Convergence of $CR$-iteration procedure for a nonlinear quasi contractive map in convex metric spaces

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Abstract

We prove that the modified $CR$-iteration procedure converges strongly to a fixed point of a nonlinear quasi contractive map in convex metric spaces which is the main result of this paper. The convergence of Picard-S iteration procedure follows as a corollary to our main result.

Keywords: Convex metric space, quasi contraction map, $CR$-iteration procedure and Picard-S iteration procedure.

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1. Introduction and preliminaries


Definition 1.1. Let $(X,d)$ be a metric space. A map $W : X \times X \times [0,1] \to X$ is said to be a ‘convex structure’ on $X$ if

$$d(u,W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y)$$

for $x,y,u \in X$ and $\lambda \in [0,1]$.

A metric space $(X,d)$ together with a convex structure $W$ is called a convex metric space and we denote it by $(X,d,W)$. We note that $W(x,y,1) = x$ and $W(x,y,0) = y$. A nonempty subset $K$ of $X$ is said to be ‘convex’ if $W(x,y,\lambda) \in K$ for $x,y \in K$ and $\lambda \in [0,1]$.

Remark 1.2. Every normed linear space $(X,||.||)$ is a convex metric space with the convex structure $W$ defined by $W(x,y,\lambda) = (1-\lambda)y + \lambda x$ for $x,y \in X$, $\lambda \in [0,1]$. But there are convex metric spaces which are not normed linear spaces $[1,8,11]$.
In 1974, Ćirić [3] introduced quasi-contraction maps in the setting of metric spaces and proved that the Picard iterative sequence converges to the fixed point in complete metric spaces.

**Definition 1.3.** Let \((X,d)\) be a metric space. A selfmap \(T : X \to X\) is said to be a quasi-contraction map if there exists a real number \(0 \leq k < 1\) such that
\[
d(Tx,Ty) \leq kM(x,y) \tag{1.2}
\]
where
\[
M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \tag{1.3}
\]
for \(x, y \in X\).

Let \(K\) be a nonempty convex subset of a normed linear space \(X\) and let \(\{\alpha_n\}_{n=0}^{\infty}\) and \(\{\beta_n\}_{n=0}^{\infty}\) be sequences in \([0,1]\). The Ishikawa iteration procedure [7] in the setting of normed linear spaces is as follows: For \(x_0 \in K\),
\[
y_n = (1 - \beta_n)x_n + \beta_n Tx_n \\
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \quad \text{for } n = 0, 1, 2, \ldots \tag{1.4}
\]

Ding [5] considered the Ishikawa iteration procedure in the setting of convex metric spaces as follows: Let \(K\) be a nonempty convex subset of a convex metric space \((X,d,W)\), and let \(\{\alpha_n\}_{n=0}^{\infty}\) and \(\{\beta_n\}_{n=0}^{\infty}\) be the sequences in \([0,1]\). For \(x_0 \in K\),
\[
y_n = W(Tx_n,x_n,\beta_n) \\
x_{n+1} = W(Ty_n,x_n,\alpha_n) \quad \text{for } n = 0, 1, 2, \ldots \tag{1.5}
\]
and proved that the Ishikawa iteration procedure (1.5) converges strongly to a unique fixed point of a quasi-contraction map in the setting of convex metric spaces, provided \(\sum_{n=0}^{\infty} \alpha_n = \infty\).

In 1999, Ćirić [4] introduced a more general quasi-contraction map and proved the convergence of an Ishikawa iteration procedure in convex metric spaces to the unique fixed point and the result is the following.

