Available online at http://www.cna-journal.com
Commun. Nonlinear Anal. 7(1) (2019), 17–35
Research Article

Communications in Nonlinear Analysis

Journal Homepage: www.cna-journal.com

Common Fixed Points of (α, ψ, φ) - Almost Generalized Weakly Contractive Maps in S-metric spaces

Gutti Venkata Ravindranadh Babu[a], Pericherla Durga Sailaja[b,*], Gadhavajjala Srichandana[c]

[a]*Department of Mathematics, Andhra University, Visakhapatnam-530 003, INDIA.*
[b]*Department of Mathematics, Lendi Institute of Engineering and Technology, Vizianagaram-535 005, INDIA.*
[c]*Department of Mathematics, Andhra University, Visakhapatnam-530 003, INDIA.*
Permanent address: Department of Mathematics, Satya Institute of Technology and Management, Vizianagaram-535 002, INDIA.

Abstract

In this paper, we introduce a pair of (α, ψ, φ) -almost generalized weakly contractive maps in S-metric spaces and prove the existence and uniqueness of common fixed points of such maps under weakly compatible property. Our results extend and generalize the results of Babu and Leta [3] to a pair of maps in S-metric spaces and generalize the result of Sedghi, Shobe and Aliouche [17]. We provide examples in support of our results.

Keywords: S-metric space, S-weakly compatible, (α, ψ, φ) -almost generalized weakly contractive maps, property (E. A.).
2010 MSC: AMS(2010) Mathematics Subject Classification: 47H10, 54H25

1. Introduction

In 2006, Mustafa and Sims [14] introduced G-metric spaces as a generalization of S-metric spaces and proved the existence of fixed points of various contraction type mappings. In 2012, Sedghi, Shobe and Aliouche [17] introduced a new concept on metric spaces, namely S-metric spaces and studied some properties of these spaces. Sedghi, Shobe and Aliouche [17] asserted that S-metric is a generalization of G-metric space. But, very recently Dung, Hieu and Radojevic [8] verified by examples that S-metric is not a generalization of G-metric and vice versa. Therefore the class of G-metric spaces and the class of S-metric spaces are different. There has been a considerable interest to study common fixed points for a pair (family) of mappings satisfying some contractive conditions in metric spaces. Jungck [10] introduced commuting mappings in 1976. Jungck

*Corresponding author

Email addresses: gvr_babu@hotmail.com (Gutti Venkata Ravindranadh Babu), sailajadurga@yahoo.com (Pericherla Durga Sailaja), sri.chandan3@gmail.com (Gadhavajjala Srichandana)

Received 2019-08-17

[11] enlarged the class of non-commuting mappings by compatible mappings in 1986. In 1998 Jungck and Rhodes [12] introduced weak compatibility. In 2002, Aamri , Moutawakil [1] introduced a new notation namely property (E.A.). It is observed that property (E.A.) requires the completeness or closedness of subspaces for the existence of common fixed point of pair of maps. In 2004, Berinde[4] introduced 'weak contractions' as a generalization of contraction maps. Berinde [5] renamed 'weak contractions' as 'almost contractions' in his later work. In 2008, Dutta and Choudhury[9] introduced (ψ, φ)- weakly contractive maps. Later in 2014, (ψ, φ)-almost weakly contractive maps in G-metric spaces were introduced by Babu and Ratna Babu[2]. Fixed points of contractive maps on S-metric spaces were studied by [3], [8], [15], [17], [18] and [19, 20]. (α, ψ, φ)-generalized weakly contractive maps in S-metric spaces were introduced by Babu and Leta [3] in 2017.
In the following we provide some basic definitions and preliminaries which we use in this paper.

2. Preliminaries

Definition 2.1. [17] Let X be a non empty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions: for each $x, y, z, a \in X$

(S1) $S(x, y, z) \geq 0$,

(S2) $S(x, y, z) = 0$ if and only if $x = y = z$ and

(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

Example 2.2. [17] Let(X, d) be a metric space. Define $S : X^3 \to [0, \infty)$ by $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y, z, \in X$. Then S is an S-metric on X and S is called the S-metric induced by the metric d.

Example 2.3. [8] Let $X = \mathbb{R}$, the set of all real numbers and let $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in X$. Then (X, S) is an S-metric space.

Example 2.4. [18] Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in R$ is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric.

Example 2.5. Let$X = [0, 1]$ and We define $S : X^3 \to [0, \infty)$ by

$$S(x, y, z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x, y, z\} & \text{otherwise} \end{cases}$$

Then S is an S-metric on X.

The following lemmas are useful in our main results.

Lemma 2.6. [17] In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.

Lemma 2.7. [8] Let (X, S) be an S-metric space. Then $S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$ and $S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$ for all $x, y, z \in X$.

Definition 2.8. [17] Let (X, S) be an S-metric space. We define the following:

(i) a sequence $\{x_n\} \in X$ converges to a point $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in N$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \epsilon$ and we denote it by $\lim_{n\to\infty} x_n = x$.

(ii) a sequence $\{x_n\} \in X$ is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in N$ such that $S(x_n, x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.

(iii) the S-metric space (X, S) is said to be complete if each Cauchy sequence in X is convergent.

Definition 2.9. [3] Let (X, S) and (Y, S') be two S-metric spaces. Then the function $f : X \to Y$ is S-continuous at a point $x \in X$ if it is S-sequentially continuous at x, that is whenever $\{x_n\}$ is S-convergent to x, we have $f(x_n)$ is S' convergent to $f(x)$.

Lemma 2.10. [17] Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then x is unique.

Lemma 2.11. [17] Let (X, S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$, then $\lim\limits_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Definition 2.12. Let (X, S) be an S-metric space and f, T be two self maps on X. A point $x \in X$ is called a common fixed point of f and T if $x = fx = Tx$.
The pair (f, T) is said to be

(i) commuting on X if $fTx = Tfx$, for all $x \in X$.

(ii) S-weakly commuting on X if $S(fTx, fTx, Tfx) \leq S(fx, fx, Tx)$ for every $x \in X$.

(iii) S-compatible if $\lim\limits_{n\to\infty} S(fTx_n, fTx_n, Tfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} Tx_n = t$ for some $t \in X$.

(iv) S-weakly compatible if they commute at their coincidence point. That is, for $x \in X$, if $fx = Tx$ holds then $fTx = Tfx$.

(v) satisfy property (E. A.), if there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} Tx_n = z$ for some $z \in X$.

Remark 2.13. (1)Every weakly commuting pair of maps in an S-metric space X is S-weakly commuting, but its converse need not be true.

(2) Every S-weakly commuting pair of maps is S-compatible, but its converse need not be true.

(3) Every S-compatible pair of maps is S-weakly compatible, but its converse need not be true.

(4) Property (E. A.) and S-weakly compatible pair of maps are independent to each other.

The following examples illustrate the above Remark.

Example 2.14. Let $X = [0, 2]$. We define $S : X^3 \to [0, \infty)$ by $S(x, y, z)$=max$\{\ |x - z|, |y - z|\}$ for all $x, y, z \in X$. Then S is an S-metric on X. We define $f, T : X \to X$ by $f(x) = \frac{x^2}{4}$ and $T(x) = \frac{x^2}{2}$. Now, $S(fTx, fTx, Tfx) = S(\frac{x^4}{16}, \frac{x^4}{16}, \frac{x^4}{32}) = \frac{x^4}{32} \leq \frac{x^2}{4} = S(\frac{x^2}{4}, \frac{x^2}{4}, \frac{x^2}{2}) = S(fx, fx, Tx)$.
Therefore the pair (f, T) is S-weakly commuting, but not commuting.
For, $fTx = \frac{x^4}{16}$ and $Tfx = \frac{x^4}{32}$. Therefore $fTx \neq Tfx$ for every $x \neq 0$.

Example 2.15. Let $X = \mathbb{R}$. We define $S : X^3 \to [0, \infty]$ by $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in X$. Then S is an S-metric on X. We define $f, g : X \to X$ by $f(x) = x^3$ and $g(x) = 2x^3$. Let $\{x_n\}$ be a sequence in X such that $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} g(x_n)$. Then $\lim\limits_{n\to\infty} x_n = 0$.
Now consider $\lim\limits_{n\to\infty} S(fgx_n, fgx_n, gfx_n) = \lim_{n\to\infty} S(8x_n^9, 8x_n^9, 2x_n^9) = 0$.
Therefore the pair (f, g) is S-compatible.
Now conider $S(fgx, fgx, gfx) = S(8x^9, 8x^9, 2x^9) = 12|x^9| \neq 2|x^3| = S(fx, fx, gx)$ for all $x \geq 1$.
Therefore the pair (f, g) is not S-weakly commuting.

Example 2.16. Let $X = [1,3]$. We define $S : X^3 \to [0,\infty]$ by $S(x,y,z)=\max\{ |x-z|, |y-z|\}$ for all $x,y,z \in X$. Then S is an S-metric on X. We define $f,T : X \to X$ by

$$fx = \begin{cases} 1 & \text{if } x = 1 \\ 2 & \text{if } x \in (1,2) \\ 1 & \text{if } x \in [2,3] \end{cases} \qquad \text{and} \qquad Tx = \begin{cases} 1 & \text{if } x = 1 \\ 1 & \text{if } x \in (1,2) \\ x-1 & \text{if } x \in [2,3] \end{cases}$$

We have $fx = Tx$ for $x = 1,2$. Implies $fTx = Tfx$ for $x = 1,2$.
Therefore the pair (f,T) is S-weakly compatible.
Now let $x_n = 2 + \frac{1}{n}, n \geq 1$. We have $\lim\limits_{n\to\infty} fx_n = 1$ and $\lim\limits_{n\to\infty} Tx_n = \lim\limits_{n\to\infty} T(2+\frac{1}{n}) = 1$.
Then $\lim\limits_{n\to\infty} S(fTx_n, fTx_n, Tfx_n) = \lim\limits_{n\to\infty} S(2,2,1) = 1 \neq 0$. Therefore the pair (f,T) is not S-compatible.

Example 2.17. Let $X = \mathbb{R}^+$. We define $S : X^3 \to [0,\infty]$ by $S(x,y,z)=\max\{ |x-z|, |y-z|\}$ for all $x,y,z \in X$. Then S is an S-metric on X. We define $f,T : X \to X$ by $f(x) = x$ and $T(x) = 1 + x$. Trivially, the pair (f,T) is S-weakly compatible.
Suppose that there is a sequence $\{x_n\} \subset X$ such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} Tx_n = z$ (say).
i.e., $\lim\limits_{n\to\infty} S(fx_n, fx_n, z) = \lim\limits_{n\to\infty} S(Tx_n, Tx_n, z) = 0$. i.e., $\lim\limits_{n\to\infty} 2|x_n - z| = \lim\limits_{n\to\infty} 2|1 + x_n - z| = 0$, so that $\lim\limits_{n\to\infty} x_n = z$ and $\lim\limits_{n\to\infty} x_n = z - 1$, which is absurd. Therefore property (E. A.) fails to hold.

Example 2.18. Let $X = [0,1]$. We define $S : X^3 \to [0,\infty]$ by $S(x,y,z)=\max\{ |x-z|, |y-z|\}$ for all $x,y,z \in X$. Then S is an S-metric on X. We define $f,T : X \to X$ by $f(x) = \frac{x}{2}$ and $T(x) = 1 - x$ for all $x \in X$. Let $\{x_n\} \subset X$ such that $x_n \to \frac{2}{3}$ *as* $n \to \infty$. Then $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} Tx_n = \frac{1}{3}$. Hence the pair (f,T) satisfies property (E. A.) but the pair (f,T) is not S-compatible. For, $\lim\limits_{n\to\infty} S(fTx_n, fTx_n, Tfx_n) = \frac{1}{2} \neq 0$.

Definition 2.19. [17] Let (X,S) be an S-metric space. A map $F : X \to X$ is said to be a contraction if there exists a constant $0 \leq \gamma < 1$ such that

$$S(F(x), F(x), F(y)) \leq \gamma S(x,x,y), \quad for \quad all \quad x,y \in X. \tag{2.1}$$

Theorem 2.20. [17] Let (X,S) be an S-metric space. A map $F : X \to X$ be a contraction. Then F has a unique fixed point u in X.

Definition 2.21. [13] An altering distance function is a function $\psi : [0,\infty) \to [0,\infty)$ which satisfies
(i)ψ is continuous (ii) ψ is non-decreasing and (iii)$\psi(t) = 0$ if and only if $t = 0$.
We denote the class of all altering distance functions by Ψ.
We denote $\mathit{\Phi} = \{\varphi : [0,\infty) \to [0,\infty)$ such that (i) φ is continuous and (ii) $\varphi(t) = 0$ if and only if $t = 0\}$.

