Fixed Point Results for Multivalued Operator in $G$–metric Space

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Abstract

In this paper, we shall give some results on fixed points of multivalued operator on $G$–metric spaces by using the method of Kikkawa [6]. Our results generalize and extend some old fixed point theorems to the multivalued case.

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1. Introduction and preliminaries

Mustafa and Sims [8] introduced the notion of G-metric space. Based on the notion of generalized metric space or $G$–metric space, many authors obtained some fixed point theorems for self mapping under some contractive conditions (e.g., [1, 9, 10, 11, 12]). Consistent with Mustafa and Sims [8], the following definitions and results will be needed in the sequel.

Definition 1.1. [8] Let $X$ be a non empty set, $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

$(G_1)$ $G(x,y,z) = 0$ if $x = y = z$,
$(G_2)$ $0 < G(x,x,y)$ for all $x, y \in X$ with $x \neq y$,
$(G_3)$ $G(x,y,z) \leq G(x,y) + G(y,z)$ for all $x, y, z \in X$ with $x \neq y$,
$(G_4)$ $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$ (symmetry in all three variables),
$(G_5)$ $G(x,y) \leq G(x,a) + G(a,y)$ for all $x, y, a \in X$ (rectangle inequality).

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Then the function $G$ is called a generalized metric, or, more specially, a $G$-metric on $X$, and the pair $(X, G)$ is called a $G$-metric space.

**Definition 1.2.** [8] Let $(X, G)$ be a $G$-metric space, and let $\{x_n\}$ be a sequence of points of $X$, therefore, we say that $\{x_n\}$ is $G$-convergent to $x \in X$ if $\lim_{n,m \to +\infty} G(x, x_n, x_m) = 0$, that is, for any $\epsilon > 0$, there exists a positive integer $N$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq N$. We call $x$ the limit of the sequence and write $x_n \to x$ or $\lim_{n \to +\infty} x_n = x$.

**Lemma 1.3.** [8] Let $(X, G)$ be a $G$-metric space. The following statements are equivalent:

1. $\{x_n\}$ is $G$-convergent to $x$,
2. $G(x_n, x_n, x) \to 0$ as $n \to +\infty$,
3. $G(x_n, x, x) \to 0$ as $n \to +\infty$,
4. $G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.

**Definition 1.4.** [8] Let $(X, G)$ be a $G$-metric space. A sequence $\{x_n\}$ is called a $G$-Cauchy sequence if, for any $\epsilon > 0$, there exists a positive integer $N$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

**Lemma 1.5.** [8] Let $(X, G)$ be a $G$-metric space. The following statements are equivalent:

1. The sequence $\{x_n\}$ is $G$-Cauchy,
2. for any $\epsilon > 0$, there exists a positive integer $N$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$.

**Definition 1.6.** [8] A $G$-metric space $(X, G)$ is called $G$-complete if every $G$-Cauchy sequence is $G$-convergent in $(X, G)$.

Every $G$-metric on $X$ defines a metric $d_G$ on $X$ given by

$$d_G = G(x, y, y) + G(y, x, x) \text{ for all } x, y \in X.$$

**Lemma 1.7.** [8] If $(X, G)$ is a $G$-metric space, then $G(x, y, y) = 2G(y, x, x)$ for all $x, y \in X$.

**Lemma 1.8.** [8] If $(X, G)$ is a $G$-metric space, then $G(x, x, y) = G(x, x, z) + G(z, z, y)$ for all $x, y, z \in X$.

Nadler [13] initiated the study of fixed points for multi-valued contraction mappings. There are many works about fixed point for multivalued mappings (cited in [7, 2, 3, 4, 5]) and weakly Picard maps (see in [15, 16, 17]).

We shall denote the set of all nonempty closed subset of $X$ by $P_d(X)$. Also, we shall denote the set of fixed points of a multifunction $T$ by $Fix(T)$. Let $X$ be a nonempty set and consider the space $\mathbb{R}^m$ endowed with the usual component-wise partial order. We denote by $M_{m,m}(\mathbb{R}^+)$ the set of all $m \times m$ matrices with positive elements and by $I$ the identity $m \times m$ matrix. A matrix $A \in M_{m,m}(\mathbb{R}^+)$ is said to be converges to zero whenever $A^\lambda \to 0$.

**Theorem 1.9.** [14] Let $A \in M_{m,m}(\mathbb{R}^+)$. The following are equivalent:

(i) $A^n \to 0$.
(ii) The eigen values of $A$ are in the open unit disc, i.e., $|\lambda| < 1$, for all $\lambda \in C$ with $\det(A - \lambda I) = 0$.
(iii) The matrix $(I - A)$ is non-singular and $(I - A)^{-1} = I + A + A^2 + \ldots + A^n + \ldots$.
(iv) The matrix $(I - A)$ is non-singular and $(I - A)^{-1}$ has nonnegative elements.
(v) $A^n q \to 0$ and $qA^n \to 0$, for all $q \in \mathbb{R}^m$.

By using Theorem 1.9(v), we have $-A$ converges to zero whenever $A$ is converges to zero. Again, Theorem 1.9 implies that $(I + A)$ is invertible and $(I + A)^{-1} \leq (I - A)^{-1}$.
2. Main Result

**Theorem 2.1.** Let \((X, G)\) be a complete G– metric space, a matrix \(A \in M_{m,m}(\mathbb{R}^+)\) converges to zero and \(T : X \times X \to P_{cl}(X)\) a multivalued operator. Suppose that for each \(x, y, z, x', y', z' \in X\),

\[
(I + A^{-1})[G(x, T(x, x')), T^2(x, x')) + G(x', T(x', x), T^2(x', x)) \leq (I - A^{-1})[G(x, y, z) + G(x', y', z')]
\]

implies that for each \(u \in T(x, x'), u' \in T(x', x), v \in T(y, y'), v' \in T(y', y)\) there exist \(w \in T(z, z')\) such that

\[
G(u, v, w) + G(u', v', w') \leq A[G(x, y, z) + G(x', y', z')].
\] (2.1)

Then \(T\) has a coupled fixed point.