**Theorem 1.4.** (Čirić [4]) Let \(K\) be a nonempty closed convex subset of a complete convex metric space \(X\) and let \(T : K \to K\) be a selfmap satisfying
\[
d(Tx,Ty) \leq w(M(x,y)), \tag{1.6}
\]
where \(M(x,y)\) is as defined in (1.3) for \(x, y \in K\) and \(w : (0,\infty) \to (0,\infty)\) is a map which satisfies (i) \(0 < w(t) < t\) for each \(t > 0\), (ii) \(w\) increases, and the following conditions:
\[
\lim_{t \to \infty} (t - w(t)) = \infty: \quad \text{and} \tag{1.7}
\]
either \(t - w(t)\) is increasing on \((0,\infty)\) \tag{1.8}
or \(w(t)\) is strictly increasing and \(\lim_{n \to \infty} w^n(t) = 0\) for \(t > 0\). \tag{1.9}

Let \(\{\alpha_n\}_{n=0}^{\infty}\) and \(\{\beta_n\}_{n=0}^{\infty}\) be sequences in \([0,1]\) such that \(\sum_{n=0}^{\infty} \alpha_n = \infty\). For \(x_0 \in K\), the Ishikawa iteration procedure \(\{x_n\}_{n=0}^{\infty}\) defined in (1.5) converges strongly to the unique fixed point of \(T\).

Sastry, Babu and Srinivasa Rao [10] improved Theorem 1.4 by replacing (1.8) and (1.9) with a single condition, namely \(0 < w(t^+) < t\) for each \(t > 0\) and proved the following theorem.
Theorem 1.5. [10] Let \((X,d,W)\) be a complete convex metric space and \(T : X \to X\) be a map that satisfies
\[
d(Tx,Ty) \leq w(M(x,y)) \tag{1.10}
\]
where \(M(x,y)\) is defined as in (1.3) for \(x, y \in X\) and \(w : (0, \infty) \to (0, \infty)\) is a map such that (i) \(w\) increases, 
(ii) \(\lim_{t \to \infty} (t - w(t)) = \infty\) (iii) \(0 < w(t^+) < t\) for \(t > 0\).

Let \(\{\alpha_n\}_{n=0}^{\infty}\) and \(\{\beta_n\}_{n=0}^{\infty}\) be sequences in \([0, 1]\) such that \(\sum_{n=0}^{\infty} \alpha_n = \infty\).

Then for any \(x_0 \in K\), the sequence \(\{x_n\}_{n=0}^{\infty}\) generated by the iteration procedure (1.5) converges strongly to a unique fixed point of \(T\).

Here we note that a map that satisfies (1.10) is said to be a nonlinear quasi contractive map on \(X\).

Remark 1.6. (i) and (iii) of Theorem 1.5 imply that \(0 < w(t) < t\) for each \(t > 0\).

Remark 1.7. If \(w(t) = kt\) for \(t \in (0, \infty)\) and \(0 \leq k < 1\) then the map \(T\) of Theorem 1.5 reduces to a quasi contraction map.

In 2012, Chugh, Kumar and Kumar [2] introduced ‘CR-iteration procedure’ as follows:

Let \(K\) be a nonempty convex subset of a normed linear space, \(X\), and let \(\{\alpha_n\}_{n=0}^{\infty}\), \(\{\beta_n\}_{n=0}^{\infty}\) and \(\{\gamma_n\}_{n=0}^{\infty}\) be sequences in \([0, 1]\). For \(x_0 \in K\),
\[
\begin{align*}
z_n &= (1 - \gamma_n)x_n + \gamma_nTx_n \\
y_n &= (1 - \beta_n)Tx_n + \beta_nTz_n \\
x_{n+1} &= (1 - \alpha_n)y_n + \alpha_nTy_n, \quad \text{for} \quad n = 0, 1, 2, \ldots .
\end{align*} \tag{1.11}
\]

By choosing \(\alpha_n \equiv 1\) for all \(n\) in (1.11), we have the following.

For \(x_0 \in K\),
\[
\begin{align*}
z_n &= (1 - \gamma_n)x_n + \gamma_nTx_n \\
y_n &= (1 - \beta_n)Tx_n + \beta_nTz_n, \\
x_{n+1} &= Ty_n, \quad \text{for} \quad n = 0, 1, 2, \ldots .
\end{align*} \tag{1.12}
\]

The iteration procedure (1.12) is called the ‘Picard-S iteration procedure’ [6].

