}_2^{(n)}(t) = \theta_2(t, \underline{z}_1^{(n)}(t), \underline{z}_2^{(n)}(t)) \geq 0,\ t \in (0,1).
\end{aligned} \tag{3.23}$$

Therefore, $(\underline{z}_1^{(n)}, \underline{z}_1^{(n)})$ is a lower solution for boundary value problem (1.1). Now from (H_6) and (H_7), $(\underline{z}_1^{(n)}, \underline{z}_2^{(n)})$ and $(\overline{z}_1^{(n)}, \overline{z}_2^{(n)})$ are upper and lower solutions under BCs (1.1), Therefore, in the view of Theorem 2.10, follow that $(\underline{z}_1^{(n)}, \underline{z}_2^{(n)}) \leq (\overline{z}_1^{(n)}, \overline{z}_2^{(n)})$, for all $n \geq 1$. □

Theorem 3.12. *[17] Under assumption (H_6) and (H_7) and condition of (1.1). Let $(\underline{z}_1^{(n)}, \underline{z}_2^{(n)})$ and $(\overline{z}_{1(n)}, \overline{z}_{2(n)})$ be lower and upper solutions, then as defined in theorem (3.11), then the sequences $(\underline{z}_1^{(n)}, \underline{z}_2^{(n)})$ and $(\overline{z}_{1(n)}, \overline{z}_{2(n)})$, $n \geq 0$, converge uniformly to $(z_{1(*)}, z_{2(*)})$ and (z_1^*, z_2^*) respectively with $(z_{1(*)}, z_{2(*)}) \leq (z_1^*, z_2^*)$.*

Proof. The sequence $x_n = (\underline{z}_1^{(n)}, \underline{z}_2^{(n)})$ is monotonically increasing and bounded above by $(\overline{z}_{1(0)}, \overline{z}_{2(0)})$ and the bounded monotonic increasing sequence shows convergence to its least upper bound say $(z_{1(*)}, z_{2(*)})$. The sequence $y_n = (\overline{z}_{1(n)}, \overline{z}_{2(n)})$ is monotonically decreasing sequence and bounded below by (z_1^*, z_2^*) and it is convergent to its greatest lower bound (z_1^*, z_2^*). The sequence x_n and y_n are continuous functions defined on the compact square $[0,1] \times [0,1]$. Thus the convergence is uniform and the view of theorem (3.11), $x_n \leq y_n$, for each $n \geq 1$, so $x_* = \lim_{n \longrightarrow \infty} y_n = y*$. □

Theorem 3.13. *[18] Under the assumptions (H_4) and (H_5), the boundary value problem (1.1) has at most one solution.*

Proof. Let (z_1, z_2) and (z_1^*, z_2^*) be the two solutions of the system (1.1)

$$\begin{cases} {}_0^R D_t^{p_1} z_1(t) = \theta_1(t, z_1(t), z_2(t)), & 1 < p_1 \leq 2, \\ {}_0^R D_t^{p_2} z_2(t) = \theta_2(t, z_2(t), z_1(t)), & 1 < p_2 \leq 2, \\ z_1(0) = 0, \quad z_1(1) = 0, \\ z_2(0) = 0, \quad z_2(1) = 0, \end{cases} \tag{3.24}$$

and

$$\begin{cases} {}_0^R D_t^{p_1} z_1^*(t) = \theta_1(t, z_1^*(t), z_2^*(t)), & 1 < p_1 \leq 2, \\ {}_0^R D_t^{p_2} z_2^*(t) = \theta_2(t, z_2^*(t), z_1^*(t)), & 1 < p_2 \leq 2, \\ z_1^*(0) = 0, \quad z_1^*(1) = 0, \\ z_2^*(0) = 0, \quad z_2^*(1) = 0. \end{cases} \tag{3.25}$$