**Proof.** For each \((x, x') \in X \times X,\)

\[
(I + A^{-1})[G(x, T(x, x'), T^2(x, x')) + G(x', T(x', x), T^2(x', x))] \\
\leq (I - A^{-1})[G(x, y, z) + G(x', y', z')].
\]

Let \((x_0, x'_0) \in X \times X\) and take \(x_1 \in T(x_0, x'_0), x'_1 \in T(x'_0, x_0), x_2 \in T(x_1, x'_1), x'_2 \in T(x'_1, x_1)\). If \(x_0 = x_1 = x_2\) and \(x'_0 = x'_1 = x'_2\) then \((x_0, x'_0)\) is a coupled fixed point of \(T\). Let any one of \(x_0, x_1, x_2\) and \(x'_0, x'_1, x'_2\) be not equal to other, from (2.1), there exist \(x_3 \in T(x_2, x'_2), x'_3 \in T(x'_2, x_2)\) such that

\[
G(x_1, x_2, x_3) + G(x'_1, x'_2, x'_3) \leq A[G(x_0, x_1, x_2) + G(x'_0, x'_1, x'_2)].
\] (2.2)

If \(x_1 = x_2 = x_3\) and \(x'_1 = x'_2 = x'_3\) then \((x_1, x'_1)\) is a coupled fixed point of \(T\). Let any one of \(x_1, x_2, x_3\) and \(x'_1, x'_2, x'_3\) be not equal to other, from (2.1) and (2.2), there exist \(x_4 \in T(x_5, x'_5), x'_4 \in T(x'_5, x_5)\) such that

\[
G(x_2, x_3, x_4) + G(x'_2, x'_3, x'_4) \leq A[G(x_1, x_2, x_3) + G(x'_1, x'_2, x'_3)] \\
\leq A^2[G(x_0, x_1, x_2) + G(x'_0, x'_1, x'_2)].
\] (2.3)

Now by induction, we construct sequences \(\{x_n\}_{n \geq 0}, \{x'_n\}_{n \geq 0}\) in \(X\) such that \(x_{n+1} \in T(x_n, x'_n), x'_{n+1} \in T(x'_n, x_n)\) and

\[
G(x_n, x_{n+1}, x_{n+2}) + G(x'_n, x'_{n+1}, x'_{n+2}) \leq A^n[G(x_0, x_1, x_2) + G(x'_0, x'_1, x'_2)]
\] (2.4)

for all \(n \geq 0\). From Theorem 1.9, for all \(m, n \in \mathbb{N}, n < m\) and by (G₃) and (G₅) we obtain

\[
G(x_n, x_m, x_m) + G(x'_n, x'_m, x'_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) \\
+ \cdots + G(x_{m-1}, x_m, x_m) + G(x'_n, x'_{n+1}, x'_{n+1}) + G(x'_{n+1}, x'_{n+2}, x'_{n+2}) \\
+ G(x'_{n+2}, x'_{n+3}, x'_{n+3}) + \cdots + G(x'_m, x'_m, x'_m)
\]

that is,

\[
[G(x_n, x_m, x_m) + G(x'_n, x'_m, x'_m)] \to 0 \text{ as } n \to \infty.
\]

Hence \(\{x_n\}_{n \geq 0}, \{x'_n\}_{n \geq 0}\) are Cauchy sequence in the complete G– metric space \((X, G)\). Choose \((x^*, x'^*) \in X \times X\) such that \(x_n \to x^*\) and \(x'_n \to x'^*\) as \(n \to \infty\). We claim that \((x, x') \in (X \times X) \setminus \{(x^*, \{x'^*\})\},\)