In 2014, Chugh and Malik [9] introduced an analogue of CR-iteration procedure (1.11) in convex metric spaces as follows:

Let \(K\) be a nonempty convex subset of a convex metric space \((X,d,W)\).

For any \(x_0 \in K\),
\[
\begin{align*}
z_n &= W(Tx_n,x_n,\gamma_n) \\
y_n &= W(Tz_n,Tx_n,\beta_n) \\
x_{n+1} &= W(Ty_n,y_n,\alpha_n) \tag{1.13}
\end{align*}
\]

where \(\{\alpha_n\}_{n=0}^{\infty}\), \(\{\beta_n\}_{n=0}^{\infty}\) and \(\{\gamma_n\}_{n=0}^{\infty}\) are in \([0, 1]\).

We call the iteration procedure \(\{x_n\}\) defined in (1.13) is a ‘modified CR-iteration procedure’ in convex metric spaces.

If \(\alpha_n \equiv 1\) then the iteration procedure (1.13) reduces to the following which is an analogue of Picard-S iteration procedure (1.12) in a convex metric space.

For \(x_0 \in K\),
\[
\begin{align*}
z_n &= W(Tx_n,x_n,\gamma_n) \\
y_n &= W(Tz_n,Tx_n,\beta_n) \\
x_{n+1} &= Ty_n \tag{1.14}
\end{align*}
\]

where \(\{\beta_n\}_{n=0}^{\infty}\) and \(\{\gamma_n\}_{n=0}^{\infty}\) are in \([0, 1]\).

We call the iteration \(\{x_n\}\) defined in (1.14) is a ‘modified Picard-S iteration procedure’.

Motivated by the results of Ćirić [4] and Sastry, Babu and Srinivasa Rao [10], in Section 2 of this paper, we prove the strong convergence of modified CR-iteration procedure to a fixed point of a nonlinear quasi contractive map (Theorem 2.2) which is the main result of this paper. The convergence of modified Picard-S iteration procedure (1.14) follows as a corollary to our main result.
2. Main results

Lemma 2.1. Let $(X,d,W)$ be a convex metric space, and let $K$ be a nonempty convex subset of $X$. Let $T : K \rightarrow K$ be a map such that

$$d(Tx, Ty) \leq w(M(x,y))$$

for $x, y \in K$, 

(2.1)

where $M(x,y)$ is defined in (1.3) with $M(x,y) > 0$ and $w : (0,\infty) \rightarrow (0,\infty)$ is a map such that (i) $w$ is increasing on $(0,\infty)$, (ii) $\lim_{t \rightarrow \infty} (t - w(t)) = \infty$, and (iii) $0 < w(t^+) < t$ for each $t > 0$. For $x_0 \in K$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences generated by the modified CR-iteration procedure (1.13). Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{Tx_n\}$, $\{Ty_n\}$ and $\{Tz_n\}$ are bounded.

Proof. For each positive integer $n$, we define the set

$$A_n = \{x_i\}_{i=0}^n \cup \{y_i\}_{i=0}^n \cup \{z_i\}_{i=0}^n \cup \{Tx_i\}_{i=0}^n \cup \{Ty_i\}_{i=0}^n \cup \{Tz_i\}_{i=0}^n.$$

We denote the diameter of $A_n$ by $a_n$. We show that $\{a_n\}_{n=1}^\infty$ is bounded. For this purpose, we define $b_n = \max\{ \sup_{0 \leq i \leq n} d(x_0, Tx_i), \sup_{0 \leq i \leq n} d(x_0, Ty_i), \sup_{0 \leq i \leq n} d(x_0, Tz_i) \}$ for $n = 1, 2, ...$.

We now show that $a_n = b_n$ for $n = 1, 2, ...$.

Clearly, $b_n \leq a_n$ for $n = 1, 2, ...$.