Definition 2.22. [3] Let (X,S) be an S-meric space. Let $f : X \to X$ be self map of X. If there exists $\alpha \in (0,1)$, $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\psi(S(fx, fy, fz)) \leq \psi(S(M_\alpha(x,y,z))) - \varphi(S(M_\alpha(x,y,z))), \tag{2.2}$$

where $M_\alpha(x,y,z) = \max\{S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz), \alpha S(fx,fx,y) + (1-\alpha)S(fy,fy,z)\}$ for all $x,y,z \in X$, then f is called an (α,ψ,φ)-generalized weakly contractive map on X.

Theorem 2.23. [3]Let (X,S) be a complete S-meric space and let f be an (α,ψ,φ)-generalized weakly contractive map. Then f has unique fixed point u(say) and f is S-continuous at u.

Lemma 2.24. [3],[7] Let (X,S) be an S-metric space and $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0. \tag{2.3}$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k$ such that

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \epsilon,\ \ S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) < \epsilon \tag{2.4}$$

and
$(i)\ \lim\limits_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \epsilon$ $\quad (ii)\ \lim\limits_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) = \epsilon$
$(iii)\ \lim\limits_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k-1}) = \epsilon$ $\quad (iv)\ \lim\limits_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) = \epsilon.$

In this paper, we define (α, ψ, φ)-almost generalized weakly contractive maps in S-metric spaces and prove the existence and uniqueness of common fixed point of such pair of maps. Also, we established a common fixed point for above maps using property (E. A.) under weakly compatible property. In section 4, we draw some corollaries and provide examples in support of our main results.

3. Main Result

Definition 3.1. Let (X, S) be an S-meric space. Let $f, T : X \to X$ be self maps on X. Suppose that there exists $\alpha \in (0, 1)$, $L \geq 0$, $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\psi(S(fx, fy, fz)) \leq \psi(M_\alpha(x, y, z)) - \varphi(M_\alpha(x, y, z)) + LN(x, y, z), \tag{3.1}$$

where
$M_\alpha(x, y, z) = \max\{S(Tx, Ty, Tz), S(Tx, Tx, fx), S(Ty, Ty, fy), S(Tz, Tz, fz),$
$\alpha S(fx, fx, Ty) + (1-\alpha)S(fy, fy, Tz)\}$ and
$N(x, y, z) = \min\{S(fx, Tx, Tx), S(fx, Ty, Ty), S(fx, Tz, Tz), S(fx, Ty, Tz)\}$, for all $x, y, z \in X$. Then the pair (f, T) is called an (α, ψ, φ)-almost generalized weakly contractive maps on X.

Example 3.2. Let$X = [0, 1]$ and We define $S : X^3 \to [0, \infty)$ by

$$S(x, y, z) = \begin{cases} 0 & \text{if } \ x = y = z \\ \max\{x, y, z\} & \text{otherwise} \end{cases}$$

Then S is an S-metric on X. Now we define $f, T : X \to X$ by

$$fx = \begin{cases} 1 & \text{if } \ x \in [0, 1] \\ 2 & \text{if } x \in (1, 2] \end{cases} \quad \text{and} \quad Tx = \begin{cases} 2 - x & \text{if } \ x \in [0, 1] \\ 0 & \text{if } x \in (1, 2]. \end{cases}$$

We now define functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t^2$ and $\varphi(t) = \frac{t}{2}$ for all $t \geq 0$. Now we verify that the pair (f, T) is an (α, ψ, φ)-almost generalized weakly contractive maps on X.
Case (i): Let $x, y, z \in [0, 1]$.
We assume without loss of generality, that $x > y > z$.
$S(fx, fy, fz) = S(1, 1, 1) = 0$, so that the inequality (3.1) holds trivially.
Case (ii): Let $x, y, z \in (1, 2]$.
We assume without loss of generality, that $x > y > z$.
$S(fx, fy, fz) = S(2, 2, 2) = 0$, so that the inequality (3.1)holds trivially.
Case (iii): Let $y, z \in [0, 1]$ and $x \in (1, 2]$.
We assume without loss of generality, that $x > y > z$.
$S(fx, fy, fz) = 2, S(Tx, Ty, Tz) = 2 - z, S(Tx, Tx, fx) = 2, S(Ty, Ty, fy) = 2 - y, S(Tz, Tz, fz) = 2 - z$ and $\alpha S(fx, fx, Ty) + (1 - \alpha)S(fy, fy, Tz) = 2 - (1 - \alpha)z$. Now, $M_\alpha(x, y, z) = 2$.
Clearly $\psi(S(fx, fy, fz)) \nleq \psi(M_\alpha(x, y, z)) - \varphi(M_\alpha(x, y, z))$ for any ψ and φ.
We have $S(fx, Tx, Tx) = 2, S(fx, Ty, Ty) = 2,\ S(fx, Tz, Tz) = 2$ and $S(fx, Ty, Tz) = 2$, so that $N(x, y, z) = 2$. Now $\psi(S(fx, fy, fz)) = \psi(2) = 2^2 = 4 \leq L(2) = LN(x, y, z)$ for any $L \geq 2$.

Case (iv): Let $z \in [0,1]$ and $x, y \in (1,2]$.
We assume without loss of generality, that $x > y > z$.
$S(fx, fy, fz) = 2,\ \ S(Tx, Ty, Tz) = 2 - z,\ \ S(Tx, Tx, fx) = 2,\ \ S(Ty, Ty, fy) = 2, S(Tz, Tz, fz) = 2 - z$ and $\alpha S(fx, fx, Ty) + (1-\alpha)S(fy, fy, Tz) = 2$, so that $M_\alpha(x, y, z) = 2$.
Clearly $\psi(S(fx, fy, fz)) \nleq \psi(M_\alpha(x, y, z)) - \varphi(M_\alpha(x, y, z))$ for any ψ and φ.
We have $S(fx, Tx, Tx) = 2, S(fx, Ty, Ty) = 2,\ S(fx, Tz, Tz) = 2$ and $S(fx, Ty, Tz) = 2$, so that $N(x, y, z) = 2$.
Now $\psi(S(fx, fy, fz)) = \psi(2) = 2^2 = 4 \leq L(2) = LN(x, y, z)$ for any $L \geq 2$.
Case (v): Let $x, y \in [0,1]$ and $z \in (1,2]$.
We assume without loss of generality, that $z > x > y$.
$S(fx, fy, fz) = 2, S(Tx, Ty, Tz) = 2 - y, S(Tx, Tx, fx) = 2 - x, S(Ty, Ty, fy) = 2 - y, S(Tz, Tz, fz) = 2$ and $\alpha S(fx, fx, Ty) + (1-\alpha)S(fy, fy, Tz) = 1 + \alpha(1 - y)$, so that $M_\alpha(x, y, z) = 2$.
Clearly $\psi(S(fx, fy, fz)) \nleq \psi(M_\alpha(x, y, z)) - \varphi(M_\alpha(x, y, z))$ for any ψ and φ.
We have $S(fx, Tx, Tx) = 2 - x,\ \ S(fx, Ty, Ty) = 2 - y,\ S(fx, Tz, Tz) = 1$ and $S(fx, Ty, Tz) = 2 - y$, so that $N(x, y, z) = 1$. Now $\psi(S(fx, fy, fz)) = \psi(2) = 2^2 = 4 \leq L(1) = LN(x, y, z)$ for any $L \geq 4$.
Case (vi): Let $x \in [0,1]$ and $z, y \in (1,2]$.
We assume without loss of generality, that $z > y > x$.
$S(fx, fy, fz) = 2, S(Tx, Ty, Tz) = 2 - x, S(Tx, Tx, fx) = 2 - x, S(Ty, Ty, fy) = 2, S(Tz, Tz, fz) = 2$ and $\alpha S(fx, fx, Ty) + (1-\alpha)S(fy, fy, Tz) = 2 - \alpha$, so that $M_\alpha(x, y, z) = 2$.
Clearly $\psi(S(fx, fy, fz)) \nleq \psi(M_\alpha(x, y, z)) - \varphi(M_\alpha(x, y, z))$ for any ψ and φ.
We have $S(fx, Tx, Tx) = 2 - x, S(fx, Ty, Ty) = 1,\ S(fx, Tz, Tz) = 1$ and $S(fx, Ty, Tz) = S(1, 0, 0) = 1$, so that $N(x, y, z) = \min\{2 - x, 1\} = 1$.
Now $\psi(S(fx, fy, fz)) = \psi(2) = 2^2 = 4 \leq L(1) = LN(x, y, z)$ for any $L \geq 4$
Case (vii): Let $y \in [0,1]$ and $z, x \in (1,2]$.
We assume without loss of generality, that $z > x > y$.
$S(fx, fy, fz) = 2, S(Tx, Ty, Tz) = 2 - y, S(Tx, Tx, fx) = 2, S(Ty, Ty, fy) = 2 - y, S(Tz, Tz, fz) = 2$ and $\alpha S(fx, fx, Ty) + (1-\alpha)S(fy, fy, Tz) = 1 + \alpha$, so that $M_\alpha(x, y, z) = 2$.
Clearly $\psi(S(fx, fy, fz)) \nleq \psi(M_\alpha(x, y, z)) - \varphi(M_\alpha(x, y, z))$ for any ψ and φ.
We have $S(fx, Tx, Tx) = 2, S(fx, Ty, Ty) = 2 - y,\ S(fx, Tz, Tz) = 2$ and $S(fx, Ty, Tz) = 2 - y$, so that $N(x, y, z) = \min\{2, 2 - y\} = 2 - y$.
Now $\psi(S(fx, fy, fz)) = \psi(2) = 4 \leq L(2 - y) = LN(x, y, z)$ for any $L \geq 4$.

Hence the pair (f, T) is an (α, ψ, φ)-almost generalized weakly contractive maps on X with any $L \geq 4$. Here we observe that f does not satisfy (2.2) and therefore f is not an (α, ψ, φ)-generalized weakly contractive map on X. For, the inequality (2.2) fails at $x = 0, y = 0$ and $z = 2$ for any choice of ψ and φ.

Lemma 3.3. Let (X, S) be an S-metric space. If a sequence $\{x_n\}$ in X converges to x and $S(x_n, x_n, y_n) \to 0$ then $y_n \to x$.

Proof. Let $\{x_n\}$ be a sequence in an S-metric space X which converges to $x \in X$ and $S(x_n, x_n, y_n) \to 0$. Let $\epsilon > 0$ be given. Then there exist $n_1, n_2 \in \mathbb{N}$ such that
for all $n \geq n_1$, we have $S(x_n, x_n, y_n) < \dfrac{\epsilon}{2}$,
for all $n \geq n_2$, we have $S(x_n, x_n, x) < \dfrac{\epsilon}{4}$ or $S(x, x, x_n) < \dfrac{\epsilon}{4}$.
Let $n_0 = \max\{n_1, n_2\}$. Then for all $n \geq n_0$, we have
$S(x, x, y_n) \leq 2S(x, x, x_n) + S(y_n, y_n, x_n) \leq 2\frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon$ for all $n \geq n_0$.
Hence $y_n \to x$. This completes the proof of the lemma. □

Theorem 3.4. Let (X, S) be an S-meric space. Let $f, T : X \to X$ be self maps on X. Assume

(i) $f(X) \subseteq T(X)$

(ii) either $f(X)$ or $T(X)$ is complete

(iii) (f,T) is (α,ψ,φ)-almost generalized weakly contractive maps.

Then f and T have a unique common fixed point u in X provided (f,T) is S-weakly compatible on X. Moreover, if T is S-continuous at u then f is S-continuous at u.

