Available online at http://www.cna-journal.com
Commun. Nonlinear Anal. 6(1) (2019), 64–77
Research Article

Communications in Nonlinear Analysis

Journal Homepage: www.cna-journal.com

Stancu Type of Cheney and Sharma Operators of Pascal Rough Triple Sequences

N. Subramanian[a,*], A. Esi[b], A. Indumathi[c]

[a]School of Arts Sciences and Humanities,Department of Mathematics, SASTRA Deemed to be University, Thanjavur-613 401, India.
[b]Department of Mathematics,Adiyaman University, 02040,Adiyaman, Turkey.
[c]School of Arts Sciences and Humanities,Department of Mathematics, SASTRA Deemed to be University, Thanjavur-613 401, India.

Abstract

In this paper, we introduce stancu type extension of the well known Cheney and Sharma operators and also devoted to the definition of new rough statistical convergence with Pascal Fibonacci binomial matrix is given and some general properties of rough statistical convergence are examined. Second, approximation theory worked as a rate of the rough statistical convergence of Stancu type of Cheney and Sharma operators.

Keywords: Rough statistical convergence, natural density, triple sequences, chi sequence, Korovkin type approximation theorems, Pascal Fibonacci matrix, positive linear operator, Stancu Cheney and Sharma operators.
2010 MSC: 40F05, 40J05, 40G05.

1. Introduction

Let β be a nonnegative real number and consider the following formulas.

$$\begin{aligned}(x+y+u\beta)^u(x+y+v\beta)^v(x+y+w\beta)^w &= \sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}\binom{u}{m}\binom{v}{n}\binom{w}{k}\\ &\times u^3(u+m\beta)^{m-1}(u+n\beta)^{n-1}(u+k\beta)^{k-1}\left[v+(u-m)(v-n)(w-k)\right]^{(u-m)+(v-n)+(w-k)}\end{aligned} \tag{1}$$

*N. Subramanian
Email addresses: nsmaths@gmail.com (N. Subramanian), aesi23@hotmail.com (A. Esi), nsmaths@yahoo.com (A. Indumathi)

Received 2019-03-05

$$(x+y+u\beta)^u (x+y+v\beta)^v (x+y+w\beta)^w = \sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w} \binom{u}{m}\binom{v}{n}\binom{w}{k} (u+m\beta)^m (u+n\beta)^n$$
$$\times (u+k\beta)^k v\left[v+(u-m)(v-n)(w-k)\right]^{(u-m-1)+(v-n-1)+(w-k-1)} \tag{2}$$

$$(x+y)(x+y+u\beta)^{u-1}(x+y+v\beta)^{v-1}(x+y+w\beta)^{w-1} = \sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w} \binom{u}{m}\binom{v}{n}\binom{w}{k}$$
$$u^3 (u+m\beta)^{m-1}(u+n\beta)^{n-1}(u+k\beta)^{k-1} v\left[v+(u-m)(v-n)(w-k)\right]^{(u-m-1)+(v-n-1)+(w-k-1)}, \tag{3}$$

where $u, v \in \mathbb{R}$ and $u, v, w \geq 1$.

Cheney and Sharma generalized Bernstein polynomials by taking $\beta \geq 0, u = x$ and $v = 1-x, x \in [0,1]$, and $u = r; v = s; w = t \in \mathbb{N}$ as in the following forms:
$P^{\beta}_{rst}(f,x) = (1+r\beta)^{-r}(1+s\beta)^{-s}(1+t\beta)^{-t}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k}$
$x^3(x+m\beta)^{m-1}(x+n\beta)^{n-1}(x+k\beta)^{k-1}\left[1-x+(r-m)(s-n)(t-k)\beta\right]^{(r-m)+(s-n)+(t-k)}$
$f\left(\frac{mnk}{rst}\right) ***(4)$ and

$$G^{\beta}_{rst}(f,x) = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} P^{\beta}_{rst,mnk}(x)\, f\left(\frac{mnk}{rst}\right) \tag{1.1}$$