Without loss of generality, we assume that $a_n > 0$ for $n = 1, 2, ...$.

Case (i): $a_n = d(Tx_i, Tx_j)$ for some $0 \leq i, j \leq n$.

Now, $a_n = d(Tx_i, Tx_j) \leq w(M(x_i, x_j)) \leq w(a_n) < a_n$,

a contradiction.

Hence, $a_n \neq d(Tx_i, Tx_j)$ for any $0 \leq i, j \leq n$.

With the similar reason, it is easy to see that $a_n \neq d(Tx_i, Ty_j)$, $a_n \neq d(Tx_i, Tz_j)$, $a_n \neq d(Ty_i, Ty_j)$, $a_n \neq d(Ty_i, Tz_j)$, and $a_n \neq d(Tz_i, Tz_j)$ for any $0 \leq i, j \leq n$.

Case (ii): $a_n = d(y_i, Tx_j)$ for some $0 \leq i, j \leq n$.

$$a_n = d(y_i, Tx_j) = d(w(Tz_i, Tx_i, \beta_i), Tx_j) \leq \beta_i d(Tz_i, Tx_j) + (1 - \beta_i) d(Tx_i, Tx_j)$$

$$\leq \max\{d(Tz_i, Tx_j), d(Tx_i, Tx_j)\} \leq a_n$$

which fails to hold by Case (i).

Therefore, $a_n \neq d(y_i, Tx_j)$ for any $0 \leq i, j \leq n$.

Similarly, it is easy to see that $a_n \neq d(y_i, Ty_j)$ and $a_n \neq d(y_i, Tz_j)$ for any $0 \leq i, j \leq n$.

Case (iii): $a_n = d(y_i, y_j)$ for some $0 \leq i, j \leq n$.

$$a_n = d(y_i, y_j) \leq d(W(Tz_i, Tx_i, \beta_i), y_j) \leq \beta_i d(y_i, Tz_i) + (1 - \beta_i) d(y_j,Tx_i)$$

$$\leq \max\{d(y_j, Tz_i), d(y_j, Tx_i)\} \leq a_n$$

which fails to hold by Case (ii).

Therefore, $a_n \neq d(y_i, y_j)$ for any $0 \leq i, j \leq n$.

Case (iv): $a_n = d(x_i, Tx_j)$ for some $0 \leq i, j \leq n$.

If $i > 0$ then $a_n = d(x_i, Tx_j) = d(W(Ty_{i-1}, y_{i-1}, \alpha_{i-1}), Tx_j)$$

$$\leq \alpha_{i-1} d(Ty_{i-1}, Tx_j) + (1 - \alpha_{i-1}) d(y_{i-1}, Tx_j)$$

$$\leq \max\{d(Ty_{i-1}, Tx_j), d(y_{i-1}, Tx_j)\} \leq a_n$$

which is absurd by Case (i) and Case (ii).

Therefore $i = 0$ and hence $a_n = d(x_0, Tx_j)$ so that $a_n \leq b_n$.

Case (v): Either $a_n = d(x_i, Ty_j)$ or $d(x_i, Tz_j)$ for some $0 \leq i, j \leq n$.

By the similar argument as in Case (iv), $i = 0$ and hence $a_n \leq b_n$.

Case (vi): $a_n = d(x_i, y_j)$ for some $0 \leq i, j \leq n$.

$$a_n = d(x_i, y_j) = d(x_i, W(Tz_j, Tx_j, \beta_j)) \leq \beta_j d(x_i, Tz_j) + (1 - \beta_j) d(x_i, Tx_j)$$

$$\leq \max\{d(x_i, Tz_j), d(x_i, Tx_j)\} \leq a_n$$

which is absurd by Case (i) and Case (ii).

Therefore $a_n = d(x_i, Tz_j)$ or $d(x_i, Tx_j)$. By Case (iv) and Case (v), we have
\[ a_n = d(x_0, Tx_0) \] or \[ d(x_0, Tz_0) \] so that \( a_n \leq b_n \).