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}$ by $fx_n = Tx_{n+1}$ for $n = 0,1,2,\dots$. If $fx_n = fx_{n+1}$ for some n, then x_{n+1} is a coincidence point of f and T. Now we assume that $fx_n \neq fx_{n+1}$ for all $n = 0,1,2,\dots$.
By substituting $x = x_{n+1}, y = x_{n+1}$ and $z = x_{n+2}$ in the inequality (3.1), we have

$$\begin{aligned}
\psi(S(fx_{n+1},fx_{n+1},fx_{n+2})) &\leq \psi(M_\alpha(x,y,z)) - \varphi(M_\alpha(x,y,z)) + L\ N(x,y,z)\\
&= \psi(\max\{S(Tx_{n+1},Tx_{n+1},Tx_{n+2}),\ S(Tx_{n+1},Tx_{n+1},fx_{n+1}),\\
&\qquad S(Tx_{n+1},Tx_{n+1},fx_{n+1}), S(Tx_{n+2},Tx_{n+2},fx_{n+2}),\\
&\qquad \alpha S(fx_{n+1},fx_{n+1},Tx_{n+1}) + (1-\alpha)S(fx_{n+1},fx_{n+1},Tx_{n+2})\})\\
&\quad - \varphi(\max\{S(Tx_{n+1},Tx_{n+1},Tx_{n+2}), S(Tx_{n+1},Tx_{n+1},fx_{n+1}),\\
&\qquad S(Tx_{n+1},Tx_{n+1},fx_{n+1}), S(Tx_{n+2},Tx_{n+2},fx_{n+2}),\\
&\qquad \alpha S(fx_{n+1},fx_{n+1},Tx_{n+1}) + (1-\alpha)S(fx_{n+1},fx_{n+1},Tx_{n+2})\})\\
&\quad + L\min\{S(fx_{n+1},Tx_{n+1},Tx_{n+1}), S(fx_{n+1},Tx_{n+1},Tx_{n+1}),\\
&\qquad S(fx_{n+1},Tx_{n+2},Tx_{n+2}), S(fx_{n+1},Tx_{n+1},Tx_{n+2})\}\\
&= \psi(max\{S(fx_n,fx_n,fx_{n+1}), S(fx_n,fx_n,fx_{n+1}), S(fx_n,fx_n,fx_{n+1}),\\
&\qquad S(fx_{n+1},fx_{n+1},fx_{n+2}), \alpha S(fx_{n+1},fx_{n+1},fx_n)+(1-\alpha)S(fx_{n+1},fx_{n+1},fx_{n+1})\})\\
&\quad - \varphi(max\{S(fx_n,fx_n,fx_{n+1}), S(fx_n,fx_n,fx_{n+1}), S(fx_n,fx_n,fx_{n+1}),\\
&\qquad S(fx_{n+1},fx_{n+1},fx_{n+2}), \alpha S(fx_{n+1},fx_{n+1},fx_n)+(1-\alpha)S(fx_{n+1},fx_{n+1},fx_{n+1})\})\\
&\quad + L\ \min\{S(fx_{n+1},fx_n,fx_n), S(fx_{n+1},fx_n,fx_n), S(fx_{n+1},fx_{n+1},fx_{n+1}),\\
&\qquad S(fx_{n+1},fx_n,fx_{n+1})\}.
\end{aligned}$$

That is

$$\begin{aligned}
\psi(S(fx_{n+1},fx_{n+1},fx_{n+2})) &\leq \psi(\max\{S(fx_n,fx_n,fx_{n+1}), S(fx_{n+1},fx_{n+1},fx_{n+2})\})\\
&\quad -\varphi(\max\{S(fx_n,fx_n,fx_{n+1}), S(fx_{n+1},fx_{n+1},fx_{n+2})\}).
\end{aligned} \tag{3.2}$$

Let $M_n = \max\{S(fx_n,fx_n,fx_{n+1}), S(fx_{n+1},fx_{n+1},fx_{n+2})\}$.
Here we have two cases, either $M_n = S(fx_n,fx_n,fx_{n+1})$ or $M_n = S(fx_{n+1},fx_{n+1},fx_{n+2})$.
Suppose that, for some n, $M_n = S(fx_{n+1},fx_{n+1},fx_{n+2})$. Therefore from (3.2), it follows
$\varphi(S(fx_n,fx_n,fx_{n+1})) = 0$. Hence $fx_n = fx_{n+1}$, a contradiction since fx_n and fx_{n+1} are distinct elements.
Thus $M_n = S(fx_n,fx_n,fx_{n+1})$ for all n. Hence from (3.2), we have

$$\begin{aligned}
\psi(S(fx_{n+1},fx_{n+1},fx_{n+2})) &\leq \psi(S(fx_n,fx_n,fx_{n+1})) - \varphi(S(fx_n,fx_n,fx_{n+1})) \\
&< \psi(S(fx_n,fx_n,fx_{n+1})).
\end{aligned} \tag{3.3}$$

Now by the non decreasing property of ψ, it follows that
$S(fx_{n+1},fx_{n+1},fx_{n+2}) \leq S(fx_n,fx_n,fx_{n+1})$ for all $n \in \mathbb{N}$.
Therefore $\{S(fx_{n+1},fx_{n+1},fx_{n+2})\}$ is a decreasing sequence of positive real numbers. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} S(fx_{n+1},fx_{n+1},fx_{n+2}) = r. \tag{3.4}$$

On letting $n \to \infty$ in (3.3) and using (3.4), we obtain $\psi(r) \leq \psi(r) - \varphi(r)$, so that $\varphi(r) = 0$. Hence

$$r = 0. \tag{3.5}$$

We now show that $\{fx_n\}$ is an S-Cauchy sequence. Suppose if possible, that $\{fx_n\}$ is not S-Cauchy. Therefore by Lemma 2.24, there exists an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k$ such that

$S(fx_{m_k}, fx_{m_k}, fx_{n_k}) \geq \epsilon, S(fx_{m_k-1}, fx_{m_k-1}, fx_{n_k}) < \epsilon$ satisfying the identities (i) to (iv) of Lemma 2.24.
We now prove the the following:

$$(i) \lim_{k\to\infty} S(fx_{n_k}, fx_{m_k-1}, fx_{m_k-1}) \leq 2\epsilon \quad and \quad (ii) \lim_{k\to\infty} S(fx_{n_k}, fx_{n_k-1}, fx_{m_k-1}) \leq 2\epsilon. \tag{3.6}$$

By using condition (S3) of S-metric space and using Lemma 2.7,
we get $S(fx_{n_k}, fx_{m_k-1}, fx_{m_k-1}) \leq S(fx_{n_k}, fx_{n_k}, fx_{n_k}) + S(fx_{m_k-1}, fx_{m_k-1}, fx_{n_k}) + S(fx_{m_k-1}, fx_{m_k-1}, fx_{n_k})$
i.e., $S(fx_{n_k}, fx_{m_k-1}, fx_{m_k-1}) \leq 2S(fx_{m_k-1}, fx_{m_k-1}, fx_{n_k})$.
On taking limits as $k \to \infty$ and using condition(i) of Lemma 2.24, we get
$\lim_{n\to\infty} S(fx_{n_k}, fx_{m_k-1}, fx_{m_k-1}) \leq 2\epsilon$.
Consider $S(fx_{n_k}, fx_{n_k-1}, fx_{m_k-1}) \leq S(fx_{n_k}, fx_{n_k}, fx_{m_k}) + S(fx_{n_k-1}, fx_{n_k-1}, fx_{m_k}) + S(fx_{m_k-1}, fx_{m_k-1}, fx_{m_k})$.
On taking limits as $k \to \infty$, we have
$\lim_{k\to\infty} S(fx_{n_k}, fx_{n_k-1}, fx_{m_k-1}) \leq \epsilon + \epsilon + 0 = 2\epsilon$.

Now taking $x = fx_{m_k}, y = fx_{m_k}, z = fx_{n_k}$ and applying the inequality (3.1), we have

$$\begin{aligned}\psi(S(fx_{n_k}, fx_{n_k}, fx_{m_k})) \leq \psi(max\{&S(Tx_{n_k}, Tx_{n_k}, Tx_{m_k}), S(Tx_{n_k}, Tx_{n_k}, fx_{m_k}), S(Tx_{n_k}, Tx_{n_k}, fx_{m_k}),\\ &S(Tx_{m_k}, Tx_{m_k}, fx_{m_k}), \alpha S(fx_{n_k}, fx_{n_k}, Tx_{n_k}) + (1-\alpha) S(fx_{n_k}, fx_{n_k}, Tx_{m_k})\})\\ &- \varphi(\max\{S(Tx_{n_k}, Tx_{n_k}, Tx_{m_k}), S(Tx_{n_k}, Tx_{n_k}, fx_{m_k}), S(Tx_{n_k}, Tx_{n_k}, fx_{m_k}),\\ &S(Tx_{m_k}, Tx_{m_k}, fx_{m_k}), \alpha S(fx_{n_k}, fx_{n_k}, Tx_{n_k}) + (1-\alpha) S(fx_{n_k}, fx_{n_k}, Tx_{m_k})\}\\ &+ Lmin\{S(fx_{n_k}, Tx_{n_k}, Tx_{n_k}), S(fx_{n_k}, Tx_{n_k}, Tx_{n_k}), S(fx_{n_k}, Tx_{m_k}, Tx_{m_k}),\\ &S(fx_{n_k}, Tx_{n_k}, Tx_{m_k})\}.\end{aligned}$$

$$\begin{aligned}= \psi(\max\{&S(fx_{n_{k-1}}, fx_{n_{k-1}}, fx_{m_{k-1}}), S(fx_{n_k-1}, fx_{n_{k-1}}, fx_{n_k}), S(fx_{n_k-1}, fx_{n_{k-1}}, fx_{n_k}),\\ &S(fx_{m_k-1}, fx_{m_k-1}, fx_{m_k}), \alpha S(fx_{n_k}, fx_{n_k}, fx_{n_{k-1}}) + (1-\alpha) S(fx_{n_k}, fx_{n_k}, fx_{m_{k-1}})\})\\ &-\varphi(\max\{S(fx_{n_{k-1}}, fx_{n_{k-1}}, fx_{m_{k-1}}), S(fx_{n_k-1}, fx_{n_{k-1}}, fx_{n_k}), S(fx_{n_k-1}, fx_{n_{k-1}}, fx_{n_k}),\\ &S(fx_{m_k-1}, fx_{m_k-1}, fx_{m_k}), \alpha S(fx_{n_k}, fx_{n_k}, fx_{n_{k-1}}) + (1-\alpha) S(fx_{n_k}, fx_{n_k}, fx_{m_{k-1}})\})\\ &+L\min\{S(fx_{n_k}, fx_{n_{k-1}}, fx_{n_{k-1}}), S(fx_{n_k}, fx_{n_{k-1}}, fx_{n_{k-1}}), S(fx_{n_k}, fx_{m_{k-1}}, fx_{m_{k-1}}),\\ &S(fx_{n_k}, fx_{n_{k-1}}, fx_{m_{k-1}})\}.\end{aligned}$$

On taking limits as $k \to \infty$ in the above inequality and using Lemma 2.24,
we get $\psi(\epsilon) \leq \psi(\epsilon) - \varphi(\epsilon) + L(0) < \psi(\epsilon)$, a contradiction.
Hence $\{fx_n\}$ is S-Cauchy.
Now suppose that $T(X)$ is complete. Then there exists $u \in X$ such that $\lim_{n\to\infty} Tx_n = u$. Hence $\lim_{n\to\infty} fx_n = u$.
Since $u \in T(X)$ there exists $z \in X$ such that $u = Tz$. We now show that $fz = u$.
We now consider

$$\begin{aligned}\psi(S(fx_n, fx_n, fz)) \leq \psi(max\{&S(Tx_n, Tx_n, Tz), S(Tx_n, Tx_n, fx_n), S(Tx_n, Tx_n, fx_n),\\ &S(Tz, Tz, fz), \alpha S(fx_n, fx_n, Tx_n) + (1-\alpha) S(fx_n, fx_n, Tz)\})\\ &- \varphi(max\{S(Tx_n, Tx_n, Tz), S(Tx_n, Tx_n, fx_n), S(Tx_n, Tx_n, fx_n),\\ &S(Tz, Tz, fz), \alpha S(fx_n, fx_n, Tx_n) + (1-\alpha) S(fx_n, fx_n, Tz)\})\\ &+ L \ min\{S(fx_n, Tx_n, Tx_n), S(fx_n, Tx_n, Tx_n), S(fx_n, Tz, Tz), S(fx_n, Tx_n, Tz)\}.\end{aligned}$$

On letting $n \to \infty$ in the above inequaity, we get

$$\begin{aligned}\psi(S(u, u, fz) \leq \psi(&max\{S(u, u, u), S(u, u, u), S(u, u, u), S(u, u, fz), 0\})\\ &- \varphi(max\{S(u, u, u), S(u, u, u), S(u, u, u), S(u, u, fz), 0\}).\end{aligned}$$

Therefore

$$\psi(S(u, u, fz)) \leq \psi(S(u, u, fz)) - \varphi(S(u, u, fz)) \tag{3.7}$$

so that $\varphi(S(u, u, fz)) = 0$. Hence $u = fz$. Therefore $fz = Tz = u$.
Since the pair (f, T) is S-weakly compatible, we have $fTz = Tfz$. Thus $fu = Tu$.
Now, suppose that $f(X)$ is complete. Then there exists $u \in f(X)$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = u$. Since $f(X) \subseteq T(X)$ we have $u \in T(X)$, so that there exists $z \in X$ such that $u = Tz$. Therefore proceeding as above, we get $fz = u$.
We now show that u is the common fixed point of f and T.