where

$$P^{\beta}_{rst,mnk}(x) = \frac{\binom{r}{m}\binom{s}{n}\binom{t}{k}x^3(x+m\beta)^{m-1}(x+n\beta)^{n-1}(x+k\beta)^{k-1}(1-x)\left[1-x+(r-m)(s-n)(t-k)\beta\right]^{(r-m-1)+(s-n-1)+(t-k-1)}}{(1+m\beta)^{m-1}(1+n\beta)^{n-1}(1+k\beta)^{k-1}}$$

for $f \in C[0,1]$, the space of real valued continuous functions on $[0,1]$. Denoting $e_\gamma(t) := t^\gamma, t \in [0,1], \gamma = 0,1,2,\cdots$. it is obvious that

$$G^{\beta}_{rst}(e_0;x) = 1 \tag{1.2}$$

we have

$$G^{\beta}_{rst}(e_1;x) = x \tag{1.3}$$

since $\beta \geq 0$, these operators are linear and positive and called as Bernstein type Cheney and Sharma operators. The reduction formula

$$S(mnk,rst,x,y) = x\, S((m-1)(n-1)(k-1), rst, x, y) + (rst)\beta$$
$$S(mnk,(r-1)(s-1)(t-1), x+\beta y),$$

where

$$S(mnk,rst,x,y) := \sum_{\alpha=0}^{r}\sum_{\beta=0}^{s}\sum_{\gamma=0}^{t}\left(r\alpha\right)\binom{s}{\beta}\binom{t}{\gamma}(x+\alpha)^{\alpha+m-1}(x+\beta)^{\beta+n-1}(x+\gamma)^{\gamma+k-1}$$
$$(y+(r-\alpha)(s-\beta)(t-\gamma)\beta)^{(r-\alpha)+(s-\beta)+(t-\gamma)},$$

the authors proved uniform convergence of each sequence operators $P^{\beta}_{rst}(f)$ and $G^{\beta}_{rst}(f)$ to f on $[0,1]$ by taking β as a sequence of nonnegative real numbers satisfying $\beta = O\left(\frac{1}{rst}\right), r,s,t \to \infty$. It is obvious that $P^0_{rst} = G^0_{rst} = B_{rst}$, where B_{rst} is the $(r,s,t)^{th}$ Bernstein operator.

In the present paper, we consider Stancu operators $L_{rst,uvw}$ in the basis of the Bernstein type Cheney and Sharma operators G^{β}_{rst} given by

$$L^{\beta}_{rst.uvw}(f;x) = \sum_{m=0}^{r-u}\sum_{n=0}^{s-v}\sum_{k=0}^{t-w} P^{\beta}_{(r-u,m)+(s-v,n)+(t-w,k)}(x)\left[(1-x)f\left(\frac{mnk}{rst}\right) + xf\left(\frac{(m+u)(n+v)(k+w)}{rst}\right)\right],$$

for $f \in C[0,1]$ and (u,v,w) is a nonnegative integer parameter with $r \geq 2u, s \geq 2v, t \geq 2w, r,s,t \in \mathbb{N}$, where $P^{\beta}_{(r-u,m)+(s-v,n)+(t-w,k)}$ with $(r-u)(s-v)(t-w)$ in places of (r,s,t). We shall call these operators as stancu type extension of Cheney and Sharma operators.

The triple pascal matrix is an infinite matrix containing the binomial coefficients as its elements. There are three ways to achieve this as either an upper-triangular matrix, a lower-triangular matrix or a symmetric matrix. The 4×4 truncation of these are shown below.
The triple upper triangular

$$U_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 27 & 96 \\ 0 & 0 & 1 & 500 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

Triple lower triangular

$$L_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 27 & 1 & 0 \\ 1 & 96 & 500 & 1 \end{pmatrix};$$

Symmetric

$$A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 27 & 500 & 8575 \\ 1 & 96 & 3375 & 87808 \\ 1 & 250 & 15435 & 592704 \end{pmatrix}$$