\[
[G(x^*, T(x, x'), T(x, x')) + G(x'^*, T(x', x'), T(x', x'))] \leq A[G(x^*, x, x) + G(x'^*, x', x')].
\] (2.5)
Let \((x, x') \in (X \times X) \setminus \{(x^*), \{x'^*\}\} \). Choose a natural number \(N\) such that
\[ [G(x_n, x^*, x^*) + G(x_n', x'^*, x'^*)] < \frac{1}{4} [G(x, x^*, x^*) + G(x', x'^*, x'^*)] \]
for all \(n \geq N\). Hence, for each \(n \geq N\) we have
\[
G(x_n, T(x_n, x'_n), T(x_n, x'_n)) + G(x'_n, T(x_n, x'_n), T(x_n, x'_n)) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x'_n, x'_{n+1}, x'_{n+1}) \\
\leq G(x_n, x^*, x^*) + G(x^*, x_{n+1}, x_{n+1}) \\
+ G(x'_n, x'^*, x'^*) + G(x'^*, x'_{n+1}, x'_{n+1}) \\
\leq G(x_n, x^*, x^*) + 2G(x_{n+1}, x^*, x^*) \\
+ G(x'_n, x'^*, x'^*) + 2G(x'_{n+1}, x'^*, x'^*) \\
\leq \frac{3}{4} [G(x, x^*, x^*) + G(x', x'^*, x'^*)] \\
\leq G(x, x^*, x^*) + G(x', x'^*, x'^*) \\
- \frac{1}{4} [G(x, x^*, x^*) + G(x', x'^*, x'^*)] \\
\leq G(x, x^*, x^*) + G(x', x'^*, x'^*) \\
- [G(x_n, x^*, x^*) + G(x'_n, x'^*, x'^*)] \\
G(x_n, T(x_n, x'_n), T(x_n, x'_n)) + G(x'_n, T(x_n, x'_n), T(x_n, x'_n)) \leq G(x_n, x^*, x^*) + G(x'_n, x'^*, x'^*). \\
\]
Thus
\[
(I + A)^{-1} [G(x_n, T(x_n, x'_n), T(x_n, x'_n)) + G(x'_n, T(x_n, x'_n), T(x_n, x'_n))] \leq \leq (I - A)^{-1} [G(x_n, x, x) + G(x'_n, x', x')] \\
\leq (I - A)^{-1} [G(x_n, x, x) + G(x'_n, x', x')] \\
\]
for \(n \geq N\).
Since \(x_{n+1} \in T(x_n, x'_n)\), \(x'_{n+1} \in T(x'_n, x_n)\), by using (2.1), for each \(n \geq N\) there exist \(u_n \in T(x, x')\) and \(u'_n \in T(x', x)\) such that
\[
G(u_n, x_{n+1}, x_{n+1}) + G(u'_n, x'_{n+1}, x'_{n+1}) \leq A[G(x_n, x, x) + G(x'_n, x', x')]. \\
\]
Hence
\[
G(x_{n+1}, T(x, x'), T(x, x')) + G(x'_{n+1}, T(x', x), T(x', x)) \leq A[G(x_n, x, x) + G(x'_n, x', x')] \\
\]
and so
\[
\lim_{n \to \infty} [G(x_{n+1}, T(x, x'), T(x, x')) + G(x'_{n+1}, T(x', x), T(x', x))] \leq \lim_{n \to \infty} [G(x_n, x, x) + G(x'_n, x', x')]. \\
\]
Thus
\[
G(x^*, T(x, x'), T(x, x')) + G(x'^*, T(x', x), T(x', x)) \leq A[G(x^*, x, x) + G(x'^*, x', x')] \\
\]
for all \((x, x') \in (X \times X) \setminus \{(x^*), \{x'^*\}\} \).
Now we show that for each \((x, x') \in X \times X\) and \(u \in T(x, x')\), \(u' \in T(x', x)\) there exist \(v \in T(x^*, x'^*), v' \in T(x'^*, x^*)\) such that
\[
G(u, v, v') \leq A[G(x, x^*, x^*) + G(x', x'^*, x'^*)]. \\
\]
If \( x_n \to x^* \) and \( x'_n \to x'^* \) we have nothing to prove. Let \( x \neq x^* \) and \( x' \neq x'^* \). By definition of \( G(x^*, T(x, x'), T(x, x')) \), \( G(x'^*, T(x', x), T(x', x)) \) and for each \( n \geq 1 \) there exist \( y_n \in T(x, x') \) and \( y'_n \in T(x', x) \) such that
\[
G(x^*, y_n, y_n) + G(x'^*, y'_n, y'_n) \leq G(x^*, T(x, x'), T(x, x')) + G(x'^*, T(x', x), T(x', x)) + \frac{1}{n}[G(x, x^*, x^*) + G(x', x'^*, x'^*)].
\]
Hence we have
\[
G(x, T(x, x'), T(x, x')) + G(x', T(x', x), T(x', x)) \leq G(x^*, y_n, y_n) + G(x'^*, y'_n, y'_n) + G(x, x^*, x^*) + G(x', x'^*, x'^*) + \frac{1}{n}[G(x, x^*, x^*) + G(x', x'^*, x'^*)].
\]
From (2.5),
\[
(I + A)^{-1}[G(x, T(x, x'), T(x, x')) + G(x', T(x', x), T(x', x))] \leq G(x, x^*, x^*) + \frac{1}{n}(I + A)^{-1}G(x, x^*, x^*) + G(x', x'^*, x'^*) + \frac{1}{n}(I + A)^{-1}G(x', x'^*, x'^*)
\]
for all \( n \geq 1 \).
Thus
\[
(I + A)^{-1}[G(x, T(x, x'), T(x, x')) + G(x', T(x', x), T(x', x))] \leq G(x, x^*, x^*) + G(x'^*, x'^*) + \frac{1}{n}(I - A)^{-1}[G(x, x^*, x^*) + G(x', x'^*, x'^*)].
\]
Now by using (2.1), for each \( u \in T(x, x') \), \( u' \in T(x', x) \), there exist \( v \in T(x^*, x'^*) \), \( v' \in T(x'^*, x^*) \) such that
\[
G(u, v, v) + G(u', v', v') \leq A[G(x, x^*, x^*) + G(x'^*, x'^*)].
\]
Since \( x_{n+1} \in T(x_n, x'_n) \) and \( x'_{n+1} \in T(x'_n, x_n) \) for all \( n \geq 1 \), there exist \( v_n \in T(x^*, x'^*) \) and \( v'_n \in T(x'^*, x^*) \) such that
\[
G(v, x_n, x_{n+1}) + G(v', x'_n, x'_{n+1}) \leq A[G(x_n, x^*, x^*) + G(x'_n, x'^*, x'^*)].
\]
Hence
\[
G(v, x^*, x^*) + G(v', x'^*, x'^*) \leq G(v, x_{n+1}, x_{n+1}) + G(x_{n+1}, x^*, x^*) + G(v', x'_{n+1}, x'_{n+1}) + G(x'_{n+1}, x'^*, x'^*) + AG(x_n, x^*, x^*) + AG(x'_n, x'^*, x'^*) + AG(x_{n+1}, x^*, x^*) + AG(x'_{n+1}, x'^*, x'^*)
\]
for all \( n \geq 1 \). Therefore \( v_n \to x^* \) and \( v'_n \to x'^* \).
Since \( v_n \in T(x^*, x'^*) \) and \( v'_n \in T(x'^*, x^*) \) for all \( n \geq 1 \) and \( T(x^*, x'^*) \) is a closed subset of \( X \times X \), \( x^* \in T(x^*, x'^*) \) and \( x'^* \in T(x'^*, x^*) \).