Consider $\psi(S(fu,fu,fx_n)) \leq \psi(max\{S(Tu,Tu,Tx_n), S(Tu,Tu,fu), S(Tu,Tu,fu),$
$S(Tx_n,Tx_n,fx_n), \alpha S(fu,fu,Tu)+(1-\alpha)S(fu,fu,Tx_n)$
$- \varphi(max\{S(Tu,Tu,Tx_n), S(Tu,Tu,fu), S(Tu,Tu,fu),$
$S(Tx_n,Tx_n,fx_n), \alpha S(fu,fu,Tu)+(1-\alpha)S(fu,fu,Tx_n)\})$
$+Lmin\{S(fu,Tu,Tu), S(fu,Tu,Tu), S(fu,Tx_n,Tx_n), S(fu,Tu,Tx_n)\}$

Taking the limit as $n \to \infty$ in the above inequality, we have

$$\begin{aligned}\psi(S(fu,fu,u)) &\leq \psi(max\{S(Tu,Tu,u), S(fu,fu,fu), S(fu,fu,fu), S(u,u,u), \alpha(0)+(1-\alpha)S(fu,fu,u)\}) \\ &-\varphi(max\{S(Tu,Tu,u), S(fu,fu,fu), S(fu,fu,fu), S(u,u,u), \alpha(0)+(1-\alpha)S(fu,fu,u))\}) \\ &+ Lmin\{S(fu,fu,fu), S(fu,fu,fu), S(fu,u,u), S(fu,fu,u)\} \\ &= \psi(S(fu,fu,u)) - \varphi(S(fu,fu,u)) + L(0).\end{aligned}$$

This implies $\psi(S(fu,fu,u)) = 0$. So that $fu = u$. Therefore $fu = Tu = u$. Hence u is the common fixed point of f and T.

We now prove uniqueness of common fixed point. Suppose if possible u and v are two common fixed points of f and T with $u \neq v$, then we consider

$$\begin{aligned}\psi(S(fu,fu,fv)) \leq \psi(\max\{&S(Tu,Tu,Tv), S(Tu,Tu,fu), S(Tu,Tu,fu), S(Tv,Tv,fv), \\ &\alpha(fu,fu,Tu)+(1-\alpha)S(fu,fu,Tv)\}) \\ - \varphi(max\{&S(Tu,Tu,Tv), S(Tu,Tu,fu), S(Tu,Tu,fu), S(Tv,Tv,fv), \\ &\alpha(fu,fu,Tu)+(1-\alpha)S(fu,fu,Tv)\}) \\ + L\min\{&S(fu,Tu,fT), S(fu,Tu,Tu), S(fu,Tv,Tv), S(fu,Tu,Tv)\}.\end{aligned}$$

Implies $\psi(S(u,u,v)) \leq \varphi(S(u,u,v)) + \ L\ (0)$.

Thus $\varphi(S(u,u,v)) = 0$, so that $S(u,u,v) = 0$. Therefore $u = v$. Hence uniqueness follows.

Now suppose that T is S-continuous at u. We show that f is also S-continuous at u.

Consider the sequence $\{x_n\}$ in X such that $x_n \to u$ as $n \to \infty$. Then

$$\psi(S(fu,fu,fx_n)) \leq \psi(M_\alpha(u,u,x_n)) - \varphi(M_\alpha(u,u,x_n)) + L \quad N(u,u,x_n) \tag{3.8}$$

where $$\begin{aligned}M_\alpha(u,u,x_n) &= \max\{S(Tu,Tu,Tx_n), S(Tu,Tu,fu), S(Tu,Tu,fu), S(Tx_n,Tx_n,fx_n), \\ &\qquad \alpha S(fu,fu,Tu)+(1-\alpha)S(fu,fu,Tx_n)\} \\ &= \max\{S(Tu,Tu,Tx_n), S(u,u,u), S(u,u,u), S(Tx_n,Tx_n,fx_n), \\ &\qquad \alpha S(Tu,Tu,Tu)+(1-\alpha)S(Tu,Tu,Tx_n)\}\end{aligned}$$

Now taking the limits as $n \to \infty$

$\lim\limits_{n\to\infty} M_\alpha(u,u,x_n) = \max\{0,0,0, \lim\limits_{n\to\infty} S(Tu,Tu,fx_n), \alpha(0)+(1-\alpha)0\} = \lim\limits_{n\to\infty} S(fu,fu,fx_n)$

and $N(u,u,x_n) = \min\{S(fu,Tu,Tu), S(fu,Tu,Tu), S(fu,Tx_n,Tx_n), S(fu,Tu,Tx_n)\}$

Now taking the limits as $n \to \infty$, we get $\lim\limits_{n\to\infty} N(u,u,x_n) = 0$.

On taking the limits as $n \to \infty$ in (3.8), we get

$\lim\limits_{n\to\infty} \psi(S(fu,fu,fx_n)) \leq \psi(\lim\limits_{n\to\infty} S(fu,fu,fx_n)) - \varphi(\lim\limits_{n\to\infty} S(fu,fu,fx_n))$.

Therefore $\varphi(\lim\limits_{n\to\infty} S(fu,fu,fx_n)) = 0$. This implies $\lim\limits_{n\to\infty} S(fu,fu,fx_n) = 0$. i.e., fx_n is S-convergent to fu. This completes the proof of the theorem. □

In the following theorem we use property (E. A.) to relax condition $f(X) \subseteq T(X)$.

Theorem 3.5. Let X be an S-meric space. Let f, T be two self maps on (X,S). Assume that $T(X)$ is a closed subspace of X and let the pair (f,T) be (α,ψ,φ)-almost generalized weakly contractive maps. If the pair (f,T) satisfies property (E. A.) then f and T have a unique common fixed point in X provided the pair (f,T) is S-weakly compatible on X.

Proof. Since f and T satisfy property (E.A.), there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} Tx_n = u$ for some $u \in X$. Since $T(X)$ is a closed sub space of X, there exists $a \in X$ such that $u = Ta$. Now

$\psi(S(fx_n,fx_n,fa)) \leq \psi(max\{S(Tx_n,Tx_n,Ta), S(Tx_n,Tx_n,fx_n), S(Tx_n,Tx_n,fx_n),$

$$S(Ta,Ta,fa),\alpha S(fx_n,fx_n,Tx_n)+(1-\alpha)S(fx_n,fx_n,Ta))\})$$
$$-\varphi(\max\{S(Tx_n,Tx_n,Ta),S(Tx_n,Tx_n,fx_n),S(Tx_n,Tx_n,fx_n),$$
$$S(Ta,Ta,fa),\alpha S(fx_n,fx_n,Tx_n)+(1-\alpha)S(fx_n,fx_n,Ta))\})$$
$$+L\min\{S(fx_n,Tx_n,Tx_n),S(fx_n,Tx_n,Tx_n),S(fx_n,Ta,Ta),S(fx_n,Tx_n,Ta)\}.$$

On letting $n\to\infty$ on both sides, we get

$$\psi(S(u,u,fa))\le\psi(max\{S(u,u,u),S(u,u,u),S(u,u,fa),S(u,u,u),\alpha(u,u,u)+(1-\alpha)S(u,u,u)\})$$
$$-\varphi(\max\{S(u,u,u),S(u,u,u),S(u,u,u),S(u,u,fa),S(u,u,u),\alpha(0)+(1-\alpha)S(u,u,u)\})$$
$$+L\min\{S(u,u,u),S(u,u,u),S(u,u,u),S(u,u,u)\}.$$
$$=\psi(S(fu,fu,u))-\varphi(S(fu,fu,u))+\ L\ (0).$$

This implies $\varphi(S(u,u,fa)=0$, so that $S(u,u,fa)=0$. Implies $u=fa$. Hence a is the coincidence point of f and T. Since the pair (f,T) is S-weakly compatible, $fu=fTa=Tfa=Tu$.

Now consider $\psi(S(fa,fa,fu))\le\psi(\max\{S(Ta,Ta,Tu),S(Ta,Ta,fa),S(Ta,Ta,fa),S(Tu,Tu,fu),$

$$\alpha S(fa,fa,Ta)+(1-\alpha)S(fa,fa,Tu)\})$$
$$-\varphi(\max\{S(Ta,Ta,Tu),S(Ta,Ta,fa),,S(Ta,Ta,fa),S(Tu,Tu,fu),$$
$$\alpha S(fa,fa,Ta)+(1-\alpha)S(fa,fa,Tu)\})$$
$$+L\min\{S(fa,Ta,Ta),S(fa,Ta,Ta),S(fa,Tu,Tu),S(fa,Ta,Tu)\}$$

That is $\psi(S(u,u,fu))\le\psi(max\{S(u,u,fu),S(u,u,u),S(fu,fu,fu),(Tu,Tu,fu),\alpha S(u,u,u)+(1-\alpha)S(u,u,fu)\})$

$$-\varphi(max\{S(u,u,fu),S(u,u,u),S(fu,fu,fu),S(Tu,Tu,fu),\alpha S(u,u,u)+(1-\alpha)S(u,u,fu)\})$$

This gives $\varphi(S(u,u,fu))=0$. By the property of φ, $S(u,u,fu)=0$. Therefore $u=fu$. This implies $u=fu=Tu$. Hence u is the common fixed point of f and T. Uniqueness of common fixed point follows as in the proof of Theorem 3.4. □

4. Corollaries and Examples

Corollary 4.1. Let (X,S) be an S-meric space. Let $f,T:X\to X$ be self maps on X. Assume

(i) $f(X)\subseteq T(X)$

(ii) either $f(X)$ or $T(X)$ is complete.

(iii) the pair (f,T) be (α,ψ,φ)-almost generalized weakly contractive maps.

Then f and T have a unique common fixed point u in X provided the pair (f,T) is S-compatible on X. Moreover, if T is S-continuous at u then f is S-continuous at u.

By choosing $\psi(t)=t$ for all $t\ge 0$ in Theorem 3.4, we obtain the following corollary.

Corollary 4.2. Let (X,S) be an S- meric space. Let $f,T:X\to X$ be two self maps on X. Assume that

(1) $f(X)\subseteq T(X)$

(2) either $f(X)$ or $T(X)$ is complete.

(3) there exists $\alpha\in(0,1)$, $L\ge 0$, $\psi\in\Psi$ and $\varphi\in\Phi$ such that
$S(fx,fy,fz))\le M_\alpha(x,y,z)-\varphi(M_\alpha(x,y,z))+LN(x,y,z)$,
where $M_\alpha(x,y,z)=\max\{S(Tx,Ty,Tz),S(Tx,Tx,fx),S(Ty,Ty,fy),S(Tz,Tz,fz)$
$\alpha S(fx,fx,Ty)+(1-\alpha)S(fy,fy,Tz)\}$
and $N(x,y,z)=\min\{S(fx,Tx,Tx),S(fx,Ty,Ty),S(fx,Tz,Tz),S(fx,Ty,Tz)\}$ for all $x,y,z\in X$.

Then f and T have a unique common fixed point u in X provided the pair (f,T) is S-weakly compatible on X. Moreover, if T is S-continuous at u then f is S-continuous at u.

By taking $\varphi(t)=(1-\lambda)t,t\ge 0$ in the Corollary 4.2, we have the following.

Corollary 4.3. Let (X, S) be a complete S-metric space and f, T be two self mappings on X. Assume that $f(X) \subseteq T(X)$ and if there exist $\lambda, \alpha \in (0,1)$, $L \geq 0$, such that $S(fx, fy, fz) \leq \lambda M_\alpha(x,y,z) + LN(x,y,z)$ where $M_\alpha(x,y,z)$ and $N(x,y,z)$ are same as in the Corollary 4.2. Then f and T have a unique common fixed point u provided the pair (f, T) is S -weakly compatible.

If $\alpha = \frac{1}{2}$ in the inequality (3.1), we have the following.

Corollary 4.4. Let (X, S) be an S- meric space. Let $f, T : X \to X$ be two self maps on X. Assume that

(i) $f(X) \subseteq T(X)$

(ii) either $f(X)$ or $T(X)$ is complete

(iii) there exists $L \geq 0$, $\psi \in \Psi$ and $\varphi \in \Phi$ such that
$$\begin{aligned}\psi(S(fx,fy,fz)) \leq\ & \psi(\max\{S(Tx,Ty,Tz), S(Tx,Tx,fx), S(Ty,Ty,fy), S(Tz,Tz,fz),\\ & \tfrac{1}{2}(S(fx,fx,Ty)+S(fy,fy,Tz))\})\\ & -\varphi(\max\{S(Tx,Ty,Tz), S(Tx,Tx,fx), S(Ty,Ty,fy), S(Tz,Tz,fz),\\ & \tfrac{1}{2}(S(fx,fx,Ty)+S(fy,fy,Tz))\})\\ & + \ L \ \min\{S(fx,Tx,Tx), S(fx,Ty,Ty), S(fx,Tz,Tz), S(fx,Ty,Tz)\}\end{aligned}$$
for all $x, y, z \in X$.