These matrices have the pleasing relationship $A_n = L_n U_n$. It is easily seen that all three matrices have determinant 1. The elements of the symmmetric triple pascal matrix are the binomial coefficients.
(i.e) $A_{ijk} = \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \begin{pmatrix} t \\ k \end{pmatrix} = \frac{r!}{m!(r-m)!} \frac{s!}{n!(n-s)!} \frac{t!}{k!(k-t)!}$, where $r, s, t = i + j + k$ and $m = i, n = j, k = t$.
In other words

$$A_{ijk} =_{i+j+k} C_{ijk} = \frac{(i+j+k)!}{i!j!k!}.$$

Thus the trace of A_n is given by

$$tr\,(A_n) = \sum_{m=0}^{r-1} \sum_{n=0}^{s-1} \sum_{k=0}^{t-1} \frac{(2m)!}{(m!)^2} \frac{(2n)!}{(n!)^2} \frac{(2k)!}{(k!)^2}$$

with the first few terms given by the sequence 1,27,729,24389, $\cdots$.
Let A_n be $n \times n \times n$ matrix whose skew diagonals are successively the rows (truncated where necessary) of pascals triangle. In general, $A_n = (a_{ijk})$, where

$$a_{ijk} = \begin{pmatrix} i+j+k \\ i \end{pmatrix} \begin{pmatrix} i+j+k \\ j \end{pmatrix} \begin{pmatrix} i+j+k \\ k \end{pmatrix} \text{ for } i, j, k = 0, 1, 2, \cdots, n-1.$$

An possesses the factorization

$$A_n = L_n L_n^T \tag{1.4}$$

where L_n^T denotes the transpose of L_n. For the $[ijk]^{th}$ secton of element of this product is
=coefficient of x^{ijk} in $(1+x)^i (1+x)^j (1+x)^k$.
$= a_{ijk} = \begin{pmatrix} i+j+k \\ i \end{pmatrix} \begin{pmatrix} i+j+k \\ j \end{pmatrix} \begin{pmatrix} i+j+k \\ k \end{pmatrix}$
clearly

$$|L_n| = 1 \tag{1.5}$$

so that

$$|A_n| = \left|L_n L_n^T\right| = |L_n|^2 = 1$$

we observe that L_n^{-1} is simply related to L_n.
For example

$$L_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -27 & 1 & 0 \\ 1 & 96 & -500 & 1 \end{pmatrix};$$

and in general

$$L_n^{-1} = (-1)^{i+j-2k} I_{ijk} \tag{1.6}$$

In addition, 1 is an eigen value of A_n when n is odd and that if λ is an eigen value of A_n then so is λ^{-1}. These conjectures are readily verified for small values of n. In general
Let

$$P_n(\lambda) = |\lambda I_n - A_n|$$

where I_n is the $n \times n \times n$ identity matrix. Then by (1.1),(1.2) and (1.3)

$$\begin{aligned} P_n(\lambda) &= \left|\lambda\, L_n L_n^{-1} - L_n L_n^T\right| \\ &= |L_n| \left|\lambda\, L_n^{-1} - L_n^T\right| \\ &= \left|\left((-1)^{i+j-2k}\, \lambda\, l_{ijk} - l_{kji}\right)\right| \\ &= (-\lambda)^n \left|\left(\lambda^{-1} l_{kji} - (-1)^{i+j-2k}\, l_{ijk}\right)\right|. \end{aligned}$$

Multiplying odd numbered rows and columns of the matrix by -1 and transposing, we get

$$P_n(\lambda) = (-\lambda)^n \left|\left((-1)^{i+j-2k}\, \lambda^{-1} l_{ijk} - l_{kji}\right)\right|$$

$$P_n(\lambda) = (-\lambda)^n P_n\left(\frac{1}{\lambda}\right) \tag{1.7}$$

But eigenvalues of A_n are the roots of $P_n(\lambda) = 0$ and thus it follows from (1.4) that if λ is an eigenvalue of A_n then so is λ^{-1}.