We subtract the first equation of (3.24) from the first equation of (3.25) and similarly, we subtract the second equation of (3.24) from the second equation of (3.25),

$$\begin{aligned} &{}_0^R D_t^{p_1}(z_1^*(t) - z_1(t)) - \theta_1((t, z_2^*(t), z_1^*(t))) - \theta_1(t, z_1(t), z_2(t)),\ t \in (0,1), \\ &{}_0^R D_t^{p_2}(z_2^*(t) - z_2(t)) - \theta_2((t, z_2^*(t), z_1^*(t))) - \theta_2(t, z_1(t), z_2(t)),\ t \in (0,1), \end{aligned} \tag{3.26}$$

we using $x = (z_1^*(t) - z_1(t))$, $y = (z_2^*(t) - z_2(t))$ and we applying the Mean value theorem,

$$\begin{aligned} &{}_0^R D_t^{p_1} x + \frac{\partial \theta_1}{\partial z_1}(m) = 0,\ \text{where } m \in (0,1), \\ &{}_0^R D_t^{p_2} y + \frac{\partial \theta_2}{\partial z_2}(n) = 0,\ \text{where } n \in (0,1), \end{aligned} \tag{3.27}$$

with boundary conditions $z_1(0) = 0,\ z_1(1) = 0$ and $z_2(0) = 0,\ z_2(1) = 0$, by Theorem 2.10, $x = 0,\ y = 0$, also the system is satisfied by using $-x, -y$, Theorem 3.17, so we have $-x > 0, -y > 0$. Thus $x = 0$ and $y = 0$, which implies that $z_1^*(t) = z_1(t)$ and $z_2^*(t) = z_2(t)$. Which implies that the coupled system under consideration has at most one solution. □

3.5. Examples corresponding to iterative solutions

To demonstrate the results of previous section, we provide the following example.

Example 3.14. Taking the system of boundary value problem as

$$\begin{cases} {}_0^R D_t^{\frac{3}{2}} z_1(t) - z_1^3(t) + z_2(t) + t^5 = 0,\ t \in (0,1), \\ {}_0^R D_t^{\frac{3}{2}} z_2(t) - z_1(t) + z_2^3(t) + t^4 = 0,\ t \in (0,1), \\ z_1(0) = 0,\ z_2(0) = 0,\ z_1(1) = 0,\ z_2(1) = 0. \end{cases} \tag{3.28}$$

We have from the above system (3.28)

$$\begin{aligned} \theta_1(t, z_1(t), z_2(t)) &= -z_1^3(t) + z_2(t) + t^5, \\ \theta_2(t, z_2(t), z_1(t)) &= -z_1(t) + z_2^3(t) + t^4. \end{aligned} \tag{3.29}$$

Taking $(-1,-1) = (\underline{z}_1^{(0)}, \underline{z}_2^{(0)})$ and $(1,1) = (\overline{z}_{1(0)}, \overline{z}_{2(0)})$, be initial value of lower and upper solutions respectively. Also $\theta_1(t, z_1, z_2)$ is strictly decreasing such that,

$$-3 \leq \frac{\partial \theta_1(t, z_1, z_2)}{\partial z_1} = -3z_1^2 < 0,$$

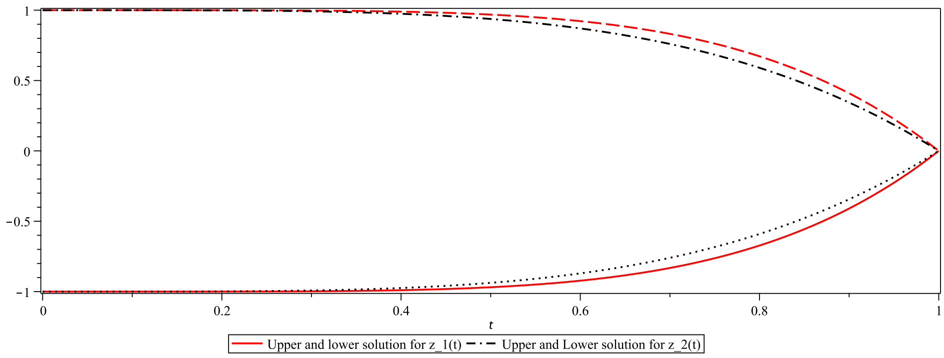

Figure 3: Plot of upper and lower solutions of Example 3.14.

and $\theta_2(t,z_1,z_2)$ is strictly decreasing with

$$-3 \leq \frac{\partial \theta_2(t,z_1,z_2)}{\partial z_2} = -3z_2^2 < 0,$$

for all $(z_1,z_2) \in [\underline{z}_1^{(0)}, \underline{z}_2^{(0)}] \times [\overline{z}_{1(0)}, \overline{z}_{2(0)}]$. Also $c = 3,\ d = 3$ and hence $(-1,-1)$ and $(1,1)$ are the initial values of lower and upper solutions respectively for the coupled system under BCs (3.28). The plot of the given extremal solutions have been expressed in Figure 3.