Then f and T have a unique common fixed point u in X provided the pair (f, T) is S-weakly compatible on X. Moreover, if T is S-continuous at u then f is S-continuous at u.

If $L = 0$ in the inequality (3.1), we have the following.

Corollary 4.5. Let (X, S) be an S-meric space. Let $f, T : X \to X$ be two self maps on X. Assume that

(i) $f(X) \subseteq T(X)$

(ii) either $f(X)$ or $T(X)$ is complete.

(iii) there exists $L \geq 0$, $\psi \in \Psi$ and $\varphi \in \Phi$ such that
$\psi(S(fx,fy,fz)) \leq \psi(M_\alpha(x,y,z)) - \varphi(M_\alpha(x,y,z))$ for all $x, y, z, \in X$.

Then f and T have a unique common fixed point u in X provided the pair (f, T) is S-weakly compatible on X.

If ψ is the identity map in the Corollary 4.5, then we have the following.

Corollary 4.6. Let (X, S) be a complete S-meric space. Let f, T be two self maps on X. Assume that $f(X) \subseteq T(X)$. Suppose there exists $\psi \in \Psi$ and $\varphi \in \mathit{\Phi}$, $\alpha \in (0,1)$ such that $S(fx,fy,fz) \leq M_\alpha(x,y,z) - \varphi(M_\alpha(x,y,z))$, for all $x, y, z \in X$. Then T and f have a unique common fixed point in X provided the pair (f, T) is S- weakly compatible.

By choosing $\varphi(t) = (1-\lambda)t, t \geq 0$ in the Corollary 4.6, we have the following.

Corollary 4.7. Let (X, S) be a complete S-metric space and f, T be two self mappings on X. Assume that $f(X) \subseteq T(X)$ and there exist $\lambda, \alpha \in (0,1)$ such that $S(fx,fy,fz) \leq \lambda M_\alpha(x,y,z)$ for all $x, y, z \in X$. Then f and T has a unique common fixed point u provided the pair (f, T) is S-weakly compatible.

Remark 4.8. Theorem 2.20 follows as a corollary to Corollary 4.7 by taking T as the identity map on X.

By choosing T as the identity map of X in Theorem 3.4, we have the following.

Corollary 4.9. Let (X,S) be a complete S-meric space. Let $f: X \to X$ be a mapping. Suppose there exists $\alpha \in (0,1)$, $L \geq 0$, $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\begin{aligned}\psi(S(fx,fy,fz)) \leq\ & \psi(\max\{S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz), \alpha S(fx,fx,y)+(1-\alpha)S(fy,fy,z)\}) \\ & -\varphi(\max\{S(x,y,z), S(x,x,fx), S(y,y,fy), S(z,z,fz), \alpha S(fx,fx,y)+(1-\alpha)S(fy,fy,z)\}) \\ & + L\min\{S(fx,x,x), S(fx,y,y), S(fx,z,z), S(fx,y,z)\}\end{aligned}$$

for all $x,y,z \in X$. Then f has a unique common fixed point u and f is S-continuous at u.

Remark 4.10. Theorem 2.23 follows as a corollary to Corollary 4.9 by taking $L = 0$.

The following is an example in support of Theorem 3.4 .

Example 4.11. Let $X = [0,1]$ and we define $S: X^3 \to [0,\infty)$ by $S(x,y,z) = max\{|x-z|, |y-z|\}$ for all $x,y,z \in X$. Then S is an S-metric on X. Now we define $f, T: X \to X$ by

$$fx = \begin{cases} \frac{x}{8} & \text{if } x \in [0,\frac{1}{2}] \\ \frac{x}{4} & \text{if } x \in (\frac{1}{2},1] \end{cases} \qquad \text{and} \qquad Tx = \frac{x}{2}.$$

Here $f(X) \subseteq T(X)$. We now define functions $\psi, \varphi: [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \frac{t}{2} \quad for \quad all \quad t \geq 0 \quad and \qquad \varphi(t) = \begin{cases} \frac{t^2}{4} & \text{if } t \in [0,1] \\ \frac{t}{4} & \text{if } t \in [1,\infty). \end{cases}$$

Now we verify that (f,T) is an (α,ψ,φ)- almost generalized weakly contractive map.
<u>Case</u> (i): Let $x,y,z \in [0,\frac{1}{2}]$.
We assume, without loss of generality, that $x > y > z$.
$S(fx,fy,fz) = S(\frac{x}{8},\frac{y}{8},\frac{z}{8}) = \frac{x}{8} - \frac{z}{8}$ and $S(Tx,Ty,Tz) = S(\frac{x}{2},\frac{y}{2},\frac{z}{2}) = \frac{x}{2} - \frac{z}{2}$. We have

$$\begin{aligned}\psi(S(fx,fy,fz)) &= \tfrac{x}{16} - \tfrac{z}{16} \leq \tfrac{x-z}{4} - \tfrac{(\frac{x-z}{2})^2}{4} \\ &\leq \tfrac{S(Tx,Ty,Tz)}{2} - \tfrac{S(Tx,Ty,Tz)^2}{4} \\ &\leq \tfrac{M_\alpha(x,y,z)}{2} - \tfrac{(M_\alpha(x,y,z))^2}{4} \\ &= \psi(M_\alpha(x,y,z)) - \varphi(M_\alpha(x,y,z)) + LN(x,y,z) \text{ for any } L \geq 0.\end{aligned}$$

<u>Case</u> (ii): Let $x,y,z \in (\frac{1}{2},1]$.
We assume, without loss of generality, that $x > y > z$.
$S(fx,fy,fz) = S(\frac{x}{4},\frac{y}{4},\frac{z}{4}) = \frac{x}{4} - \frac{z}{4}$ and $S(Tx,Ty,Tz) = S(\frac{x}{2},\frac{y}{2},\frac{z}{2}) = \frac{x}{2} - \frac{z}{2}$.

$$\begin{aligned}\text{Now } \psi(S(fx,fy,fz)) &= \tfrac{1}{2}(\tfrac{x}{4} - \tfrac{z}{4}) \leq \tfrac{x}{4} - \tfrac{z}{4} - \tfrac{(\frac{x-z}{2})^2}{4} \\ &= \tfrac{S(Tx,Ty,Tz)}{2} - \tfrac{(S(Tx,Ty,Tz))^2}{4} \\ &\leq \tfrac{M_\alpha(x,y,z)}{2} - \tfrac{(M_\alpha(x,y,z))^2}{4} \\ &= \psi(M_\alpha(x,y,z)) - \varphi(M_\alpha(x,y,z)) + LN(x,y,z) \text{ for any } L \geq 0.\end{aligned}$$

<u>Case</u> (iii): Let $y,z \in [0,\frac{1}{2}]$ and $x \in (\frac{1}{2},1]$.
We assume, without loss of generality, that $x > y > z$.
$S(fx,fy,fz) = S(\frac{x}{4},\frac{y}{8};\frac{z}{8}) = \frac{x}{4} - \frac{z}{8}, S(Tx,Tx,fx) = S(\frac{x}{2},\frac{x}{2},\frac{x}{4}) = \frac{x}{4}$ and $S(Tx,Ty,Tz) = S(\frac{x}{2},\frac{y}{2},\frac{z}{2}) = \frac{x}{2} - \frac{z}{2}$.

<u>Sub Case</u> (a): Let $\frac{x}{2} < z < x$.

$$\begin{aligned}\psi(S(fx,fy,fz)) &= \tfrac{x}{8} - \tfrac{z}{16} \leq \tfrac{x}{8} - \tfrac{(\frac{x}{4})^2}{4} \\ &= \tfrac{S(Tx,Tx,fx)}{2} - \tfrac{(S(Tx,Tx,fx))^2}{4} \\ &\leq \tfrac{1}{2}M_\alpha(x,y,z) - \tfrac{1}{4}(M_\alpha(x,y,z)) \\ &= \psi(M_\alpha(x,y,z)) - \varphi(M_\alpha(x,y,z)) + LN(x,y,z) \text{ for any } L \geq 0.\end{aligned}$$

<u>Sub Case</u> (b): Let $z \leq \frac{x}{2} < x$. Then
$S(fx,fy,fz) = \frac{x}{4} - \frac{z}{8}$ and $S(Tx,Ty,Tz) = \frac{x}{2} - \frac{z}{2}$.

Now, $\psi(S(fx,fy,fz) = \frac{x}{8} - \frac{z}{16} \leq \frac{x-z}{4} - \frac{x-z}{16}$
$= \frac{\frac{x-z}{2}}{2} - \frac{(\frac{x-z}{2})^2}{4}$
$= \frac{S(Tx,Ty,Tz)}{2} - \frac{(S(Tx,Ty,Tz))^2}{4}$
$= \psi(M_\alpha(x,y,z)) - \varphi(M_\alpha(x,y,z)) + LN(x,y,z)$ for any $L \geq 0$.

<u>Case</u> (iv): Let $z \in [0, \frac{1}{2}]$ and $x, y \in (\frac{1}{2}, 1]$.
We assume, without loss of generality, that $x > y > z$.
$S(fx,fy,fz) = S(\frac{x}{4}, \frac{y}{4}, \frac{z}{8}) = \frac{x}{4} - \frac{z}{8}$, $S(Tx,Tx,fx) = S(\frac{x}{2}, \frac{x}{2}, \frac{x}{4}) = \frac{x}{4}$, $S(Tx,Ty,Tz) = S(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}) = |\frac{x}{2} - \frac{z}{2}|$.

<u>Sub Case</u> (a): Let $\frac{x}{2} < z < x$.

$\psi(S(fx,fy,fz)) = \frac{x}{8} - \frac{z}{16} \leq \frac{x}{8} - \frac{(\frac{x}{4})^2}{4}$
$= \frac{S(Tx,Tx,fx)}{2} - \frac{(S(Tx,Tx,fx))^2}{4}$
$\leq \frac{M_\alpha(x,y,z)}{2} - \frac{(M_\alpha(x,y,z))^2}{4}$
$= \psi(M_\alpha(x,y,z)) - \varphi(M_\alpha(x,y,z)) + LN(x,y,z)$ for any $L \geq 0$.

<u>Sub Case</u> (b): Let $z \leq \frac{x}{2} < x$.

$\psi(S(fx,fy,fz)) = \frac{x}{8} - \frac{z}{16} \leq \frac{x-z}{4} - \frac{(x-z)^2}{16}$
$= \frac{x-z}{4} - \frac{(\frac{x-z}{2})^2}{4}$
$= \frac{S(Tx,Ty,Tz)}{2} - \frac{(S(Tx,Ty,Tz))^2}{4}$
$\leq \frac{M_\alpha(x,y,z)}{2} - \frac{(M_\alpha(x,y,z))^2}{4}$
$= \psi(M_\alpha(x,y,z)) - \varphi(M_\alpha(x,y,z)) + LN(x,y,z)$ for any $L \geq 0$.

<u>Case</u> (v): Let $x, y \in [0, \frac{1}{2}]$ and $z \in (\frac{1}{2}, 1]$.
We assume, without loss of generality, that $x > y$.
$S(fx,fy,fz) = S(\frac{x}{8}, \frac{y}{8}, \frac{z}{4}) = \frac{z}{4} - \frac{x}{8}$, $S(Tx,Ty,Tz) = S(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}) = \frac{z}{2} - \frac{x}{2}$ and $S(Tz,Tz,fz) = S(\frac{z}{2}, \frac{z}{2}, \frac{z}{4}) = \frac{z}{4}$.

<u>Sub Case</u> (a): Let $\frac{z}{2} < x < z$.

$\psi(S(fx,fy,fz)) = \frac{z}{8} - \frac{x}{16} \leq \frac{z}{8} - \frac{(\frac{z}{4})^2}{4}$
$= \frac{S(Tz,Tz,fz)}{2} - \frac{(S(Tz,Tz,fz))^2}{4}$
$\leq \frac{M_\alpha(x,y,z)}{2} - \frac{(M_\alpha(x,y,z))^2}{4}$
$= \psi(M_\alpha(x,y,z)) - \varphi(M_\alpha(x,y,z)) + LN(x,y,z)$ for any $L \geq 0$.

<u>Sub Case</u> (b): Let $x \leq \frac{z}{2} < z$.

$\psi(S(fx,fy,fz)) = \frac{z}{8} - \frac{x}{16} \leq \frac{z-x}{4} - \frac{(z-x)^2}{16}$
$= \frac{z-x}{4} - \frac{(\frac{z-x}{2})^2}{4}$
$= \frac{S(Tx,Ty,Tz)}{2} - \frac{(S(Tx,Ty,Tz))^2}{4}$
$\leq \frac{M_\alpha(x,y,z)}{2} - \frac{(M_\alpha(x,y,z))^2}{4}$
$= \psi(M_\alpha(x,y,z)) - \varphi(M_\alpha(x,y,z)) + LN(x,y,z)$ for any $L \geq 0$.