**Theorem 2.2.** Let \((X, G)\) be a complete \( G^-\) metric space, a matrix \( A \in M_{m,m}(\mathbb{R}^+) \) converges to zero and \( T: X \times X \to P_{\mathcal{D}}(X) \) a multivalued operator. Suppose that for each \( x, y, z, x', y', z' \in X \),
\[
(I + A)^{-1} \max\{G(x, T(x, x')), T^2(x, x'))\} \leq (I - A)^{-1} \max\{G(x, y, z), G(x', y', z')\}
\]

implies that for each \( u \in T(x, x'), u' \in T(x', x), v \in T(y, y'), v' \in T(y', y) \) there exist \( w \in T(z, z'), w' \in T(z', z) \) such that

\[
\max\{G(u, v, w), G(u', v', w')\} \leq A \max \left\{ \begin{array}{l}
G(x, y, z), G(x, T(x, x'), T(x, x')),
G(y, T(y, y'), T(y, y')), G(z, T(z, z'), T(z, z')),
G(x', y', z'), G(x', T(x', x), T(x', x)),
G(y', T(y', y), T(y', y)), G(z', T(z', z), T(z', z))
\end{array} \right\}.
\] (2.6)

Then \( T \) has a coupled fixed point.

**Proof.** For each \( (x, x') \in X \times X \),

\[
(I + A^{-1})[G(x, T(x, x'), T^2(x, x')) + G(x', T(x', x), T^2(x', x'))] \\
\leq (I - A^{-1})[G(x, T(x, x'), T^2(x, x')) + G(x', T(x', x), T^2(x', x'))].
\]

Let \( (x_0, x_0') \in X \times X \) and take \( x_1 \in T(x_0, x_0'), x_1' \in T(x_0', x_0), x_2 \in T(x_1, x_1'), x_2' \in T(x_1', x_1). \) If \( x_0 = x_1 = x_2 \) and \( x_0' = x_1' = x_2' \) then \( (x_0, x_0') \) is a coupled fixed point of \( T \). Let any one of \( x_0, x_1, x_2 \) and \( x_0', x_1', x_2' \) be not equal to other. From (2.6) there exist \( x_3 \in T(x_2, x_2'), x_3' \in T(x_2', x_2) \) such that

\[
\max\{G(x_1, x_2, x_3), G(x_1', x_2', x_3')\} \leq A \max \left\{ \begin{array}{l}
G(x_0, x_1, x_2), G(x_0, T(x_0, x_0'), T(x_0, x_0')),
G(x_1, T(x_1, x_1'), T(x_1, x_1')),
G(x_2, T(x_2, x_2'), T(x_2, x_2')),
G(x_0', x_1', x_2'), G(x_0', T(x_0', x_0), T(x_0', x_0)),
G(x_1', T(x_1', x_1), T(x_1', x_1)),
G(x_2', T(x_2', x_2), T(x_2', x_2))
\end{array} \right\}.
\] (2.7)

\[
\max\{G(x_1, x_2, x_3), G(x_1', x_2', x_3')\} \leq A \max \left\{ \begin{array}{l}
G(x_0, x_1, x_2), G(x_0, x_1, x_2), G(x_1, x_2, x_2), G(x_2, x_3, x_3),
G(x_0', x_1', x_2'), G(x_0', x_1', x_1'), G(x_1', x_2', x_2'), G(x_2', x_3', x_3')
\end{array} \right\}.
\] (2.8)

If \( x_1 = x_2 = x_3 \) and \( x_1' = x_2' = x_3' \) then \( (x_1, x_1') \) is a coupled fixed point of \( T \). Let any one of \( x_1, x_2, x_3 \) and \( x_1', x_2', x_3' \) be not equal to other, from (2.1) and (2.8) there exist \( x_4 \in T(x_5, x_5'), x_4' \in T(x_5', x_5) \) such that
Now by induction we construct sequences \( \{x_n\}_{n \geq 0} \), \( \{x'_n\}_{n \geq 0} \) in \( X \) such that \( x_{n+1} \in T(x_n, x'_n) \), \( x'_{n+1} \in T(x'_n, x_n) \) and

\[
\max\{G(x_n, x_{n+1}, x_{n+2}), G(x'_n, x'_{n+1}, x'_{n+2})\} \leq A^n \max\{G(x_0, x_1, x_2), G(x'_0, x'_1, x'_2)\} \quad (2.10)
\]

which gives

\[
G(x_n, x_{n+1}, x_{n+2}) \leq A^n G(x_0, x_1, x_2) \quad (2.11)
\]

and

\[
G(x'_n, x'_{n+1}, x'_{n+2}) \leq A^n G(x'_0, x'_1, x'_2) \quad (2.12)
\]

for all \( n \geq 0 \).

From Theorem 1.9, for all \( m, n \in N \), \( n < m \) and by \((G_3)\) and \((G_5)\) we obtain

\[
\max\{G(x_n, x_m, x_m), G(x'_n, x'_m, x'_m)\} \leq (A^n + A^{n+1} + A^{n+2} + \cdots + A^{m-1}) \max\{G(x_0, x_1, x_2), G(x'_0, x'_1, x'_2)\} \\
\leq A^n (I + A + A^2 + \cdots + A^{m-1-n}) \max\{G(x_0, x_1, x_2), G(x'_0, x'_1, x'_2)\} \\
\leq A^n (I - A)^{-1} \max\{G(x_0, x_1, x_2), G(x'_0, x'_1, x'_2)\}
\]

that is, \( \max\{G(x_0, x_1, x_2), G(x'_0, x'_1, x'_2)\} \to 0 \) as \( n \to \infty \).