2. The triple Pascal matrix of inverse and triple Pascal sequence spaces

Let P denote the Pascal means defined by the Pascal matrix as is defined by

$$P = \left[P_{mnk}^{rst}\right] = \begin{cases} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \begin{pmatrix} t \\ k \end{pmatrix} & \text{if } (0 \le (m \le r, n \le s, k \le t)) \\ 0, & \text{if } ((m > r, n > s, k > t), r, s, t, m, n, k \in \mathbb{N}) \end{cases}$$

and the inverse of Pascal's matrix

$$P = \left[P_{mnk}^{rst}\right]^{-1} = \begin{cases} (-1)^{(r-m)+(s-n)+(t-k)} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \begin{pmatrix} t \\ k \end{pmatrix} & \text{if } (0 \le (m \le r, n \le s, k \le t)) \\ 0, if\, ((m > r, n > s, k > t), r, s, t, m, n, k \in \mathbb{N}) \end{cases}$$

... ... (*)
There is some interesting properties of Pascal matrix. For example, we can form three types of matrix;

symmetric, lower triangular and upper triangular; for any integer $i, j, k > 0$. The symmetric Pascal matrix of order $n \times n \times n$ is defined by

$$A_{ijk} = a_{ijk} = \begin{pmatrix} i+j+k \\ i \end{pmatrix} \begin{pmatrix} i+j+k \\ j \end{pmatrix} \begin{pmatrix} i+j+k \\ k \end{pmatrix} fori, j, k = 0, 1, 2, \cdots, n. \tag{2.1}$$

We can define the lower triangular Pascal matrix of order $n \times n \times n$ by

$$L_{ijk} = (L_{ijk}) = \frac{1}{(-1)^{i+j-2k} I_{ijk}}; i, j, k = 1, 2, \cdots n. \tag{2.2}$$

and the upper triangular Pascal matrix of order $n \times n \times n$ is defined by

$$U_{ijk} = (U_{ijk}) = \frac{1}{(-1)^{k-(i+j)} I_{ijk}}; i, j, k = 1, 2, \cdots n. \tag{2.3}$$

We know that $U_{ijk} = (L_{ijk})^T$ for any positive integer i, j, k.
(i) Let A_{ijk} be the symmetric Pascal matrix of order $n \times n \times n$ defined by $*, L_{ijk}$ be the lower triangular Pascal matrix of order $n \times n \times n$ defined by (2.2), then $A_{ijk} = L_{ijk}U_{ijk}$ and $det\,(A_{ijk}) = 1$.
(ii) Let A and B be $n \times n \times n$ matrices. We say that A is similar to B if there is an invertible $n \times n \times n$ matrix P such that $P^{-1}AP = B$.
(iii) Let A_{ijk} be the symmetric Pascal matrix of order $n \times n \times n$ defined by (2.1), then A_{ijk} is similar to its inverse A^{-1}_{ijk}.
(iv) Let L_{ijk} be the lower triangular Pascal matrix of order $n \times n \times n$ defined by (2.2), then $L_n^{-1} = L_{ijk}^{-1} = (-1)^{i+j-2k} I_{ijk}$.
We wish to introduce the Cheney and Sharma Pascal triple sequence spaces P_{Λ^3} and P_{χ^3} as the set of all sequences such that $P-$ transforms of them are in the spaces Λ^3 and χ^3, respectively, that is

$$\Lambda^3_P = \eta_{mnk} = \left\{ sup_{rst} \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \begin{pmatrix} t \\ k \end{pmatrix} \left| L^{\beta}_{rst,uvw}\,(f, x) - (f, x) \right|^{\frac{1}{m+n+k}} < \infty \right\},$$

and

$$\chi^3_P = \mu_{mnk} = \left\{ lim_{r,s,t\to\infty} \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \begin{pmatrix} t \\ k \end{pmatrix} \left((m+n+k)! \left| L^{\beta}_{rst,uvw}\,(f, x) - (f, x) \right| \right)^{\frac{1}{m+n+k}} = 0 \right\}.$$

We may redefine the spaces Λ^3_P, χ^3_P as follows:$\Lambda^3_P = P^3_\Lambda, \chi^3_P = P^3_\chi$.
If λ is an normed or paranormed sequence space; then matrix domain λ_P is called an Pascal triple sequence space. We define the triple sequence $y = (y_{rst})$ which will be frequently used, as the $P-$ transform of a triple sequence $x = (x_{rst})$
ie.,

$$y_{rst} = \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} \begin{pmatrix} r \\ m \end{pmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \begin{pmatrix} t \\ k \end{pmatrix} \left(L^{\beta}_{rst,uvw}\,(f, x) - (f, x) \right), (r, s, t \in \mathbb{N}). \tag{2.4}$$