<u>Case</u> (vi): Let $x \in [0, \frac{1}{2}]$ and $z, y \in (\frac{1}{2}, 1]$.
We assume, without loss of generality, that $z > y > x$.
$S(fx,fy,fz) = S(\frac{x}{8}, \frac{y}{4}, \frac{z}{4}) = \frac{z}{4} - \frac{x}{8}, S(Tx,Ty,Tz) = S(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}) = \frac{z}{2} - \frac{x}{2}, S(Tz,Tz,fz) = S(\frac{z}{2}, \frac{z}{2}, \frac{z}{4}) = \frac{z}{4}$.

<u>Sub Case</u> (a): Let $\frac{z}{2} < x < z$.

$$\begin{aligned}\psi(S(fx, fy, fz)) &= \tfrac{z}{8} - \tfrac{x}{16} \leq \tfrac{z}{8} - \tfrac{z^2}{64}\\ &= \tfrac{z}{8} - \tfrac{(\frac{z}{4})^2}{4}\\ &= \tfrac{S(Tz,Tz,fz)}{2} - \tfrac{(S(Tz,Tz,fz))^2}{4}\\ &\leq \tfrac{M_\alpha(x,y,z)}{2} - \tfrac{(M_\alpha(x,y,z))^2}{4}\\ &= \psi(M_\alpha(x, y, z)) - \varphi(M_\alpha(x, y, z)) + LN(x, y, z) \text{ for any } L \geq 0.\end{aligned}$$

Sub Case (b): Let $x \leq \frac{z}{2} < z$.

$$\begin{aligned}\psi(S(fx, fy, fz)) &= \tfrac{z}{8} - \tfrac{x}{16} \leq \tfrac{z-x}{4} - \tfrac{(z-x)^2}{16}\\ &= \tfrac{z-x}{4} - \tfrac{(\frac{z-x}{2})^2}{4}\\ &= \tfrac{S(Tx,Ty,Tz)}{2} - \tfrac{(S(Tx,Ty,Tz))^2}{4}\\ &\leq \tfrac{M_\alpha(x,y,z)}{2} - \tfrac{(M_\alpha(x,y,z))^2}{4}\\ &= \psi(M_\alpha(x, y, z)) - \varphi(M_\alpha(x, y, z)) + LN(x, y, z) \text{ for any } L \geq 0.\end{aligned}$$

Case (vii): Let $y \in [0, \frac{1}{2}]$ and $z, x \in (\frac{1}{2}, 1]$.

We assume, without loss of generality, that $z > x > y$.

$S(fx, fy, fz) = S(\frac{x}{4}, \frac{y}{8}, \frac{z}{4}) = \frac{z}{4} - \frac{y}{8}$, $S(Tx, Ty, Tz) = S(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}) = \frac{z}{2} - \frac{y}{2}$ and $S(Tz, Tz, fz) = S(\frac{z}{2}, \frac{z}{2}, \frac{z}{4}) = \frac{z}{4}$.

Sub Case (a): Let $\frac{z}{2} < y < z$.

$$\begin{aligned}\psi(S(fx, fy, fz)) &= \tfrac{z}{8} - \tfrac{y}{16} \leq \tfrac{z}{8} - \tfrac{z^2}{64}\\ &= \tfrac{z}{8} - \tfrac{(\frac{z}{4})^2}{4}\\ &= \tfrac{S(Tz,Tz,fz)}{2} - \tfrac{(S(Tz,Tz,fz))^2}{4}\\ &\leq \tfrac{M_\alpha(x,y,z)}{2} - \tfrac{(M_\alpha(x,y,z))^2}{4}\\ &= \psi(M_\alpha(x, y, z)) - \varphi(M_\alpha(x, y, z)) + LN(x, y, z) \text{ for any } L \geq 0.\end{aligned}$$

Sub Case (b): Let $y \leq \frac{z}{2} < z$.

$$\begin{aligned}\psi(S(fx, fy, fz)) &= \tfrac{z}{8} - \tfrac{y}{16} \leq \tfrac{z-y}{4} - \tfrac{(z-y)}{16}\\ &= \tfrac{z-y}{4} - \tfrac{(\frac{z-y}{2})^2}{4}\\ &= \tfrac{S(Tx,Ty,Tz)}{2} - \tfrac{(S(Tx,Ty,Tz))^2}{4}\\ &\leq \tfrac{M_\alpha(x,y,z)}{2} - \tfrac{(M_\alpha(x,y,z))^2}{4}\\ &= \psi(M_\alpha(x, y, z)) - \varphi(M_\alpha(x, y, z)) + LN(x, y, z) \text{ for any } L \geq 0.\end{aligned}$$

Hence (f, T) is an (α, ψ, φ)- almost generalized weakly contractive map on X. Also, the pair (f, T) is S-weakly compatible and f and T have unique common fixed point 0.

The following example is in support of Theorem 3.5

Example 4.12. Let$X = [0, 1]$ and We define $S : X^3 \to [0, \infty)$ by $S(x, y, z) = max\{|x - z|, |y - z|\}$ for all $x, y, z \in X$. Then S is an S-metric on X. Now we define $f, T : X \to X$ by

$$fx = \begin{cases} \frac{1+x}{2} & \text{if } x \in [0, \frac{1}{2}) \\ \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad Tx = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}) \\ 1 - x & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

We now define functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = \frac{t}{2} \quad \textit{and} \quad \varphi(t) = \begin{cases} \frac{t^2}{4} & \text{if } t \in [0, 1] \\ \frac{t}{4} & \text{if } t \in [1, \infty). \end{cases}$$

We now verify that (f,T) is an (α,ψ,φ)- almost generalized weakly contractive maps on X.

<u>Case</u> (i): Let $x,y,z\in[0,\frac{1}{2})$.

We assume, without loss of generality, that $x>y>z$.

$S(fx,fy,fz)=S(\frac{1+x}{2},\frac{1+y}{2},\frac{1+z}{2})=\frac{x-z}{2},\quad S(Tx,Ty,Tz)=0$ and $S(Tz,Tz,fz)=S(0,0,\frac{1+z}{2})=\frac{1+z}{2}$.

$$\begin{aligned}\psi(S(fx,fy,fz))=\tfrac{x-z}{4}&\le\tfrac{1}{4}+\tfrac{z}{4}-\tfrac{(\frac{1+z}{2})^2}{4}\\&=\tfrac{S(Tz,Tz,fz)}{2}-\tfrac{(S(Tz,Ty,fz))^2}{4}\\&\le\tfrac{M_\alpha(x,y,z)}{2}-\tfrac{(M_\alpha(x,y,z))^2}{4}\\&=\psi(M_\alpha(x,y,z))-\varphi(M_\alpha(x,y,z))+LN(x,y,z)\text{ for any }L\ge0.\end{aligned}$$

<u>Case</u> (ii): Let $x,y,z\in[\frac{1}{2},1]$.

We assume, without loss of generality, that $x>y>z$.

$S(fx,fy,fz)=S(\frac{1}{2},\frac{1}{2},\frac{1}{2})=0$, so that the inequality (3.1) holds trivially.

<u>Case</u> (iii): Let $y,z\in[0,\frac{1}{2})$ and $x\in[\frac{1}{2},1]$.

We assume, without loss of generality, that $x>y>z$.

$S(fx,fy,fz)=S(\frac{1}{2},\frac{1+y}{2},\frac{1+z}{2})=\max\{\frac{z}{2},|\frac{y}{2}-\frac{z}{2}|\}$.

$S(Tx,Ty,Tz)=1-x,\quad S(Ty,Ty,fy)=\frac{1+y}{2}$, and $S(Tz,Tz,fz)=\frac{1+z}{2}$.

If $S(fx,fy,fz)=\frac{z}{2}$ then

$$\begin{aligned}\psi(S(fx,fy,fz))=\tfrac{z}{4}&\le\tfrac{z}{4}+\tfrac{1}{4}-\tfrac{(\frac{1+z}{2})^2}{4}\\&=\tfrac{S(Tz,Tz,fz)}{2}-\tfrac{(S(Tz,Tz,fz))^2}{4}\\&\le\tfrac{M_\alpha(x,y,z)}{2}-\tfrac{(M_\alpha(x,y,z))^2}{4}\\&=\psi(M_\alpha(x,y,z))-\varphi(M_\alpha(x,y,z))+LN(x,y,z)\text{ for any }L\ge0.\end{aligned}$$

If $S(fx,fy,fz)=\frac{y-z}{2}$ then

$$\begin{aligned}\psi(S(fx,fy,fz))=\tfrac{y-z}{4}&\le\tfrac{1+z}{4}-\tfrac{(\frac{1+z}{2})^2}{4}\\&=\tfrac{S(Tz,Tz,fz)}{2}-\tfrac{(S(Tz,Ty,fz))^2}{4}\\&\le\tfrac{M_\alpha(x,y,z)}{2}-\tfrac{(M_\alpha(x,y,z))^2}{4}\\&=\psi(M_\alpha(x,y,z))-\varphi(M_\alpha(x,y,z))+LN(x,y,z)\text{ for any }L\ge0.\end{aligned}$$

If $S(fx,fy,fz)=\frac{z-y}{2}$ then

$$\begin{aligned}\psi(S(fx,fy,fz))=\tfrac{z-y}{4}&\le\tfrac{1+y}{4}-\tfrac{(\frac{1+y}{2})^2}{4}\\&=\tfrac{S(Ty,Ty,fy)}{2}-\tfrac{(S(Ty,Ty,fy))^2}{4}\\&\le\tfrac{M_\alpha(x,y,z)}{2}-\tfrac{(M_\alpha(x,y,z))^2}{4}\\&=\psi(M_\alpha(x,y,z))-\varphi(M_\alpha(x,y,z))+LN(x,y,z)\text{ for any }L\ge0.\end{aligned}$$

<u>Case</u> (iv): Let $z\in[0,\frac{1}{2})$ and $x,y\in[\frac{1}{2},1]$.

We assume, without loss of generality, that $x>y>z$.

$S(fx,fy,fz)=S(\frac{1}{2},\frac{1}{2},\frac{1+z}{2})=\frac{z}{2}$ and $S(Tz,Tz,fz)=S(0,0,\frac{1+z}{2})=\frac{1+z}{2}$.

$$\begin{aligned}\psi(S(fx,fy,fz))=\tfrac{z}{4}&\le\tfrac{1+z}{4}-\tfrac{(\frac{1+z}{2})^2}{4}\\&=\tfrac{S(Tz,Tz,fz)}{2}-\tfrac{(S(Tz,Tz,fz))^2}{4}\\&\le\tfrac{M_\alpha(x,y,z)}{2}-\tfrac{(M_\alpha(x,y,z))^2}{4}\\&=\psi(M_\alpha(x,y,z))-\varphi(M_\alpha(x,y,z))+LN(x,y,z)\text{ for any }L\ge0.\end{aligned}$$

<u>Case</u> (v): Let $x,y\in[0,\frac{1}{2})$ and $z\in(\frac{1}{2},1]$.

We assume, without loss of generality, that $x>y$.

$S(fx,fy,fz)=S(\frac{1+x}{2},\frac{1+y}{2},\frac{1}{2})=\max\{\frac{x}{2},\frac{y}{2}\},S(Tx,Tx,fx)=\frac{1+x}{2}$ and $S(Ty,Ty,fy)=S(0,0,\frac{1+y}{2})=\frac{1+y}{2}$.

If $S(fx,fy,fz)=\frac{x}{2}$ then

$\psi(S(fx,fy,fz)) = \frac{x}{4} \leq \frac{1+x}{4} - \frac{\frac{1+x}{2}}{4}$

$\leq \frac{1+x}{4} - \frac{(\frac{1+x}{2})^2}{4}$

$= \frac{S(Tx,Tx,fx)}{2} - \frac{(S(Tx,Tx,fx))^2}{4}$

$\leq \frac{M_\alpha(x,y,z)}{2} - \frac{(M_\alpha(x,y,z))^2}{4}$

$= \psi(M_\alpha(x,y,z)) - \varphi(M_\alpha(x,y,z)) + LN(x,y,z)$ for any $L \geq 0$.

If $S(fx,fy,fz) = \frac{y}{2}$ then

$\psi(S(fx,fy,fz)) = \frac{y}{4} \leq \frac{1+y}{4} - \frac{(\frac{1+y}{2})^2}{4}$

$\leq \frac{S(Ty,Ty,fy)}{2} - \frac{(S(Ty,Ty,fy))^2}{4}$

$= \psi(M_\alpha(x,y,z)) - \varphi(M_\alpha(x,y,z)) + LN(x,y,z)$ for any $L \geq 0$.