Hence \( \{x_n\}_{n \geq 0}, \{x'_n\}_{n \geq 0} \) are Cauchy sequence in the complete \( G \)-metric space \((X, G)\). Choose \((x^*, x'^*) \in X \times X\) such that \( x_n \to x^* \) and \( x'_n \to x'^* \) as \( n \to \infty \). We claim that \((x, x') \in (X \times X) \setminus \{\{x^*\}, \{x'^*\}\}\),

\[
\max\{G(x^*, T(x, x'), T(x, x'))\}, G(x'^*, T(x, x'), T(x, x'))\} \leq A \max\{G(x^*, x, x), G(x'^*, x, x')\} \quad (2.13)
\]

Let \((x, x') \in (X \times X) \setminus \{\{x^*\}, \{x'^*\}\}\). Choose a natural number \( N \) such that

\[
\max\{G(x_n, x^*, x^*), G(x'_n, x'^*, x'^*)\} < \frac{1}{4} \max\{G(x, x^*, x^*), G(x', x'^*, x'^*)\}
\]

for all \( n \geq N \). Hence, for each \( n \geq N \) we have

\[
\max\{G(x_n, T(x_n, x_n'), T(x_n, x_n'))\}, G(x'_n, T(x_n, x'_n), T(x_n, x'_n))\} \leq \max\{G(x, x^*, x^*), G(x', x'^*, x'^*)\} \\
\quad - \max\{G(x_n, x^*, x^*), G(x'_n, x'^*, x'^*)\} \\
\max\{G(x_n, T(x_n, x_n'), T(x_n, x_n'))\}, G(x'_n, T(x_n, x'_n), T(x_n, x'_n))\} \leq \max\{G(x_n, x^*, x^*), G(x'_n, x'^*, x'^*)\}
\]
Thus
\[(I + A)^{-1} \max\{G(x_n, T(x_n, x'_n), T(x_n, x'_n)), G(x'_n, T(x_n, x'_n), T(x_n, x'_n))\} \leq (I - A)^{-1} \max\{G(x_n, T(x_n, x'_n), T(x_n, x'_n)), G(x'_n, T(x_n, x'_n), T(x_n, x'_n))\} \leq (I - A)^{-1} \max\{G(x_n, x, x'), G(x'_n, x, x')\}\]

for \(n \geq N\). Since \(x_{n+1} \in T(x_n, x'_n), x'_{n+1} \in T(x'_n, x_n)\), by using (2.1) for each \(n \geq N\) there exist \(u_n \in T(x, x')\) and \(u'_n \in T(x', x)\) such that

\[G(u_n, x_{n+1}, x_{n+1}) + G(u'_n, x'_{n+1}, x'_{n+1}) \leq A[G(x_n, x, x) + G(x'_n, x', x')].\]

Hence
\[
\max\{G(x_{n+1}, T(x, x')), (T(x, x')), G(x'_{n+1}, T(x', x), T(x', x'))\} \leq A \max\{G(x_n, x, x), G(x'_n, x', x')\}
\]

and so
\[
\lim_{n \to \infty} \max\{G(x_{n+1}, T(x, x'), (T(x, x')), G(x'_{n+1}, T(x', x), T(x', x'))\} \leq \lim_{n \to \infty} \max\{G(x_n, x, x), G(x'_n, x', x')\}.
\]

Thus
\[
\max\{G(x^*, T(x, x'), T(x, x'))\}, G(x'^*, T(x', x), T(x', x'))\} \leq A \max\{G(x^*, x, x), G(x'^*, x', x')\}
\]

for all \((x, x') \in X \times X \setminus \{(x^*, \{x'^*\})\}).

Now we show that for each \((x, x') \in X \times X\) and \(u \in T(x, x')\), \(u' \in T(x', x)\) there exist \(v \in T(x^*, x'^*)\), \(v' \in T(x'^*, x^*)\) such that

\[
\max\{G(u, v, v), G(u', v', v')\} \leq A \max\{G(x^*, x^*), G(x'^*, x'^*)\}.
\]

If \(x_n \to x^*\) and \(x'_n \to x'^*\) we have nothing to prove. Let \(x \neq x^*\) and \(x' \neq x'^*\). By definition of \(G(x^*, T(x, x'), T(x, x'))\), \(G(x'^*, T(x', x), T(x', x'))\) and for each \(n \geq 1\) there exist \(y_n \in T(x, x')\) and \(y'_n \in T(x', x)\) such that

\[
\max\{G(x^*, y_n, y_n), G(x'^*, y'_n, y'_n)\} \leq \max\{G(x^*, T(x, x'), T(x, x')), G(x'^*, T(x', x), T(x', x))\} + \frac{1}{n} \max\{G(x^*, x^*), G(x'^*, x'^*)\}.
\]

Hence from 2.13, we have

\[
(I + A)^{-1} \max\{G(x, T(x, x'), T(x, x')), G(x', T(x', x), T(x', x'))\} \leq \max\{G(x, x^*, x^*), G(x', x'^*, x'^*)\} + \frac{1}{n} (I + A)^{-1} \max\{G(x, x^*, x'^*), G(x, x^*), G(x', x'^*, x'^*)\},
\]

for all \(n \geq 1\). Thus
\[
(I + A)^{-1} \max\{G(x, T(x, x'), T(x, x')), G(x', T(x', x), T(x', x'))\} \leq \max\{G(x, x^*, x^*), G(x', x'^*, x'^*)\} \leq (I - A)^{-1} \max\{G(x, x^*, x^*), G(x', x'^*, x'^*)\}
\]

Now by using (2.1) for each \(u \in T(x, x')\), \(u' \in T(x', x)\) there exist \(v \in T(x^*, x'^*)\), \(v' \in T(x'^*, x^*)\) such that

\[
\max\{G(u, v, v), G(u', v', v')\} \leq A \max\{G(x^*, x^*), G(x'^*, x'^*)\}.
\]
Since $x_{n+1} \in T(x_n, x'_n)$ and $x'_{n+1} \in T(x'_n, x_n)$ for all $n \geq 1$, there exist $v_n \in T(x^*, x'^*)$ and $v'_n \in T(x'^*, x^*)$ such that

$$\max \{G(v, x_n, x_{n+1}), G(v', x'_n, x'_{n+1})\} \leq A \max \{G(x_n, x^*, x^*), G(x'_n, x'^*, x'^*)\}.$$ 

Hence

$$\max \{G(v_n, x^n, x'^*), G(v'_n, x'^n, x'^*)\} \leq A \max \{G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x^*, x^*), G(v'_n, x'_{n+1}, x'_{n+1}) + G(x'_{n+1}, x'^*, x'^*)\} \leq A \max \{G(x_n, x^n, x'^*), G(x'_n, x'^n, x'^*)\} \leq A \max \{G(x_n, x^*, x^*), G(x'_n, x'^*, x'^*)\}$$

for all $n \geq 1$. Therefore $v_n \to x^*$ and $v'_n \to x'^*$.