Pascal sequence spaces P_{Λ^3} and P_{χ^3} as the set of all sequences such that $P-$ transforms of them are in the spaces Λ^3 and χ^3, respectively, that is
it can be shown easily that P_{χ^3} are linear and metric space by the following metric:

$$d\,(x, y)_{P_{\chi^3}} = d\,(Px, Py) = sup_{mnk} \left\{ ((m+n+k)!\,|x_{mnk} - y_{mnk}|)^{\frac{1}{m+n+k}} : m, n, k = 1, 2, 3, \cdots \right\}.$$

2.1. Densities and rough statistical convergence

In the theory of numbers, there are many different definitions of density. It is well known that the most popular of these definitions is asymptotic density. But, asymptotic density does not exist for all sequences. Now densities have been defined to fill those gaps and to serve different purposes.

The asymptotic density is one of the possibilities to measure how large a subset of the set of natural number. We know intuitively that positive integers are much more than perfect squares. Because, every perfect square is positive and many other positive integers exist besides. However, the set of positive integers is not in fact larger than the set of perfect squares: both sets are infinite and countable and can therefore be put in one-to-one correspondence. Nevertheless if one goes through the natural numbers, the squares become increasingly sarce. It is precisely in this case, natural density help us and makes this intuition precise.

Let α is a subset of positive integer. We consider the interval $[1, n]$ and select an integer in this interval, randomly. Then, the ratio of the number of elements of $\alpha \in [1, n]$ to the total number of elements in $[1, n]$ is belong to α, probably. For $n \to \infty$, if this probability exists, that is this probability tends to some limit, then this limit is used to as the asymptotic density of the set α. This mentions us that the asymptotic density is a kind of probability of choosing a number from the set α.

The set of positive integers will be denoted by $\mathbb{Z}^+$. Let α and β be subsets of $\mathbb{Z}^+$. If the symmetric difference $\alpha \Delta \beta$ is finite, then we can say α is asymptotically equal to β and denote $\alpha \approx \beta$. Freedman and Sember have introduced the concept of a lower asymptotic density and defined a concept of convergence in density.

2.2. Definition

Let f be a function which defined for all sets of natural numbers and take values in the interval $[0, 1]$. Then, the function f is said to a lower asymptotic density, if the following conditions hold:
(i) $f(\alpha) = f(\beta)$, if $\alpha \approx \beta$,
(ii)$f(\alpha) + f(\beta) \leq f(\alpha \bigcup \beta)$, if $\alpha \bigcap \beta = \phi$,
(iii)$f(\alpha) + f(\beta) \leq 1 + f(\alpha \bigcap \beta)$, for all α,
(iv) $f(\mathbb{Z}^+) = 1$.
We can define the upper density based on the definition of lower density as follows:
Let f be any density. Then, for any set of natural numbers α, the function $\bar{f}$ is said to upper density associated with f, if $\bar{f}(\alpha) = 1 - f(\mathbb{Z}^+ \setminus \alpha)$.

Consider the set $\alpha \subset \mathbb{Z}^+$. If $f(\alpha) = \bar{f}(\alpha)$, then we can say that the set α has natural density with respect to α. The term asymptotic density if often used for the function

$$d(\alpha) = lim_{u,v,w \to \infty} inf \tfrac{\alpha(u,v,w)}{u,v,w}, \text{ where } \alpha \subset \mathbb{N} \text{ and } \alpha(u, v, w) = \textstyle\sum_{(a,b,c) \leq (u,v,w),(a,b,c) \in \alpha} 1.$$

Also the natural density of α is given by

$$d(\alpha) = lim_{u,v,w} \tfrac{1}{uvw} |\alpha(u, v, w)|$$

where $|\alpha(u, v, w)|$ denotes the number of elements in $\alpha(u, v, w)$.

The idea of statistical convergence was introduced by *Steinhaus [24]* and also independently by *Fast [13]* for real or complex sequences. Statistical convergence is a generalization of the usual notion of convergence, which parallels the theory of ordinary convergence.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Bipan Hazarika et al. [2]*, *Sahiner et al. [17,18]*, *Esi et al. [3-10]*, *Datta et al. [11]*, *Subramanian et al. [19-23]*, *Debnath et al. [12]* and many others.

