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Fixed points of involution mappings in convex uniform spaces

Joy C. Umudu[a,*], Johnson O. Olaleru[b], Adesanmi A. Mogbademu[b]

[a] *Department of Mathematics, Faculty of Natural Sciences, University of Jos*
[b] *Department of Mathematics, Faculty of Science, University of Lagos*

Abstract

In this paper, we study some fixed point theorems for self-mappings satisfying certain contraction principles on a S-complete convex Hausdorff uniform space, these theorems generalize previously obtained results in convex metric space and convex partial metric space.

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1. Introduction

In 1970, Takahashi [16] introduced the notion of convexity in metric spaces and studied some fixed point theorems for nonexpansive mappings in such spaces. A convex metric space is a generalization of some spaces. For example, every normed space and Banach space is a convex metric space and complete convex metric space respectively. Subsequently, Beg [2], Beg and Abbas [3, 4], Chang, Kim and Jin [8], Ciric [9], Shimizu and Takahashi [14], Tian [15], Ding [10], Moosaei [12] and others studied fixed point theorems in convex metric spaces.
One of the abstract spaces in literature that generalises the metric and pseudometric spaces is the uniform space. Weil [18] was the first to characterise uniform spaces in terms of a family of pseudometrics and Bourbaki [7] provided the definition of a uniform structure in terms of entourages. Aamri and El Moutawakil [1] gave some results on common fixed point for some contractive and expansive maps in uniform spaces and provided the definition of A-distance and E-distance. Olisama et al. [13] introduced the concept of J_{AV}-distance (an analogue of b-metric), in Hausdorff uniform spaces and investigated the existence and uniqueness of best proximity points for these modified contractive mappings.
The purpose of this paper is to study the existence of a fixed point for a self-mapping defined on a nonempty

*Joy C. Umudu
Email addresses: umuduj@unijos.edu.ng (Joy C. Umudu), jolaleru@unilag.edu.ng (Johnson O. Olaleru), amogbademu@unilag.edu.ng (Adesanmi A. Mogbademu)

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closed convex subset of a S-complete convex Hausdorff uniform space that satisfies certain conditions. Since the uniform space is a generalization of metric space, our results improve and extend some of Beg's results in [2], Beg and Olatinwo's results in [5] from a complete convex metric space to an S-complete convex Hausdorff uniform space.

2. Preliminaries

Definition 2.1. [7] A uniform space (X, Γ) is a nonempty set X equipped with a uniform structure which is a family Γ of subsets of Cartesian product $X \times X$ which satisfy the following conditions:

(i) If $U \in \Gamma$, then U contains the diagonal $\Delta = \{(x, x) : x \in X\}$.
(ii) If $U \in \Gamma$, then $U^{-1} = \{(y, x) : (x, y) \in U\}$ is also in Γ.
(iii) If $U, V \in \Gamma$, then $U \cap V \in \Gamma$.
(iv) If $U \in \Gamma$, and $V \subseteq X \times X$, which contains U, then $V \in \Gamma$.
(v) If $U \in \Gamma$, then there exists $V \in \Gamma$ such that whenever (x, y) and (y, z) are in V, then (x, z) is in U.

Γ is called the uniform structure or uniformity of U and its elements are called entourages, neighbourhoods, surroundings, or vicinities.

Definition 2.2. [1] Let (X, Γ) be a uniform space. A function $p : X \times X \to \Re^+$ is said to be an

(a) A-distance if, for any $V \in \Gamma$, there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$;
(b) E-distance if p is an A-distance and $p(x, z) \leq p(x, y) + p(y, z)$, $\forall x, y, z \in X$.

Definition 2.3. [13] Let (X, Γ) be a uniform space. A function $p : X \times X \to \Re^+$ is said to be a J_{AV} - distance if

(a) p is an A - distance,
(b) $p(x, y) \leq s[p(x, z) + p(z, y)], \forall x, y, z \in X,\ s \geq 1$.

Note that the function p reduces to an E-distance if the constant s is taken to be 1.

Definition 2.4. [7] Let (X, Γ) be a uniform space and p an A-distance on X.