Since $v_n \in T(x^*, x'^*)$ and $v'_n \in T(x'^*, x^*)$ for all $n \geq 1$ and $T(x^*, x'^*)$ is a closed subset of $X \times X$, $x^* \in T(x^*, x'^*)$ and $x'^* \in T(x'^*, x^*)$, that is, $G(x_n, x_m, x_m) \to 0$ as $n \to \infty$.

Hence $\{x_n\}_{n \geq 0}$ is Cauchy sequence in the complete $G-$ metric space $(X, G)$. Choose $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. We claim that $x \in X \setminus \{x^*\}$.
Theorem 2.3. Let \((X, G)\) be a complete \(G\)-metric space, a matrix \(A \in M_{m,m}(\mathbb{R}^+\)) converges to zero and 
\(T : X \times X \to P_d(X)\) a multivalued operator and \(F : \mathbb{R}^m_+ \to \mathbb{R}^m_+\) an increasing sublinear continuous function such that 
\(F(0) = 0\) and \(F(t) > 0\) for all \(t = (t_i)_{i=1}^m \in \mathbb{R}^m_+\) where 
\[
\mathbb{R}^m_+ = \{(t_1, ..., t_m) : t_i > 0, \text{ for } i = 1, 2, 3, ..., m\}. 
\]
Suppose that for each \(x, y, z, x', y', z' \in X,\) 
\((I + A^{-1})F(\max\{G(u, v, w), G(u', v', w')\}) \leq (I - A^{-1})F(\max\{G(x, y, z), G(x', y', z')\})\) implies that for each \(u \in T(x, x'), u' \in T(x', v), v \in T(y, y'), v' \in T(y', y)\) there exist \(w \in T(z, z'), w' \in T(z', z)\) such that
\[
F(\max\{G(u, v, w), G(u', v', w')\}) \leq AF\left(\max\left\{\begin{array}{l}
G(x, y, z), G(x, T(x, x'), T(x, x')),
G(y, T(y, y'), T(y, y')), G(z, T(z, z'), T(z, z')),
G(x', y', z'), G(x', T(x', x), T(x', x)),
G(y', T(y', y), T(y', y'), G(z', T(z', z'), T(z', z'))
\end{array}\right\}\right).
\]
Then \(T\) has a coupled fixed point.

Proof. For each \((x, x') \in X \times X,\) 
\((I + A^{-1})F(\max\{G(x, T(x, x'), T^2(x, x'))\}) \leq (I - A^{-1})F(\max\{G(x', T(x', x), T^2(x', x'))\}).\)
Let \((x_0, x_0') \in X \times X\) and take \(x_1 \in T(x_0, x_0'), x_1' \in T(x_0', x_0), x_2 \in T(x_1, x_1'), x_2' \in T(x_1', x_1).\) If \(x_0 = x_1 = x_2\) and \(x_0' = x_1' = x_2'\) then \((x_0, x_0')\) is a coupled fixed point of \(T.\) Let any one of \(x_0, x_1, x_2, x_0', x_1', x_2'\) be not equal to other. From (2.6), there exist \(x_3 \in T(x_2, x_2'), x_3' \in T(x_2, x_2)\) such that
\[
F(\max\{G(x_1, x_2, x_3), G(x_1', x_2, x_3')\}) \leq AF\left(\max\left\{\begin{array}{l}
G(x_0, x_1, x_2), G(x_0, T(x_0, x_0'), T(x_0, x_0')),
G(x_1, T(x_1, x_1'), T(x_1, x_1')), G(x_2, T(x_2, x_2'), T(x_2, x_2')),
G(x_0', x_1', x_2'), G(x_0, T(x_0', x_0), T(x_0', x_0)),
G(x_1', T(x_1', x_1), T(x_1', x_1')), G(x_2', T(x_2', x_2), T(x_2', x_2))
\end{array}\right\}\right).
\]
If \(x_1 = x_2 = x_3\) and \(x_1' = x_2' = x_3'\) then \((x_1, x_1')\) is a coupled fixed point of \(T.\) Let any one of \(x_1, x_2, x_3\) and \(x_1', x_2', x_3'\) be not equal to other, from (2.1) and (2.8) there exist \(x_4 \in T(x_5, x_5), x_4' \in T(x_5', x_5')\) such that
\[
F(\max\{G(x_2, x_3, x_4), G(x_2', x_3', x_4')\}) \leq AF(\max\{G(x_1, x_2, x_3), G(x_1', x_2', x_3')\}) \leq AF(\max\{G(x_0, x_1, x_2), G(x_0', x_1', x_2')\}).
\]
Now by induction we construct sequences \(\{x_n\}_{n \geq 0}, \{x'_n\}_{n \geq 0}\) in \(X\) such that \(x_{n+1} \in T(x_n, x_n'), x_{n+1}' \in T(x_n', x_n)\) and
\[
F(\max\{G(x_n, x_{n+1}, x_{n+2}), G(x_n', x_{n+1}', x_{n+2}')\}) \leq A^n F(\max\{G(x_0, x_1, x_2), G(x_0', x_1', x_2')\})
\]
for all \(n \geq 0.\) Since \(A\) converges to zero,
\[
F(\max\{G(x_n, x_{n+1}, x_{n+2}), G(x_n', x_{n+1}', x_{n+2}')\}) \to 0.
\]
We claim that
\[
\max\{G(x_n, x_{n+1}, x_{n+2}), G(x_n', x_{n+1}', x_{n+2}')\} \to 0.
\]
If
\[
\max\{G(x_n, x_{n+1}, x_{n+2}), G(x_n', x_{n+1}', x_{n+2}')\} \to 0
\]
is not true, then there exists $\gamma \in \mathbb{R}_+^n$ such that for each $k > 0$ there is an integer number $n_k \geq k$ such that

$$\max \{G(x_{n_k}, x_{n_k+1}, x_{n_k+2}), G(x_{n_k}', x_{n_k+1}', x_{n_k+2}')\} \geq \gamma.$$ 