Let K be a subset of the set $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, and let us denote the set
$\{(m, n, k) \in K : m \leq u, n \leq v, k \leq w\}$ by K_{uvw}. Then the natural density of K is given by

$$\delta(K) = \texttt{lim}_{uvw \to \infty} \frac{|K_{uvw}|}{uvw},$$

where $|K_{uvw}|$ denotes the number of elements in K_{uvw}. Clearly, a finite subset has natural density zero, and we have $\delta(K^c) = 1 - \delta(K)$ where $K^c = \mathbb{N} \backslash K$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$.

Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}, m, n, k \in \mathbb{N}$.

A triple sequence $x = (x_{mnk})$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st - lim\, x = 0$, provided that the set

$$\left\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk} - 0| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence x.

If a triple sequence is statistically convergent, then for every $\epsilon > 0$, infinitely many terms of the sequence may remain outside the $\epsilon-$ neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence $x = (x_{mnk})$ satisfies some property P for all m, n, k except a set of natural density zero, then we say that the triple sequence x satisfies P for "almost all (m, n, k)" and we abbreviate this by "a.a. (m, n, k)".

Let $\left(x_{m_i n_j k_\ell}\right)$ be a sub sequence of $x = (x_{mnk})$. If the natural density of the set

$$K = \left\{(m_i, n_j, k_\ell) \in \mathbb{N}^3 : (i, j, \ell) \in \mathbb{N}^3\right\}$$

is different from zero, then $\left(x_{m_i n_j k_\ell}\right)$ is called a non thin sub sequence of a triple sequence x.

$c \in \mathbb{R}$ is called a statistical cluster point of a triple sequence $x = (x_{mnk})$ provided that the natural density of the set

$$\left\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk} - c| < \epsilon\right\}$$

is different from zero for every $\epsilon > 0$. We denote the set of all statistical cluster points of the sequence x by Γ_x.

A triple sequence $x = (x_{mnk})$ is said to be statistically analytic if there exists a positive number M such that

$$\delta\left(\left\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk}|^{1/m+n+k} \geq M\right\}\right) = 0$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

In this paper, we define the Pascal Fibonacci binomial matrix $F = \left(f_{ij\ell}^{mnk}\right)_{m,n,k=1}^{\infty}$, which differs from existing Pascal Fibonacci binomial matrix by using Fibonacci numbers $f_{ij\ell}$ and introduce some new triple sequence space of P_{χ^3} and P_{Λ^3}. Now, we define the Pascal Fibonacci binomial matrix $Ab^{rs} = Ab^{rs}_{uvw,mnk}$, where

$$b^{rs} = b^{rs}_{uvw,mnk} = \begin{cases} \frac{f_{sr}}{f_{(s+r)^{u+v+w}}} \binom{u}{m} \binom{v}{n} \binom{w}{k} s^{(u-m)+(v-n)+(w-k)} & \text{if } m \leq u, n \leq v, k \leq w \\ 0 & \text{if} m > u, n > v, k > w \end{cases}$$

The idea of rough convergence was introduced by Phu [16], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. [15] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

In this paper, we introduce the notion of rough statistical convergence of triple sequences. we obtain Cheney and Sharma operators of Pascal Fibonacci binomial matrix criteria associated with this set of rough statistical convergence. Throughout the paper r be a nonnegative real number.

2.3. Definition

A Cheney and Sharma operators of Pascal triple sequence $\mu = (\mu_{mnk})$ is said to be rough convergent ($r - convergent$) to l (Pringsheim's sense), denoted as $\mu_{mnk} \to^{r} l$, provided that

$$\forall \epsilon > 0, \exists\, i_{\epsilon} \in \mathbb{N} : m, n, k \geq i_{\epsilon} \Longrightarrow |\mu_{mnk} - l| < r + \epsilon, \tag{2.5}$$

or equivalently, if

$$\texttt{lim sup}\, |\mu_{mnk} - l| \leq r. \tag{2.6}$$

Here r is called the roughness of degree. If we take $r = 0$, then we obtain the ordinary convergence of a Cheney and Sharma operators of Pascal triple sequence.