Case (vi): Let $x \in [0, \frac{1}{2})$ and $z, y \in (\frac{1}{2}, 1]$.
We assume, without loss of generality, that $z > y > x$.
$S(fx,fy,fz) = S(\frac{1+x}{2}, \frac{1}{2}, \frac{1}{2}) = \frac{x}{2}$ and $S(Tx,Tx,fx) = S(0,0,\frac{1+x}{2}) = \frac{1+x}{2}$.
Now $\psi(S(fx,fy,fz)) = \frac{x}{4} \leq \frac{1+x}{4} - \frac{\frac{1+x}{2}}{4}$

$\leq \frac{1+x}{4} - \frac{(\frac{1+x}{2})^2}{4}$

$= \frac{S(Tx,Tx,fx)}{2} - \frac{(S(Tx,Tx,fx))^2}{4}$

$\leq \frac{M_\alpha(x,y,z)}{2} - \frac{(M_\alpha(x,y,z))^2}{4}$

$= \psi(M_\alpha(x,y,z)) - \varphi(M_\alpha(x,y,z)) + LN(x,y,z)$ for any $L \geq 0$.

Case (vii): Let $y \in [0, \frac{1}{2})$ and $z, x \in (\frac{1}{2}, 1]$.
We assume, without loss of generality, that $z > x > y$.
$S(fx,fy,fz) = S(\frac{1}{2}, \frac{1+y}{2}, \frac{1}{2}) = \frac{y}{2}$ and $S(Ty,Ty,fy) = S(0,0,\frac{1+y}{2}) = \frac{1+y}{2}$.
If $S(fx,fy,fz) = \frac{y}{2}$ then

$\psi(S(fx,fy,fz)) = \frac{y}{4} \leq \frac{1+y}{4} - \frac{(\frac{1+y}{2})^2}{4}$

$\leq \frac{S(Ty,Ty,fy)}{2} - \frac{(S(Ty,Ty,fy))^2}{4}$

$= \psi(M_\alpha(x,y,z)) - \varphi(M_\alpha(x,y,z)) + LN(x,y,z)$ for any $L \geq 0$.

Hence (f,T) is an (α,ψ,φ)-almost generalized weakly contractive maps on X. More over (f,T) is S-weakly compatible and (f,T) satisfies property (E. A.) and f and T have a unique common fixed point $\frac{1}{2}$.

If the maps f and T fail to satisfy property (E. A.) in Theorem 3.5, then f and T may not have a common fixed point.The following example illustrates this fact.

Example 4.13. Let $X = [0,1]$ and We define $S : X^3 \to [0,\infty)$ by $S(x,y,z) = max\{|x-z|, |y-z|\}$ for all $x,y,z \in X$. Then S is an S-metric on X. Now we define $f, T : X \to X$ by

$$fx = \begin{cases} 1 & \text{if } x \in [0, \frac{2}{3}] \\ \frac{2}{3} & \text{if } x \in (\frac{2}{3}, 1] \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{3}{4} & \text{if } x \in (0, \frac{2}{3}] \\ \frac{1}{2} & \text{if } x \in (\frac{2}{3}, 1]. \end{cases}$$

We now define functions $\psi, \varphi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \frac{t}{2} \quad and \quad \varphi(t) = \frac{t^2}{4} \quad \text{for all} \quad t \geq 0$$

We now verify (f,T) is an (α,ψ,φ)- almost generalized weakly contractive maps on X.
Case (i): Let $x, y, z \in [0, \frac{2}{3}]$.
We assume, without loss of generality, that $x > y > z$.

$S(fx, fy, fz)) = S(1, 1, 1) = 0$, so that the inequality (3.1) holds trivially.
<u>Case</u> (ii): Let $x, y, z \in (\frac{2}{3}, 1]$.
We assume, without loss of generality, that $x > y > z$.
$S(fx, fy, fz)) = S(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) = 0$, so that the inequality (3.1) holds trivially.
<u>Case</u> (iii): Let $y, z \in [0, \frac{2}{3}]$ and $x \in (\frac{2}{3}, 1]$.
We assume, without loss of generality, that $x > y > z$.
$S(fx, fy, fz) = S(\frac{2}{3}, 1, 1) = \frac{1}{3}$, $S(fx, Tx, Tx) = S(\frac{2}{3}, \frac{1}{2}, \frac{1}{2}) = \frac{1}{6}, S(fx, Ty, Ty) = S(\frac{2}{3}, \frac{3}{4}, \frac{3}{4}) = \frac{1}{12}$,
$S(fx, Tz, Tz) = S(\frac{2}{3}, \frac{3}{4}, \frac{3}{4}) = \frac{1}{12}$ and $S(fx, Ty, Tz)) = S(\frac{2}{3}, \frac{3}{4}, \frac{3}{4}) = \frac{1}{12}$, so that $N(x, y, z) = \frac{1}{12}$.

$$\begin{aligned}\psi(S(fx, fy, fz)) = \psi(\tfrac{1}{3}) = \tfrac{1}{6} &\leq \tfrac{L}{12} \text{ for any } L \geq 2\\ &= LN(x, y, z) \text{ for any } L \geq 2\\ &\leq \psi(M_\alpha(x, y, z)) - \varphi(M_\alpha(x, y, z)) + LN(x, y, z) \text{ for any } L \geq 2.\end{aligned}$$

<u>Case</u> (iv): Let $z \in [0, \frac{2}{3}]$ and $x, y \in (\frac{2}{3}, 1]$.
We assume, without loss of generality, that $x > y > z$.
$S(fx, fy, fz) = S(\frac{2}{3}, \frac{2}{3}, 1) = \frac{1}{3}$, $S(fx, Tx, Tx) = S(\frac{2}{3}, \frac{1}{2}, \frac{1}{2}) = \frac{1}{6}, S(fx, Ty, Ty) = S(\frac{2}{3}, \frac{1}{2}, \frac{1}{2}) = \frac{1}{6}$,
$S(fx, Tz, fz) = S(\frac{2}{3}, \frac{3}{4}, \frac{3}{4}) = \frac{1}{12}$, and $S(fx, Ty, Tz)) = S(\frac{2}{3}, \frac{1}{2}, \frac{3}{4}) = \frac{1}{4}$, so that $N(x, y, z) = \frac{1}{12}$.

$$\begin{aligned}\text{Now, } \psi(S(fx, fy, fz)) = \tfrac{1}{6} &\leq \tfrac{L}{12} \text{ for any } L \geq 2\\ &= L\ \ N(x, y, z) \text{ for any } L \geq 2\\ &\leq \psi(M_\alpha(x, y, z)) - \varphi(M_\alpha(x, y, z)) + L\ \ N(x, y, z) \text{ for any } L \geq 2.\end{aligned}$$

<u>Case</u> (v): Let $x, y \in [0, \frac{2}{3}]$ and $z \in (\frac{2}{3}, 1]$.
We assume, without loss of generality, that $z > y > z$.
$S(fx, fy, fz) = S(1, 1, \frac{2}{3}) = \frac{1}{3}, S(fx, Tx, Tx) = S(\frac{3}{4}, \frac{3}{4}, 1) = \frac{1}{4}, S(fx, Ty, Ty) = S(1, \frac{3}{4}, \frac{3}{4}) = \frac{1}{4}$,
$S(fx, Tz, Tz) = S(1, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$ and $S(fx, Ty, Tz)) = S(1, \frac{3}{4}, \frac{1}{2}) = \frac{1}{2}$, so that $N(x, y, z) = \frac{1}{4}$.

$$\begin{aligned}\text{Now, } \psi(S(fx, fy, fz)) = \psi(\tfrac{1}{3}) = \tfrac{1}{6} &\leq \tfrac{L}{4} \text{ for any } L \geq \tfrac{2}{3}\\ &= L\ \ N(x, y, z) \text{ for all } L \geq \tfrac{2}{3}\\ &\leq \psi(M_\alpha(x, y, z)) - \varphi(M_\alpha(x, y, z)) + L\ \ N(x, y, z) \text{ for any } L \geq \tfrac{2}{3}.\end{aligned}$$

<u>Case</u> (vi): Let $x \in [0, \frac{2}{3}]$ and $z, y \in (\frac{2}{3}, 1]$.
$S(fx, fy, fz) = S(1, \frac{2}{3}, \frac{2}{3}) = \frac{1}{3}$, $S(fx, Tx, Tx) = S(1, \frac{3}{4}, \frac{3}{4}) = \frac{1}{4}, S(fx, Ty, Ty) = S(1, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$,
$S(fx, Tz, Tz) = S(1, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$, and $S(fx, Ty, Tz)) = S(1, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$, so that $N(x, y, z) = \frac{1}{4}$.

$$\begin{aligned}\text{Now, } \psi(S(fx, fy, fz)) = \tfrac{1}{6} &\leq \tfrac{L}{4} \text{ for any } L \geq \tfrac{2}{3}.\\ &= L\ \ N(x, y, z) \text{ for any } L \geq \tfrac{2}{3}\\ &\leq \psi(M_\alpha(x, y, z)) - \varphi(M_\alpha(x, y, z)) + L\ \ N(x, y, z) \text{ for any } L \geq \frac{2}{3}.\end{aligned}$$

<u>Case</u> (vii): Let $x, y \in [0, \frac{2}{3}]$ and $z \in (\frac{2}{3}, 1]$.
We assume, without loss of generality, that $z > x > y$.
$S(fx, fy, fz) = S(\frac{2}{3}, 1, \frac{2}{3}) = \frac{1}{3}$, $S(fx, Tx, Tx) = S(\frac{2}{3}, \frac{1}{2}, \frac{1}{2}) = \frac{1}{6}, S(fx, Ty, Ty) = S(\frac{2}{3}, \frac{3}{4}, \frac{3}{4}) = \frac{1}{12}$,
$S(fx, Tz, Tz) = S(\frac{2}{3}, \frac{1}{2}, \frac{1}{2}) = \frac{1}{6}$ and $S(fx, Ty, Tz)) = S(\frac{2}{3}, \frac{3}{4}, \frac{3}{4}) = \frac{1}{12}$, so that $N(x, y, z) = \frac{1}{12}$.

$$\begin{aligned}\text{Now, } \psi(S(fx, fy, fz)) = \tfrac{1}{6} &\leq \tfrac{L}{12} \text{ for any } L \geq 2\\ &= L\ \ N(x, y, z) \text{ for any } L \geq 2\\ &\leq \psi(M_\alpha(x, y, z)) - \varphi(M_\alpha(x, y, z)) + L\ \ N(x, y, z) \text{ for any } L \geq 2.\end{aligned}$$

Therefore from all the above cases (f, T) satisfies inequality (3.1). Hence the pair (f, T) is (α, ψ, φ)-almost generalized weakly contractive map for any $L \geq 2$. Also the pair (f, T) is S-weakly compatible but the pair (f, T) do not satisfy property (E.A.) and f and T do not have a common fixed point.

The following examples are in support of Corollary 4.9 .

Example 4.14. Let $X = [0, \frac{3}{2}]$ and We define $S : X^3 \to [0, \mathbb{R}^+)$ by

$$S(x, y, z) = \begin{cases} 0 & \text{if } \ x = y = z\\ \max\{x, y, z\} & \text{otherwise}\end{cases}$$

Then S is an S-metric on X. Now we define $f :: X \to X$ by

$$fx = \begin{cases} \frac{3}{5} & \text{if } x \in [0, \frac{3}{5}) \\ \frac{3}{2} & \text{if } x \in [\frac{3}{5}, \frac{3}{2}] \end{cases}$$

We now define functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = t \quad \textit{and} \quad \varphi(t) = \frac{t^2}{2} \quad \text{for all} \quad t \geq 0$$

We now verify (f, T) satisfies inequality in corollary 3.8.

<u>*Case*</u> (i): Let $x, y, z \in [0, \frac{3}{5})$.
We assume, without loss of generality, that $x > y > z$.
$S(fx, fy, fz)) = S(\frac{3}{5}, \frac{3}{5}, \frac{3}{5}) = 0$, so that the inequality (3.1) holds trivially.

<u>*Case*</u> (ii): Let $x, y, z \in [\frac{3}{5}, \frac{3}{2}]$.
We assume, without loss of generality, that $x > y > z$.
$S(fx, fy, fz)) = S(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}) = 0$, so that the inequality (3.1) holds trivially.

<u>*Case*</u> (iii): Let $y, z \in [0, \frac{3}{5})$ and $x \in [\frac{3}{5}, \frac{3}{2}]$.
We assume, without loss of generality, that $y > z$.
$S(fx, fy, fz)) = S(\frac{3}{2}, \frac{3}{5}, \frac{3}{5}) = \frac{3}{2}$, $S(x, y, z) = x$, $S(x, x, fx) = S(x, x, \frac{3}{2}) = \frac{3}{2}$, $S(y, y, fy) = S(y, y, \frac{3}{5}) = \frac{3}{5}$, $S(z, z, fz) = S(z, z, \frac{3}{5}) = \frac{3}{5}$ and $\alpha S(fx, fx, y) + (1-\alpha) S(fy, fy, z) = \alpha S(\frac{3}{2}, \frac{3}{2}, y) + (1-\alpha) S(\frac{3}{5}, \frac{3}{5}, z) = \frac{3}{5} + \frac{9}{10}\alpha$.