Hence,

$$0 < F(\gamma) \leq F(\max \{G(x_n, x_{n+1}, x_{n+2}), G(x_n', x_{n+1}', x_{n+2}')\}) \to 0.$$ 

This contradiction shows that

$$\max \{G(x_n, x_{n+1}, x_{n+2}), G(x_n', x_{n+1}', x_{n+2}')\} \to 0.$$ 

Now, from sublinearity of $F$ Theorem (1.9), for all $m, n \in N$, $n < m$ and by $(G_3)$ and $(G_5)$ we obtain

$$F(\max \{G(x_n, x_m, x_m), G(x_n', x_m', x_m')\}) \leq A^m(I - A)^{-1}F(\max \{G(x_0, x_1, x_2), G(x_0', x_1', x_2')\})$$

that is

$$F(\max \{G(x_n, x_m, x_m), G(x_n', x_m', x_m')\}) \to 0$$

as $n \to \infty$ and so

$$\max \{G(x_n, x_m, x_m), G(x_n', x_m', x_m')\} \to 0.$$ 

If $x_n \to x^*$ and $x_n' \to x'^*$ we have nothing to prove. Let $x \neq x^*$ and $x' \neq x'^*$. By definition of $G(x^*, T(x, x'), T(x, x'))$, $G(x'^*, T(x', x), T(x', x))$ and for each $n \geq 1$ there exist $y_n \in T(x, x')$ and $y_n' \in T(x', x)$ such that

$$\max \{G(x^*, y_n, y_n), G(x'^*, y_n', y_n')\} \leq \max \{G(x^*, T(x, x'), T(x, x')), G(x'^*, T(x', x), T(x', x))\}$$

$$+ \frac{1}{n} \max \{G(x, x^*, x^*), G(x', x'^*, x'^*)\}.$$ 

Hence from (2.13), we have

$$(I + A)^{-1} \max \{G(x, T(x, x'), T(x, x')), G(x', T(x', x), T(x', x))\} \leq \max \{G(x, x^*, x^*), G(x', x'^*, x'^*)\}$$

$$+ \frac{1}{n} \max \{G(x, x^*, x^*), G(x', x'^*, x'^*)\},$$

for all $n \geq 1$. Thus

$$(I + A)^{-1} \max \{G(x, T(x, x'), T(x, x')), G(x', T(x', x), T(x', x))\} \leq \max \{G(x, x^*, x^*), G(x', x'^*, x'^*)\}$$

$$\leq (I - A)^{-1} \max \{G(x, x^*, x^*), G(x', x'^*, x'^*)\}.$$ 

Now by using (2.1) for each $u \in T(x, x')$, $u' \in T(x', x)$ there exist $v \in T(x^*, x'^*)$, $v' \in T(x'^*, x^*)$ such that

$$\max \{G(u, v, v), G(u', v', v')\} \leq A \max \{G(x, x^*, x^*), G(x', x'^*, x'^*)\}.$$ 

Since $x_{n+1} \in T(x_n, x'_n)$ and $x'_{n+1} \in T(x'_n, x_n)$ for all $n \geq 1$, there exist $v_n \in T(x^*, x'^*)$ and $v'_n \in T(x'^*, x^*)$ such that

$$\max \{G(v, x_n, x_{n+1}), G(v', x'_n, x'_{n+1})\} \leq A \max \{G(x_n, x^*, x^*), G(x'_n, x'^*, x'^*)\}.$$ 

Hence
\[
\max \{G(v_n, x^*, x^*), G(v'_n, x^*, x^*)\} \leq \max \{G(v_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x^*, x^*), G(v'_n, x'_{n+1}, x'_{n+1}) + G(x'_{n+1}, x^*, x^*)\} \\
\leq A \max \{G(x_n, x^*, x^*) + G(x_{n+1}, x^*, x^*), G(x'_n, x^*, x^*) + G(x'_{n+1}, x^*, x^*)\}
\]
for all \( n \geq 1 \). Therefore \( v_n \to x^* \) and \( v'_n \to x'^* \).

Since \( v_n \in T(x^*, x'^*) \) and \( v'_n \in T(x^*, x'^*) \) for all \( n \geq 1 \) and \( T(x^*, x'^*) \) is a closed subset of \( X \times X \), so \( x^* \in T(x^*, x'^*) \) and \( x'^* \in T(x^*, x'^*) \). That is \( G(x_n, x_{n}, x_m) \to 0 \) as \( n \to \infty \).