\text{Case (vii)} : \( a_n = d(x_i, x_j) \) for some \( 0 \leq i < j \leq n \).

\[ \frac{a_n}{d(x_i, x_j)} = d(x_i, W(Ty_{j-1}, y_{j-1}, \alpha_{j-1})) \leq \alpha_{j-1}d(x_i, Ty_{j-1}) + (1 - \alpha_{j-1})d(x_i, y_{j-1}) \]

\[ \leq \max\{d(x_i, Ty_{j-1}), d(x_i, y_{j-1})\} \leq a_n \]

so that \( a_n = d(x_i, Ty_{j-1}) \) or \( d(x_i, y_{j-1}) \).

Hence, \( a_n \leq b_n \) follows from from \text{Case (v)} and \text{Case (vi)}.

\text{Case (viii)} : \( a_n = d(x_i, z_j) \) for some \( 0 \leq i, j \leq n \).

\[ \frac{a_n}{d(x_i, z_j)} = d(x_i, W(Tx_j, x_j, \gamma_j)) \leq \gamma_jd(x_i, Tx_j) + (1 - \gamma_j)d(x_i, x_j) \]

\[ \leq \max\{d(x_i, Tx_j), d(x_i, x_j)\} \leq a_n \]

so that \( a_n = d(x_i, Tx_j) \) or \( d(x_i, x_j) \).

Hence, \( a_n \leq b_n \) follows from \text{Case (iv)} and \text{Case (vii)}.

\text{Case (ix)} : \( a_n = d(y_i, z_j) \) for some \( 0 \leq i, j \leq n \).

\[ \frac{a_n}{d(y_i, z_j)} = d(y_i, W(Tx_j, x_j, \gamma_j)) \leq \gamma_jd(y_i, Tx_j) + (1 - \gamma_j)d(y_i, x_j) \]

\[ \leq \max\{d(y_i, Tx_j), d(y_i, x_j)\} \leq a_n \]

so that \( a_n = d(y_i, Tx_j) \) or \( d(y_i, x_j) \).

By \text{Case (ii)}, \( a_n \neq d(y_i, Tx_j) \).

Therefore \( a_n = d(y_i, x_j) \) and hence \( a_n \leq b_n \) follows from \text{Case (vi)}.

\text{Case (x)} : \( a_n = d(z_i, Tx_j) \) for some \( 0 \leq i, j \leq n \).

\[ \frac{a_n}{d(z_i, Tx_j)} = d(z_i, W(Tx_i, x_i, \gamma_i), Tx_j) \leq \gamma_jd(z_i, Tx_j) + (1 - \gamma_j)d(z_i, x_j) \]

\[ \leq \max\{d(z_i, Tx_j), d(z_i, x_j)\} \leq a_n \]

so that \( a_n = d(z_i, Tx_j) \) or \( d(z_i, x_j) \). Hence it follows from \text{Case (viii)} and \text{Case (x)} that \( a_n \leq b_n \).

\text{Case (xi)} : Either \( a_n = d(z_i, Ty_j) \) or \( a_n = d(z_i, Tz_j) \).

In this case, clearly \( a_n \leq b_n \).

Hence, by considering all the above cases, it follows that \( a_n \leq b_n \) so that \( a_n = b_n \) for \( n = 1, 2, \ldots \).

Now for any \( 0 \leq i \leq n \),

\[ d(x_0, Tx_i) \leq d(x_0, Tx_0) + d(Tx_0, Tx_i) \]

\[ \leq A + w(M(x_0, x_i)) \]

\[ \leq A + w(a_n), \text{ where } A = d(x_0, Tx_0). \]

Similarly, it is easy to see that

\[ d(x_0, Ty_i) \leq A + w(a_n) \text{ for } 0 \leq i \leq n \] and

\[ d(x_0, Tz_i) \leq A + w(a_n) \text{ for } 0 \leq i \leq n. \]

Therefore \( b_n \leq A + w(a_n) \) so that

\[ a_n - w(a_n) \leq A \quad \text{(2.2)} \]

for \( n = 1, 2, \ldots \), since \( b_n = a_n \).