(a) If $V \in \Gamma, (x, y) \in V$, and $(y, x) \in V$, x and y are said to be V-close. A sequence (x_n) is a Cauchy sequence for Γ if, for any $V \in \Gamma$, there exists $N \geq 1$ such that x_n and x_m are V-close for $n, m \geq N$. The sequence $(x_n) \in X$ is a p-Cauchy sequence if for every $\epsilon > 0$ there exists $n_0 \in N$ such that $p(x_n, x_m) < \epsilon$ for all $n, m \geq N$.
(b) X is S-complete if for any p-Cauchy sequence $\{x_n\}$, there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$.
(c) $f : X \to X$ is p-continuous if $\lim_{n\to\infty} p(x_n, x) = 0$ implies $\lim_{n\to\infty} p(f(x_n), f(x)) = 0$.
(d) X is said to be p-bounded if $\delta_p(X) = \sup\{p(x, y) : x, y \in X\} < \infty$.

To guarantee the uniqueness of the limit of the Cauchy sequence for Γ, the uniform space (X, Γ) needs to be Hausdorff.

Definition 2.5. [7] A uniform space (X, Γ) is said to be Hausdorff if and only if the intersection of all the $V \in \Gamma$ reduces to the diagonal Δ of X, $\Delta = \{(x, x), x \in X\}$. In other words, $(x, y) \in V$ for all $V \in \Gamma$ implies $x = y$.

The following Lemma will be used efficiently in the sequel.

Lemma 2.6. *[17] Let (X, Γ) be a Hausdorff uniform space and p be an A-distance on X. Let $\{x_n\}_{n=\infty}^{\infty}$, $\{y_n\}_{n=\infty}^{\infty}$ be arbitrary sequences in X and $\{\alpha_n\}_{n=\infty}^{\infty}, \{\beta_n\}_{n=\infty}^{\infty}$ be sequences in $\Re^+$ converging to 0. Then, for $x, y, z \in X$ the following holds:*

(a) *If* $p(x_n, y) \leq \alpha_n$ *and* $p(x_n, z) \leq \beta_n$ $\forall n \in N$, *then* $y = z$. *In particular, if* $p(x, y) = 0$ *and* $p(x, z) = 0$, *then* $y = z$;

(b) *If* $p(x_n, y_n) = p(A, B)$ *and* $p(x_n, z_n) = p(A, B)$, *then* $y_n = z_n$;

(c) *If* $p(x_n, y_n) \leq \alpha_n$ *and* $p(x_n, z) \leq \beta_n$ $\forall n \in N$, *then* $\{y_n\}_{n=0}^{\infty}$ *converges to* z;

(d) *If* $p(x_n, x_m) \leq \alpha_n$ $\forall m > n$, *then* $\{x_n\}_{n=0}^{\infty}$ *is a* p*-Cauchy sequence in* (X, Γ).

We now define convexity in uniform spaces.

Definition 2.7. Let (X, Γ) be a uniform space such that p is an E-distance on X and $I = [0, 1]$. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$p(u, W(x, y, \lambda)) \leq \lambda p(u, x) + (1 - \lambda)p(u, y). \tag{2.1}$$

A uniform space (X, Γ) together with a convex structure W is called a convex uniform space, which is denoted by (X, Γ, W).

Definition 2.8. Let (X, Γ, W) be a convex uniform space. A nonempty subset E of (X, Γ, W) is said to be convex if $(W, x, y, \lambda) \in E$ whenever $(x, y, \lambda) \in E \times E \times [0, 1]$. Clearly, we have from (2.1) that $W(x, x, \lambda) = x$.

We give an example of a convex uniform space

Example 2.9. Consider $X = [0, \infty)$ and define for all $x, y \in X$,

$$p(x, y) = \begin{cases} x - y, & \text{if} \quad x \geq y, \\ 1 & \text{if} \quad otherwise. \end{cases}$$

Then X is a convex Hausdorff uniform space such that p is an E-distance on X. We define analogue of k-Lipschitzian [2], (k, L)-Lipschitzian [15] and involution mappings [11] in a uniform space.

Definition 2.10. Let E be a nonempty subset of a convex uniform space such that p is an E-distance on X. A mapping $T : E \to E$ is said to be k-Lipschitzian if there exists a $k \in [0, \infty)$ such that

$$p(Tx, Ty) \leq kp(x, y), \forall x, y \in E. \tag{2.2}$$

Definition 2.11. Let (X, Γ, W) be an S-complete convex Hausdorff uniform space such that p is an E-distance on X and E a nonempty closed convex subset of X. A mapping $T : E \to E$ is said to be (k, L)-Lipschitzian if there exists a $k \in [1, \infty)$, $L \in [0, 1)$ such that

$$p(Tx, Ty) \leq Lp(x, Tx) + kp(x, y), \forall x, y \in E. \tag{2.3}$$

Definition 2.12. Let (X, Γ, W) be an S-complete convex Hausdorff uniform space and E a nonempty convex subset of X. A mapping $T : E \to E$ is said to be an involution mapping if $T^2(x) = x$.