Now, $M_\alpha(x, y, z) = \max\{S(x, y, z), S(x, x, fx), S(y, y, fy), S(z, z, fz), \alpha S(fx, fx, y) + (1-\alpha) S(fy, fy, z)\}$
$= \max\{x, \frac{3}{2}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5} + \frac{9}{10}\alpha\} = \frac{3}{2}$.
$S(fx, x, x) = S(\frac{3}{2}, x, x) = \frac{3}{2}$, $S(fx, y, y) = S(\frac{3}{2}, y, y) = \frac{3}{2}$, $S(fx, z, z) = S(\frac{3}{2}, z, z) = \frac{3}{2}$,
$S(fx, y, z) = \frac{3}{2}$, so that $N(x, y, z) = \min\{S(fx, x, x), S(fx, y, y), S(fx, z, z), S(fx, y, z)\} = \frac{3}{2}$.
Now, $\psi(S(fx, fy, fz)) \leq L\frac{3}{2}$ for any $L \geq 1$.

<u>*Case*</u> (iv): Let $z \in [0, \frac{3}{5})$ and $x, y \in [\frac{3}{5}, \frac{3}{2}]$.
We assume, without loss of generality, that $x > y$.
$S(fx, fy, fz)) = S(\frac{3}{2}, \frac{3}{2}, \frac{3}{5}) = \frac{3}{2}$, $S(x, y, z) = x$, $S(x, x, fx) = \frac{3}{2}$, $S(y, y, fy) = \frac{3}{2}$, $S(z, z, fz) = \frac{3}{5}$ and $\alpha S(fx, fx, y) + (1-\alpha) S(fy, fy, z) = \frac{3}{2}$, so that $M_\alpha(x, y, z) = \max\{x, \frac{3}{2}, \frac{3}{2}, \frac{3}{5}, \frac{3}{2}\} = \frac{3}{2}$.
Now, $S(fx, x, x) = \frac{3}{2}$, $S(fx, y, y) = \frac{3}{2}$, $S(fx, z, z) = \frac{3}{2}$, $S(fx, y, z) = \frac{3}{2}$, so that $N(x, y, z) = \frac{3}{2}$.
Therefore $\psi(S(fx, fy, fz)) \leq L\ \frac{3}{2}$ for any $L \geq 1$.

<u>*Case*</u> (v): Let $x, y \in [0, \frac{3}{5})$ and $z \in [\frac{3}{5}, \frac{3}{2}]$.
We assume, without loss of generality, that $z > x > y$.
$S(fx, fy, fz) = \frac{3}{2}$, $S(x, y, z) = z$, $S(x, x, fx) = \frac{3}{5}$, $S(y, y, fy) = \frac{3}{5}$, $S(z, z, fz) = \frac{3}{2}$ and
$\alpha S(fx, fx, y) + (1-\alpha) S(fy, fy, z) = \frac{3}{5}\alpha + (1-\alpha)z$, so that $M_\alpha(x, y, z) = \max\{x, \frac{3}{5}, \frac{3}{5}, \frac{3}{2}, \frac{3}{5}\alpha + (1-\alpha)z\} = \frac{3}{2}$.
Now, $S(fx, x, x) = \frac{3}{5}$, $S(fx, y, y) = \frac{3}{5}$, $S(fx, z, z) = z$, $S(fx, y, z) = z$, so that $N(x, y, z) = \frac{3}{5}$.
Now, $\psi(S(fx, fy, fz)) = \psi(\frac{3}{2}) = \frac{3}{2} \leq L\frac{3}{5}$ for any $L \geq \frac{5}{2}$. <u>*Case*</u> (vi): Let $x \in [0, \frac{3}{5}]$ and $z, y \in (\frac{3}{5}, \frac{3}{2}]$. We assume, without loss of generality, that $z > y > x$. $S(fx, fy, fz) = S(\frac{3}{5}, \frac{3}{2}, \frac{3}{2}) = \frac{3}{2}$, $S(x, y, z) = z$, $S(x, x, fx) = \frac{3}{5}$, $S(y, y, fy) = \frac{3}{2}$, $S(z, z, fz) = S(z, z, \frac{3}{2}) = \frac{3}{2}$ and $\alpha S(fx, fx, y) + (1-\alpha) S(fy, fy, z) = \alpha S(\frac{3}{5}, \frac{3}{5}, y) + (1-\alpha) S(\frac{3}{2}, \frac{3}{2}, z) = \frac{3}{2} - \alpha\frac{9}{10}$,
so that $M_\alpha(x, y, z) = \max\{z, \frac{3}{5}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} - \alpha\frac{9}{10}\} = \frac{3}{2}$. Now, $S(fx, x, x) = \frac{3}{5}$, $S(fx, y, y) = y$, $S(fx, z, z) = z$, $S(fx, y, z) = z$, so that $N(x, y, z) = \frac{3}{5}$. Now, $\psi(S(fx, fy, fz)) = \psi(\frac{3}{2}) = \frac{3}{2} \leq L\frac{3}{5}$ for any $L \geq \frac{5}{2}$. <u>*Case*</u> (vii): Let $y \in [0, \frac{3}{5})$ and $x, z \in (\frac{3}{5}, \frac{3}{2}]$. We assume, without loss of generality, that $z > x$. $S(fx, fy, fz) = \frac{3}{2}$, $S(x, y, z) = x$, $S(x, x, fx) = \frac{3}{2}$, $S(y, y, fy) = \frac{3}{2}$, $S(z, z, fz) = \frac{3}{2}$ and
$\alpha S(fx, fx, y) + (1-\alpha) S(fy, fy, z) = z + (\frac{3}{2} - z)\alpha$, so that $M_\alpha(x, y, z) = \max\{z, \frac{3}{2}, \frac{3}{5}, \frac{3}{2}, z + (\frac{3}{2} - z)\alpha\} = \frac{3}{2}$.
Now, $S(fx, x, x) = \frac{3}{2}, S(fx, y, y) = y, S(fx, z, z) = \frac{3}{2}, S(fx, y, z) = \frac{3}{2}$, so that $N(x, y, z) = \frac{3}{2}$. Now, $\psi(S(fx, fy, fz)) = \psi(\frac{3}{2}) = \frac{3}{2} \leq L\frac{3}{2}$ for any $L \geq 1$. Hence f satisfies all the hypotheses of Corollary 4.9 for any $L \geq \frac{5}{2}$ and f has a unique fixed point $u = \frac{3}{2}$.
Here we observe that f is not an (α, ψ, φ)-generalized weakly contractive map. For, at $x = \frac{1}{3}$, $y = \frac{1}{4}$, $z = \frac{3}{4}$. Hence Theorem 2.23 is not applicable.

Also, f is not a contraction. For, at $x = \frac{1}{3}, y = \frac{3}{2}, z = \frac{3}{2}$ the inequality (2.1) fails to hold. Hence Theorem 2.20 is not applicable.

Example 4.15. Let $X = [\frac{1}{4}, 1]$ and We define $S : X^3 \to [0, \infty)$ by

$$S(x, y, z) = \begin{cases} 0 & \text{if } \ x = y = z \\ \max\{x, y, z\} & \text{otherwise} \end{cases}$$

Then S is an S-metric on X. Now we define $f : X \to X$ by

$$fx = \begin{cases} 2x & \text{if } \ x \in [\frac{1}{4}, \frac{1}{2}) \\ \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

We now define functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = \frac{t^3}{3} \qquad and \qquad \varphi(t) = \frac{t}{1+t} \qquad \text{for} \quad \text{all} \quad t \geq 0$$

We can easily verify that f satisfies all the hypotheses of Corollary 4.9 for any $L \geq 2$ and f has a unique fixed point $u = \frac{1}{2}$. Here we observe that the inequality (2.1) and the inequality (2.2) fail to hold when $x = \frac{3}{8}, y = \frac{1}{2}, z = \frac{3}{4}$. Hence Corollary 4.9 is a generalization of Theorem 2.20 and Theorem 2.23 which inturn Theorem 3.4 is a generalization of Theorem 2.20 and Theorem 2.23.

References

[1] M. Aamri, and D. EI. Moutawakil, *Some New Common Fixed Point Theorems under Strict Contractive Conditions*, J. Math. Anal. Appl. **270**(2002), 181–188. 1

[2] G. V. R. Babu, Dasari Ratna Babu, Kanuri Nageswara Rao, Bendi Venkata Siva Kumar, *Fixed Points Of (ψ, φ)-Almost Weakly Contractive Maps in G-Metric Spaces*, Applied Mathematics E- Notes, **4**(2014), 69–85. 1

[3] G. V. R. Babu, Leta Bekere Kumssa, *Fixed Points of (α, ψ, φ)- Generalized Weakly Contractive Maps and Property (P) in S-metric spaces*, Filomat **1:14**(2017),4469-4481,DOI 10.2298/FIL1714469B. (document), 1, 2.9, 2.22, 2.23, 2.24

[4] V. Berinde, *Approximating Fixed Points Of Weak Contractions Using The Picard Iteration*, Nonlinear Anal. Forum, **9**(2004),43–53. 1

[5] V. Berinde, *General Constructive Fixed Point Theorems for Ciric-type Almost Contractions in Metric Spaces*, Carpath. J. Math., **4**(2008),10-19. 1

[6] D. Doric, *Common Fixed Point For Generalized (ψ, φ)-Weak Contractions*, Appl. Math.Lett., **2**(2009),1896-1900.

[7] T. Dosenovic, S. Radenovic, A. Rezvani and S. Sedghi, *Coincidence Point Theorems In S-Metric Spaces Using Inegral Type Of Contraction*, U. P. B. Sci. Bull, Series A, vol. 79, Iss. 4, 2017, ISSN, 1223-7027. 2.24

[8] N. V. Dung, N.T. Hieu, and S.Radojevic, *Fixed Point Theorems for g-Monotone Maps on Partially Ordered S-metric spaces*, Filomat **28:9**(2014),1885-1898 DOI 10.2298/FIL 1409885D. 1, 2.3, 2.7

[9] P. N. Dutta and B. S. Choudhury, *A Generalization Contraction Principle in Metric Spaces*, Fixed Point Theory and Appl. (2008), Article ID 406368, 8 pages. 1

[10] G. Jungck, *Commuting Mappings And Fixed Points*, Amer. Math. **83**(1976), 261–263. 1

[11] G. Jungck, *Compatible Mappings and Common Fixed Points*, Int. J. Math. and Math. Sci., **9**(1986),771–779. 1

[12] G. Jungck and B. E. Rhodes, *Fixed Points For Set Valued Functions Without Continuity*, Indian J. Pure and appl. Math.**29**(1998),227–238. 1

[13] M. S. Khan, M. Swaleh and S.Sessa, *Fixed Point Theorems By Altering Distance Between Points*, Bull. Aust. Math. Soc. **30**(1)(1984),1–9. 2.21

[14] Z. Mustafa, B. Sims, *A New Approach To Generalized Metric Spaces*, J. Nonlinear and Convex Anal., **7**(2006), 289–297. 1

[15] N. Y. Ozgur and N. Tas, *Some Fixed Point Theorems on S-Metric Spaces*, Math. Vesnik, **69(1)**, 2017,39–52. 1

[16] S. Sedghi, N. Shobe , and H. Zhou, *A Common Fixed Point Theorem In D^*- Metric Spaces*, Fixed Point Theory Appl.(2007), Article ID 27906, (2007),13 pages.

[17] S. Sedghi, N. Shobe and A. Aliouche, *A Generalization Of Fixed Point Theorem In S-Metric Spaces*, Math. Vesnik *64*(2012),258–266. (document), 1, 2.1, 2.2, 2.6, 2.8, 2.10, 2.11, 2.19, 2.20

[18] S. Sedghi and N. V. Dung, *Fixed Point Theorems On S-Metric Spaces*, Math. Vesnik *66*(2014), 113–124. 1, 2.4

[19] H.H. Alsulami, R.P. Agarwal, E. Karapnar, F. Khojasteh,*A short note on C^*-valued contraction mappings* Journal of Inequalities and Applications, 2016,*50*(2016), doi:10.1186/s13660-016-0992-5 1

[20] M. Javahernia, A. Razani, F. Khojasteh, *Common fixed point of the generalized Mizoguchi-Takahashi's type contractions*, Fixed Point Theory Appl 2014, 195 (2014) doi:10.1186/1687-1812-2014-195