Since \( \lim_{t \to \infty} (t - w(t)) = \infty \), there exists \( c > 0 \) such that \( t - w(t) > A \) for all \( t > c \).

If \( a_n > c \) for some \( n \geq 1 \) then \( a_n - w(a_n) > A \),

a contradiction.

Thus \( a_n \leq c \) for all \( n \), i.e., the sequence \( \{a_n\}_n \) is bounded.

Hence the conclusion of the lemma follows.

\[ \square \]

\textbf{Theorem 2.2.} Let \( (X, d, W) \) be a complete convex metric space and \( K \) be a nonempty closed convex subset of \( X \). Let \( T : K \to K \) satisfy all the hypotheses of Lemma 2.1. Let \( \{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \) and \( \{\gamma_n\}_{n=0}^{\infty} \) be sequences in \([0, 1]\) such that \( \sum_{n=0}^{\infty} \alpha_n = \infty \). Then the sequence \( \{x_n\} \) generated by the modified CR-iteration procedure (1.13) converges strongly to a unique fixed point of \( T \).
Proof. Without loss of generality, we assume that \( x_n \neq Tx_n \) for any \( n = 0, 1, 2, \ldots \).

For each integer \( n \geq 0 \), we let
\[
C_n = \{x_i\}_{i=n}^{\infty} \cup \{y_i\}_{i=n}^{\infty} \cup \{z_i\}_{i=n}^{\infty} \cup \{Tx_i\}_{i=n}^{\infty} \cup \{Ty_i\}_{i=n}^{\infty} \cup \{Tz_i\}_{i=n}^{\infty}.
\]

By Lemma 2.1, \( C_n \) is bounded. We denote the diameter of \( C_n \) by \( c_n \).

Let \( d_n = \max\{\sup d(x_n, Tx_i), \sup d(x_n, Ty_i), \sup d(x_n, Tz_i)\} \) for \( n = 0, 1, 2, \ldots \).

Then it is easy to see that \( c_n = d_n \) for \( n = 0, 1, 2, \ldots \).

Clearly, the sequence \( \{c_n\} \) is a decreasing sequence of nonnegative real numbers so that \( \lim_{n \to \infty} c_n \) exists, we let it be \( c \).

Now we prove that \( c = 0 \). On the contrary, we assume that \( c > 0 \) so that \( c_n > 0 \) for \( n = 0, 1, 2, \ldots \).

For each positive integer \( n \) and for each \( j \geq n \), we have
\[
d(x_n, Tx_j) = d(Tx_j, W(Ty_{n-1}, y_{n-1}, \alpha_{n-1}))
\leq \alpha_{n-1}d(Tx_j, Ty_{n-1}) + (1 - \alpha_{n-1})d(Tx_j, y_{n-1})
\leq \alpha_{n-1}w(M(x_j, y_{n-1})) + (1 - \alpha_{n-1})d(Tx_j, y_{n-1})
\leq \alpha_{n-1}w(c_n) + (1 - \alpha_{n-1})c_{n-1} 
\text{ so that}
\]
\[
\sup_{j \geq n} d(x_n, Tx_j) \leq \alpha_{n-1}w(c_n) + (1 - \alpha_{n-1})c_{n-1}.
\]

Similarly, \( \sup_{j \geq n} d(x_n, Ty_j) \leq \alpha_{n-1}w(c_n) + (1 - \alpha_{n-1})c_{n-1} \) and

\[
\sup_{j \geq n} d(x_n, Tz_j) \leq \alpha_{n-1}w(c_n) + (1 - \alpha_{n-1})c_{n-1} \text{ hold.}
\]