Definition 2.13. [6] A function $\phi : \Re^+ \to \Re^+$ is called a comparison function if:

(i) ϕ is monotone increasing; and

(ii) $\lim\limits_{n\to\infty} \phi^n(t) = 0, \forall t \in \Re^+$.

Several iterative processes for approximating fixed points of various mappings are available in the literature. The equivalence of the convergence of those iterations for the quasi-contraction mappings in convex metric spaces was proved in [19]. In this paper, we intend to prove our result by using the Krasnoselskii iteration method to state the iteration process in the context of convex Hausdorff uniform space.

3. Main Result

An analogous definition in [2] is given in S-complete convex Hausdorff uniform space.
Let E be a nonempty closed convex subset of an S-complete convex Hausdorff uniform space X and $T: E \to E$. For $x_0 \in E$, we define,

$$x_{n+1} = W(x_n, Tx_n, \frac{1}{2}), \quad n \geq 0.$$

If there exists a real number $c \in [0, 1)$ such that

$$p(x_{n+2}, x_{n+1}) \leq cp(x_{n+1}, x_n), \quad n = 0, 1, 2, ...$$

Then $\{x_n\}$ converges to a point $x^* \in E$.

Similarily, if there exists a comparison function $\phi : \Re^+ \to \Re^+$ such that

$$p(x_{n+2}, x_{n+1}) \leq \phi(p(x_{n+1}, x_n))$$

then $\{x_n\}$ converges to a point $x^* \in E$.

Theorem 3.1. *Let X be an S-complete convex Hausdorff uniform space such that p is an E-distance on X. Let E be a closed convex subset of X and $T : E \to E$ be a k-Lipschitzian map. Suppose there exists real constants a, b such that $0 \leq a < 1$ and $b > 0$. If for arbitrary $x \in E$ there exists $u \in E$ such that*

(i) $p(Tu, u) \leq ap(Tx, x)$, *and*
(ii) $p(u, x) \leq bp(Tx, x)$.

Then T has a fixed point $x^* \in E$.

Proof. Let $x_0 \in E$ be an arbitrary point. Consider a sequence $\{x_n\} \subset E$ which satisfies the following conditions

$$p(Tx_{n+1}, x_{n+1}) \leq ap(Tx_n, x_n), \tag{3.1}$$

and

$$p(x_{n+1}, x_n) \leq bp(Tx_n, x_n), n = 0, 1, 2, ... \tag{3.2}$$

We get by induction in (3.1) that,

$$\begin{aligned} p(Tx_{n+1}, x_{n+1}) &\leq ap(Tx_n, x_n) \leq a^2 p(Tx_{n-1}, x_{n-1}) \\ &\leq ... \leq a^{n+1} p(Tx_0, x_0). \end{aligned} \tag{3.3}$$

Using (3.3) in (3.2) gives

$$p(x_{n+1}, x_n) \leq ba^n p(Tx_0, x_0) \to 0 \quad as \quad n \to \infty. \tag{3.4}$$

Thus $\{x_n\}$ is a Cauchy sequence in E. Since E is S-complete, there exists $x^* \in E$ such that $\lim_{n\to\infty} x_n = x^*$.
By (2.2), (3.3) and triangle inequality, we have

$$\begin{aligned} p(Tx^*, x^*) &\leq p(Tx^*, Tx_n) + p(Tx_n, x_n) + p(x_n, x^*) \\ &\leq kp(x^*, x_n) + p(Tx_n, x_n) + p(x_n, x^*) \\ &= (1 + k)p(x_n, x^*) + a^n p(Tx_0, x_0) \quad \to 0 \quad as \quad n \to \infty. \end{aligned}$$

Hence $Tx^* = x^*$ and x^* is a fixed point of T.

Theorem 3.2. *Let X be an S-complete convex Hausdorff uniform space such that p is an E-distance on X. Let E be a closed convex subset of X and $T : E \to E$ be a k-Lipschitzian involution. If $1 \leq k < 2$ then T has a fixed point $x^* \in E$.*