Hence \( \{x_n\}_{n \geq 0}, \{x'_n\}_{n \geq 0} \) are Cauchy sequence in the complete \( G \)-metric space \( (X, G) \). Choose \((x^*, x'^*) \in X \times X \) such that \( x_n \to x^* \) and \( x'_n \to x'^* \) as \( n \to \infty \). We claim that \((x, x') \in (X \times X) \setminus (\{x^*\}, \{x'^*\})\).

\[
F(\max \{G(x^*, T(x, x'), T(x', x')), G(x'^*, T(x, x'), T(x', x'))\}) \leq AF(\max \{G(x^*, x, x), G(x'^*, x', x')\})
\]

Let \((x, x') \in (X \times X) \setminus (\{x^*\}, \{x'^*\})\). Choose a natural number \( N \) such that

\[
\max \{G(x_n, x^*, x^*), G(x'_n, x'^*, x'^*)\} < \frac{1}{4} \max \{G(x, x^*, x^*), G(x', x'^*, x'^*)\}
\]

for all \( n \geq N \). Hence, for each \( n \geq N \) we have

\[
F(\max \{G(x_n, T(x_n, x'_n), T(x_n, x'_n)), G(x'_n, T(x_n, x'_n), T(x_n, x'_n))\}) \leq F(\max \{G(x_n, x^*, x^*), G(x'_n, x'^*, x'^*)\})
\]

Thus

\[
(I + A)^{-1} F(\max \{G(x_n, T(x_n, x'_n), T(x_n, x'_n)), G(x'_n, T(x_n, x'_n), T(x_n, x'_n))\})
\]

\[
\leq (I - A)^{-1} F(\max \{G(x_n, T(x_n, x'_n), T(x_n, x'_n)), G(x'_n, T(x_n, x'_n), T(x_n, x'_n))\})
\]

\[
\leq (I - A)^{-1} F(\max \{G(x_n, x, x), G(x'_n, x', x')\})
\]

for \( n \geq N \). Since \( x_{n+1} \in T(x_n, x'_n) \), \( x'_{n+1} \in T(x'_n, x_n) \), by using (2.1) for each \( n \geq N \) there exist \( u_n \in T(x, x') \) and \( u'_n \in T(x', x) \) such that

\[
F(\max \{G(u_n, x_{n+1}, x_{n+1}), G(u'_n, x'_{n+1}, x'_{n+1})\}) \leq AF(\max \{G(x_n, x, x), G(x'_n, x', x')\})
\]

Hence

\[
F(\max \{G(x_{n+1}, T(x, x'), T(x, x')), G(x'_n, T(x, x'), T(x', x'))\}) \leq AF(\max \{G(x_n, x, x), G(x'_n, x', x')\})
\]

and so

\[
\lim_{n \to \infty} F(\max \{G(x_{n+1}, T(x, x'), T(x, x')), G(x'_n, T(x', x), T(x', x'))\}) \leq \lim_{n \to \infty} F(\max \{G(x_n, x, x), G(x'_n, x', x')\})
\]

Thus

\[
F(\max \{G(x^*, T(x, x'), T(x, x')), G(x'^*, T(x', x), T(x', x'))\}) \leq AF(\max \{G(x^*, x, x), G(x'^*, x', x')\})
\]

for all \((x, x') \in (X \times X) \setminus (\{x^*\}, \{x'^*\})\).

Now we show that for each \((x, x') \in X \times X \) and \( u \in T(x, x') \), \( u' \in T(x', x) \) there exist \( v \in T(x^*, x'^*) \), \( v' \in T(x'^*, x^*) \) such that

\[
F(\max \{G(u, v, v), G(u', v', v')\}) \leq AF(\max \{G(x, x^*, x^*), G(x', x'^*, x'^*)\})
\]

From (2.20),
\[(I + A)^{-1}F(\max\{G(x, T(x, x')), T(x, x')\}, G(x', T(x', x), T(x', x')))\]
\[\leq F(\max\{G(x, T(x, x')), T(x, x')\}, G(x', T(x', x), T(x', x')))\]
\[+ \frac{1}{n} (I + A)^{-1}F(\max\{G(x, T(x, x'), T(x, x')\}, G(x', T(x', x), T(x', x')))\]

for all \(n \geq 1\).

Thus
\[(I + A)^{-1}F(\max\{G(x, T(x, x'), T(x, x')\}, G(x', T(x', x), T(x', x')))\]
\[\leq F(\max\{G(x, x^*, x^*), G(x', x^*, x^*)\})\]
\[\leq (I - A)^{-1}F(\max\{G(x, x^*, x^*), G(x', x^*, x^*)\}).\]

Now by using (2.15), for each \(u \in T(x, x'), u' \in T(x', x)\) there exist \(v \in T(x^*, x^*)\), \(v' \in T(x'^*, x^*)\) such that
\[F(\max\{G(u, v, v), G(u', v', v')\}) \leq AF(\max\{G(x, x^*, x^*), G(x', x'^*, x'^*)\}).\]

Since \(x_{n+1} \in T(x_n, x'_n)\), \(x'_{n+1} \in T(x'_n, x_n)\) for all \(n \geq 1\), there exist \(v_n \in T(x^*, x^*)\), \(v'_n \in T(x'^*, x^*)\) such that
\[F(\max\{G(v, x_n, x_{n+1}), G(v', x'_n, x'_{n+1})\}) \leq AF(\max\{G(x, x^*, x^*), G(x', x'^*, x'^*)\}).\]

Hence
\[F(\max\{G(v, x_n, x^*), G(v'_n, x'^*, x'^*)\}) \leq F(\max\{G(v, x_{n+1}, x_{n+1}), G(v'_n, x'_{n+1}, x'_{n+1})\})\]
\[+ F(\max\{G(x_{n+1}, x^*, x^*), G(x'_{n+1}, x'^*, x'^*)\})\]
\[\leq AF(\max\{G(x, x^*, x^*), G(x', x'^*, x'^*)\})\]
\[+ F(\max\{G(x_{n+1}, x^*, x^*), G(x'_{n+1}, x'^*, x'^*)\})\]

for all \(n \geq 1\). Therefore \(v_n \rightarrow x^*\) and \(v'_n \rightarrow x'^*\).

Since \(v \in T(x^*, x^*)\), \(v' \in T(x'^*, x^*)\) for all \(n \geq 1\) and \(T(x^*, x^*)\) is a closed subset of \(X \times X\), \(x^* \in T(x^*, x^*)\) and \(x'^* \in T(x'^*, x^*)\).

\[\square\]

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References