Example 3.15. For more explanation, we give another example of fractional differential equation subject to the coupled integral boundary conditions

$$\begin{cases} {}_0^R D_t^{\frac{7}{4}} z_1(t) - z_1(t)\exp(z_1(t)) + z_2(t) + 1 = 0,\ t \in (0,1), \\ {}_0^R D_t^{\frac{7}{4}} z_2(t) + z_1(t) - z_2(t)\exp(z_2(t)) + 1 = 0,\ t \in (0,1), \\ z_1(0) = z_2(0) = 0,\ z_1(1) = 0,\ z_2(1) = 0. \end{cases} \tag{3.30}$$

We have from (3.30),

$$\begin{aligned} \theta_1(t,z_1(t),z_2(t)) &= -z_1(t)\exp(z_1(t)) + z_2(t) + 1, \\ \theta_2(t,z_1(t),z_2(t)) &= z_1(t) - z_2(t)\exp(z_2(t)) + 1. \end{aligned} \tag{3.31}$$

Where $(0,0) = (\underline{z}_1^{(0)}, \underline{z}_2^{(0)})$ and $(2,2) = (\overline{z}_{1(0)}, \overline{z}_{2(0)})$ be initial values of lower and upper solutions respectively. Then from (3.31), we see that the function $\theta_1(t,z_1,z_2)$ is strictly decreasing with

$$-3\exp(2) \leq \frac{\partial \theta_1(t,z_1,z_2)}{\partial z_1} = -\exp(z_1)(z_1+1) < 0,$$

and $\theta_2(t,z_2,z_1)$ is strictly decreasing with

$$-3\exp(2) \leq \frac{\partial \theta_2(t,z_2,z_1)}{\partial z_2} = -\exp(z_2)(z_2+1) < 0,$$

for all $(z_1,z_2) \in [\underline{z_1}^{(0)}, \underline{z_2}^{(0)}] \times [\overline{z}_{1(0)}, \overline{z}_{2(0)}]$. Further the constants $c = d = 3\exp(2)$. Hence $(0,0)$ is the initial value of lower solution while $(2,2)$ is initial value of upper solution to the given coupled system under BCs (3.30). The plot of the given extremal solutions have been expressed in Figure 4.

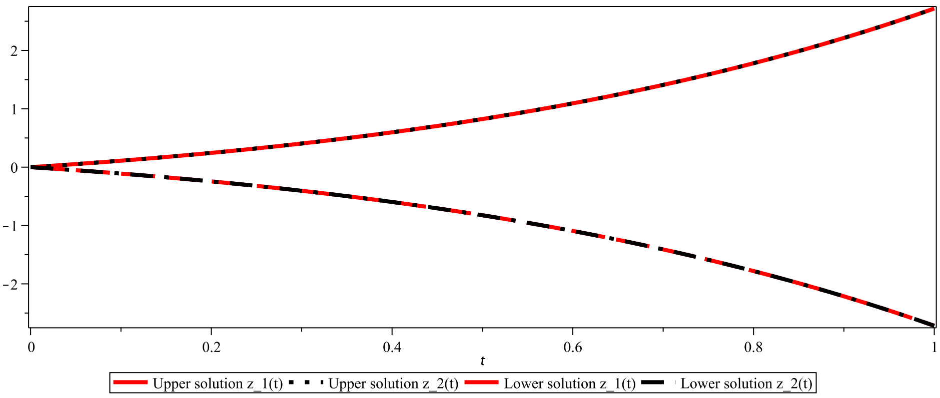

Figure 4: Plot of upper and lower solutions of Example 3.15.

4. Conclusion

In this thesis we have investigated a nonlinear coupled system of AODEs under BCs for existence theory. We have successfully established two types of results regarding our considered problem. The first type of results were devoted to existence and uniqueness for which we have applied Perove's and Schauder's fixed point theorems. Next, we have successfully applied the updated MIT which has been very rarely used for AODEs involving Riemann-Liouville fractional derivative. Necessary and sufficient results were constructed which gives multiplicity results about the iterative solutions. Suitable examples have been provided to demonstrate our results. We concluded that updated MIT works very well for AODEs involving Riemann-Liouville fractional derivative without any strict restriction on the nonlinearity of the system.

Authors contribution

All authors equally contributed this paper and approved the final version.

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Conflict of Interest

The authors have no conflict of interest regarding the publication of this article.

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