Proof. For any $x \in E$, let $u = W(x, Tx, \frac{1}{2})$. Then,

$$p(u, x) = p(W(x, Tx, \frac{1}{2})x) \leq \frac{1}{2}p(Tx, x), \tag{3.5}$$

and

$$\begin{aligned}
p(u, Tu) &= p(W(x, Tx, \frac{1}{2}), Tu)\\
&\leq \frac{1}{2}\left[p(x, Tu) + p(Tx, Tu)\right]\\
&= \frac{1}{2}\left[p(T^2x, Tu) + p(Tx, Tu)\right]\\
&\leq \frac{k}{2}\left[p(Tx, u) + p(x, u)\right]\\
&= \frac{k}{2}\left[p(Tx, W(x, Tx, \frac{1}{2})) + p(x, W(x, Tx, \frac{1}{2}))\right]\\
&\leq \frac{k}{2}p(Tx, x),
\end{aligned} \tag{3.6}$$

where $\frac{k}{2} < 1$. For $x_0 \in E$, we define a sequence $\{x_n\} \subset E$ by

$$x_{n+1} = W(x_n, Tx_n, \frac{1}{2}), \quad n = 0, 1, 2, ...$$

By induction using (3.6) and as in Theorem 3.1 above, we have

$$\begin{aligned}
p(Tx_{n+1}, x_{n+1}) &\leq \frac{k}{2}\left(p(Tx_n, x_n)\right) \leq \left(\frac{k}{2}\right)^2 p(Tx_{n-1}, x_{n-1})\\
&\leq ... \leq \left(\frac{k}{2}\right)^n p(Tx_0, x_0).
\end{aligned} \tag{3.7}$$

Using (3.7) in (3.5) gives

$$p(x_{n+1}, x_n) \leq \left(\frac{1}{2}\right)\left(\frac{k}{2}\right)^n p(Tx_0, x_0) \to 0 \quad as \quad n \to \infty.$$

Therefore, the sequence $\{x_n\}$ is Cauchy in E. Since E is S-complete, there exists $x^* \in E$ such that $\lim_{n\to\infty} x_n = x^*$. By (2.2), (3.7) and triangle inequality we have,

$$\begin{aligned}
p(Tx^*, x^*) &\leq p(Tx^*, Tx_n) + p(Tx_n, x_n) + p(x_n, x^*)\\
&\leq kp(x^*, x_n) + p(Tx_n, x_n) + p(x_n, x^*)\\
&= (1 + k)p(x_n, x^*) + \left(\frac{k}{2}\right)^n p(Tx_0, x_0) \quad \to 0 \quad as \quad n \to \infty.
\end{aligned}$$

Hence $Tx^* = x^*$ and x^* is a fixed point of T.

Theorem 3.3. *Let X be an S-complete convex Hausdorff uniform space such that p is an E-distance on X. Let E be a closed convex subset of X, $T : E \to E$ be a (k, L)-Lipschitzian map and $\phi : \Re^+ \to \Re^+$ be a comparison function such that for arbitrary $x \in E$ there exist $u \in E$ such that;*

(i) $p(Tu, u) \leq \phi(p(Tx, x))$,
(ii) $p(u, x) \leq bp(Tx, x)$, $b > 0$.

Then T has a fixed point in E.

Proof. Let $x_0 \in E$ be an arbitrary point. Consider a sequence $\{x_n\} \subset E$ which satisfies conditions (i) and (ii), we have,

$$p(Tx_{n+1}, x_{n+1}) \leq \phi(p(Tx_n, x_n)), \tag{3.8}$$

and

$$p(x_{n+1}, x_n) \leq bp(Tx_n, x_n), n = 0, 1, 2, ... \tag{3.9}$$

We get by induction in (3.7) that,

$$\begin{aligned} p(Tx_{n+1}, x_{n+1}) &\leq \phi(p(Tx_n, x_n)) \leq \phi^2(p(Tx_{n-1}, x_{n-1})) \\ &\leq ... \leq \phi^{n+1}(p(Tx_0, x_0)). \end{aligned} \tag{3.10}$$

Using (3.10) in (3.9) gives

$$p(x_{n+1}, x_n) \leq b\phi^n(p(Tx_0, x_0)) \to 0 \quad as \quad n \to \infty. \tag{3.11}$$

Thus $\{x_n\}$ is a Cauchy sequence in E. Since E is S-complete, there exists $x^* \in E$ such that $\lim_{n\to\infty} x_n = x^*$. By (2.3), (3.10) and triangle inequality, we have

$$\begin{aligned} p(Tx^*, x^*) &\leq p(Tx^*, Tx_n) + p(Tx_n, x_n) + p(x_n, x^*) \\ &\leq Lp(Tx_n, x_n) + kp(x_n, x^*) + p(Tx_n, x_n) + p(x_n, x^*) \\ &= (1 + L)p(Tx_n, x_n) + (1 + k)p(x_n, x^*) \\ &\leq (1 + L)\phi^n(p(Tx_0, x_0)) + (1 + k)p(x_n, x^*) \quad \to 0 \quad as \quad n \to \infty. \end{aligned}$$

Hence $Tx^* = x^*$ and x^* is a fixed point of T.