Therefore
\[
d_n \leq \alpha_{n-1}w(c_n) + (1 - \alpha_{n-1})c_{n-1} \quad \text{for} \quad n = 1, 2, \ldots.
\]

Since \( c_n = d_n \), we have
\[
\alpha_{n-1}(c_n - w(c_n)) \leq c_{n-1} - c_n \quad \text{for} \quad n = 1, 2, \ldots.
\]  \( \text{(2.3)} \)

Let \( s = \inf\{c_n - w(c_n) : n \geq 0\}. \) If \( s = 0 \) then there exists a subsequence \( \{c_{n(k)}\} \) of the sequence \( \{c_n\} \) such that \( \lim_{k \to \infty} (c_{n(k)} - w(c_{n(k)})) = 0 \), i.e., \( c - w(c^+) = 0 \),

a contradiction, from (iii) of Lemma 2.1.

Therefore \( s > 0 \) so that there exists a real number \( \eta > 0 \) such that \( c_n - w(c_n) \geq \eta \) for \( n = 0, 1, 2, \ldots \).

It follows from the inequality (2.3) that \( \eta \alpha_{n-1} \leq c_{n-1} - c_n \) for \( n = 1, 2, \ldots \).

Since the sequence \( \{c_n\} \) is convergent, we have the series \( \sum \alpha_n < \infty \),

a contradiction.

Therefore \( c = 0 \) so that the sequence \( \{x_n\} \) is Cauchy and hence there exists \( x \in K \) such that \( \lim_{n \to \infty} x_n = x \).

Since \( c = 0 \), we have \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \) so that \( \lim_{n \to \infty} Tx_n = x \).

Now, we prove that \( x \) is a fixed point of \( T \).

Since \( T \) satisfies the inequality (2.1), we have
\[
d(Tx_n, Tx) \leq w(M(x_n, x)) \quad \text{for} \quad n = 0, 1, 2, \ldots
\]

(2.4)

Since \( M(x_n, x) \geq d(x, Tx) \) for \( n = 0, 1, 2, \ldots \) and \( \lim_{n \to \infty} M(x_n, x) = d(x, Tx) \), we have

\[
\lim_{n \to \infty} w(M(x_n, x)) = w(d(x, Tx)^+) \quad \text{so that} \quad d(x, Tx) \leq w(d(x, Tx)^+).
\]

Hence \( x \) is a fixed point of \( T \) by using (iii) of Lemma 2.1.

Now from the inequality (2.1) and Remark 1.6, clearly the uniqueness of fixed point of \( T \) follows. \( \square \)

If \( \alpha_n \equiv 1 \) in the modified CR-iteration procedure (1.13) then we have the following corollary from Theorem 2.2.

**Corollary 2.3.** Let \( X, K, T \) be as in Theorem 2.2. Let \( \{\beta_n\}_{n=0}^{\infty} \) and \( \{\gamma_n\}_{n=0}^{\infty} \) be sequences in \( [0, 1] \). For \( x_0 \in K \), let the sequence \( \{x_n\}_{n=0}^{\infty} \) be generated by the modified Picard-S iteration procedure (1.14). Then \( \{x_n\}_{n=0}^{\infty} \) converges to a unique fixed point of \( T \).
In the following, we prove that $CR$-iteration procedure (1.11) and Picard-S iteration procedure (1.12) converge to a unique fixed point of a quasi-contraction map under certain hypotheses in the setting of Banach spaces.

**Corollary 2.4.** Let $X$ be a Banach space, $K$ be a nonempty closed convex subset of $X$, and $T : K \to K$ be a quasi-contraction map. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. For $x_0 \in K$, let $\{x_n\}$ be the sequence generated by either $CR$-iteration procedure (1.11) or by Picard-S iteration procedure (1.12). Then $\{x_n\}$ converges strongly to a unique fixed point of $T$.

**Proof.** Follows from Remark 1.7, Theorem 2.2 and Corollary 2.3. \hfill \square

**References**


1, 1.2