2.4. Definition

It is obvious that the $r-$ limit set of a Cheney and Sharma operators of Pascal triple sequence is not unique. The $r-$ limit set of the Cheney and Sharma operators of Pascal triple sequence $\mu = (\mu_{mnk})$ is defined as $LIM^{r}\mu_{mnk} := \{l \in \mathbb{R} : \mu_{mnk} \to^{r} l\}$.

2.5. Definition

A Cheney and Sharma operators of Pascal triple sequence $\mu = (\mu_{mnk})$ is said to be $r-$ convergent if $LIM^{r}\mu \neq \phi$. In this case, r is called the convergence degree of the Cheney and Sharma operators of Pascal triple sequence $\mu = (\mu_{mnk})$. For $r = 0$, we get the ordinary convergence.

2.6. Definition

A Cheney and Sharma operators of Pascal triple sequence (μ_{mnk}) is said to be $r-$ statistically convergent to l, denoted by $\mu_{mnk} \to^{rst} l$, provided that the set

$$\left\{(m, n, k) \in \mathbb{N}^3 : |\mu_{mnk} - l| \geq r + \epsilon\right\}$$

has natural density zero for every $\epsilon > 0$, or equivalently, if the condition

$$st - \texttt{lim sup}\ |\mu_{mnk} - l| \leq r$$

is satisfied.

In addition, we can write $\mu_{mnk} \to^{rst} l$ if and only if the inequality

$$|\mu_{mnk} - l| < r + \epsilon$$

holds for every $\epsilon > 0$ and almost all (m, n, k). Here r is called the roughness of degree. If we take $r = 0$, then we obtain the statistical convergence of Cheney and Sharma operators of Pascal triple sequences.

2.7. Definition

A Cheney and Sharma operators of Pascal triple sequence $\mu = (\mu_{mnk})$ is said to be rough statistically Cauchy sequence if for every $\epsilon > 0$ and r be a positive number there is positive integer $N = N(r + \epsilon)$ such that
$d\left(\left\{(m, n, k) \in \mathbb{N} : \left|\mu_{mnk} - \mu_{N(r+\epsilon)}\right| \geq r + \epsilon\right\}\right) = 0.$

In a similar fashion to the idea of classical rough convergence, the idea of rough statistical convergence of a Cheney and Sharma operators of Pascal triple sequence can be interpreted as follows:

Assume that a Cheney and Sharma operators of Pascal triple sequence $\gamma = (\gamma_{mnk})$ is statistically convergent and cannot be measured or calculated exactly; one has to do with an approximated (or statistically approximated) triple sequence $\mu = (\mu_{mnk})$ satisfying $|\mu_{mnk} - \gamma_{mnk}| \leq r$ for all m, n, k (or for almost all (m, n, k), i.e.,

$$\delta\left(\left\{(m, n, k) \in \mathbb{N}^3 : |\mu_{mnk} - \gamma_{mnk}| > r\right\}\right) = 0.$$

Then the Cheney and Sharma operators of Pascal triple sequence μ is not statistically convergent any more, but as the inclusion

$$\left\{(m,n,k)\in\mathbb{N}^3 : |\gamma_{mnk}-l|\geq\epsilon\right\}\supseteq\left\{(m,n,k)\in\mathbb{N}^3 : |\mu_{mnk}-l|\geq r+\epsilon\right\} \tag{2.7}$$

holds and we have

$$\delta\left(\left\{(m,n,k)\in\mathbb{N}^3 : |\gamma_{mnk}-l|\geq\epsilon\right\}\right)=0,$$

i.e., we get

$$\delta\left(\left\{(m,n,k)\in\mathbb{N}^3 : |\gamma_{mnk}-l|\geq r+\epsilon\right\}\right)=0,$$

i.e., the Cheney and Sharma operators of Pascal triple sequence spaces μ is $r-$ statistically convergent.

2.8. Approximation theory

Korovkin type approximation theorems are practical tools to check whether a given Pascal triple sequence $(\alpha_{mnk})_{mnk\geq 1}$ of positive linear operators on $C[a,b]$ of all continuous functions on the real interval $[a,b]$ is an approximation process. That is, these theorems present a variety of test functions which provide that the approximation property holds on the whole space if it holds for them such a property was determined by Korovkin in 1953 for the functions $1,x,x^2$ in the space $C[a,b]$ as well as for the functions $1, cosx$ and $sinx$ in the space of all continuous 2π periodic functions on the real line.