Theorem 3.4. *Let X be an S-complete convex Hausdorff uniform space such that p is an E-distance on X. Let E be a closed convex subset of X and $T : E \to E$ be a (k, L)-Lipschitzian involution. If $1 \leq k < 2$, then T has a fixed point in E.*

Proof. For any $x \in E$, let $u = W(x, Tx, \frac{1}{2})$, then

$$p(u, x) = p(W(x, Tx, \frac{1}{2})x) \leq \frac{1}{2}p(Tx, x), \tag{3.12}$$

and

$$\begin{aligned} p(u, Tu) &= p(W(x, Tx, \frac{1}{2}), Tu) \\ &\leq \frac{1}{2}\left[p(x, Tu) + p(Tx, Tu)\right] \\ &= \frac{1}{2}\left[p(T^2x, Tu) + p(Tx, Tu)\right] \\ &\leq \frac{1}{2}\left[Lp(Tx, Tx^2) + kp(Tx, u) + Lp(x, Tx) + kp(x, u)\right] \\ &\leq \frac{1}{2}\left[L\left(Lp(x, Tx) + kp(x, u)\right) + kp(Tx, u) + Lp(x, Tx) + kp(x, u)\right] \\ &= \frac{1}{2}\left(L^2 + L\right)p(Tx, x) + (kL + k)\,p(x, u) + kp(Tx, u) \\ &\leq \frac{1}{4}\left[2(L^2 + L) + (k(L + 2)\right]p(Tx, x) \end{aligned}$$

$$= \phi\left(p(Tx,x)\right), \tag{3.13}$$

where $\phi = \frac{1}{4}\left[2(L^2+L)+(k(L+2)\right]$ and $1 \leq k < 2$. For arbitrary $x_0 \in E$, we define a sequence $\{x_n\} \subset E$ by

$$x_{n+1} = W(x_n, Tx_n, \frac{1}{2}), \quad n = 0,1,2,...$$

By induction using (3.13) and as in Theorem 3.2, we have

$$\begin{aligned} p(Tx_{n+1}, x_{n+1}) &\leq \beta p(Tx_n, x_n) \leq \beta^2 p(Tx_{n-1}, x_{n-1}) \\ &\leq ... \leq \phi^{n+1} p(Tx_0, x_0). \end{aligned} \tag{3.14}$$

Substitute (3.14) in (3.12) to get

$$p(x_{n+1}, x_n) \leq b\phi^n p(Tx_0, x_0) \to 0 \quad as \quad n \to \infty.$$

Therefore, the sequence $\{x_n\}$ is Cauchy in E. Since E is S-complete, there exists $x^* \in E$ such that $\lim_{n\to\infty} x_n = x^*$. By (2.3), (3.14) and triangle inequality we have

$$\begin{aligned} p(Tx^*, x^*) &\leq p(Tx^*, Tx_n) + p(Tx_n, x_n) + p(x_n, x^*) \\ &\leq (1+L)p(Tx_n, x_n) + (1+k)p(x_n, x^*) \\ &\leq (1+L)\phi^n p(Tx_0, x_0) + (1+k)p(x_n, x^*) \quad \to 0 \quad as \quad n \to \infty. \end{aligned}$$

Hence $Tx^* = x^*$ and x^* is a fixed point of T.

Example 3.5. Let $X = [0, \infty)$ be a uniform space such that p is an E distance on X. Let p be defined by

$$p(x,y) = \begin{cases} x - y, & x \geq y, \\ y, & otherwise. \end{cases}$$

Consider the mapping $T : X \to X$ defined by $T(x) = \frac{1}{x}$ for all $x \geq 1$. We note that X is an S-complete convex Hausdorff uniform space such that p is an E-distance on X. T is a (k, L)-Lipschitzian map and if we define the function $\phi : \Re^+ \to \Re^+$ by $\phi(t) = \frac{t}{2}$. All conditions of Theorem 3.3 and Theorem 3.4 are satisfied and $x = 1$ is a fixed point in X.

Remark 3.6. Note that in the example given, X is a uniform space but not a metric space. Thus, these results are generalizations of Beg ([2], Theorem 3.1), Beg and Olatinwo ([5], Theorem 2.1 and 2.3). We are also able to improve Theorem 2.3 in [5] by giving less restrictions.

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