3. A Cheney and Sharma operators of Pascal Fibonacci Binomial of rough statistical convergence

A Cheney and Sharma operators of Pascal sequence $\eta=(\eta_{mnk})$is said to be triple analytic if

$$sup_{m,n,k}|\eta_{mnk}|^{\frac{1}{m+n+k}}<\infty.$$

The vector space of all Cheney and Sharma operators of Pascal triple analytic sequences are usually denoted by P_{Λ^3}. The Cheney and Sharma operators of Pascal triple sequence space P_{Λ^3} is a metric space with the metric

$$d(x,y)=sup_{m,n,k}\left\{|\mu_{mnk}-\gamma_{mnk}|^{\frac{1}{m+n+k}} : m,n,k : 1,2,3,...\right\}, \tag{3.1}$$

forall $\mu=\{\mu_{mnk}\}$ and $\gamma=\{\gamma_{mnk}\}\, in\, P_{\Lambda^3}$. Then,

$$P_{\chi^3}\left(Ab^{rs}_{uvw,mnk}\right)=\left\{\mu=(\mu_{mnk})\in w : \left(Ab^{rs}_{uvw,mnk}\mu_{mnk}\right)\in P_{\chi^3}\right\}.$$

It is clear that if P_{χ^3} is a linear space then $P_{\chi^3}\left(Ab^{rs}_{uvw,mnk}\right)$ is also a linear space.

If P_{χ^3} is a complete metric space, then, $P_{\chi^3}\left(Ab^{rs}_{uvw,mnk}\right)$ is also a complete metric space with the metric

$$d(x,y)=sup\left\{|Ab^{rs}\mu-Ab^{rs}\gamma| : m,n,k=1,2,3,\cdots\right\}_{P_{\chi^3}}$$

3.1. Lemma

If $P_{\chi^3_\mu}\subset P_{\chi^3_\gamma}$ then $P_{\chi^3}\left(Ab^{rs}\mu\right)\subset P_{\chi^3}\left(Ab^{rs}\gamma\right)$.

Proof: It is trivial.

4. Main Results

4.1. Theorem

Consider that P_{χ^3} is a complete metric space and α is closed subset of P_{χ^3}. Then $\alpha\left(Ab^{rs}\right)$ is also closed in $P_{\chi^3\left(Ab^{rs}\right)}$.
Proof: Since α is a closed subset of P_{χ^3} from Lemma 3.1, the we can write

$$\alpha\left(Ab^{rs}\right) \subset P_{\chi^3}\left(Ab^{rs}\right).$$

$\overline{\alpha\left(Ab^{rs}\right)}, \bar{\alpha}$ denote the closure of $\alpha\left(Ab^{rs}\right)$ and α respectively.
It is enough to prove that $\overline{\alpha\left(Ab^{rs}\right)} = \bar{\alpha}\left(Ab^{rs}\right)$.
Firstly, we take $\mu \in \overline{\alpha\left(Ab^{rs}\right)}$, there exists a sequence $\left(\mu^{uvw}\right) \in \alpha\left(Ab^{rs}\right)$ such that
$\left|\mu^{uvw} - x\right|_{Ab^{rs}} \to 0$ in $\alpha\left(Ab^{rs}\right)$ for $u, v, w \to \infty$. Thus,
$\left|\mu^{uvw}_{mnk} - \mu_{mnk}\right|_{Ab^{rs}} \to 0$ as $u, v, w \to \infty$ in $\mu \in \alpha\left(Ab^{rs}\right)$ so that
$\sum_{r=1}^{i}\sum_{s=1}^{j}\sum_{t=1}^{\ell}\left|\mu^{uvw}_{rst} - \mu_{rst}\right| + \left|Ab^{rs}\mu^{uvw}_{mnk} - Ab^{rs}\mu_{mnk}\right| \to 0$ for $(u, v, w) \to \infty$, in α.
Therefore, $Ab^{rs}\mu \in \bar{\alpha}$ and so $\mu \in \bar{\alpha}\left(Ab^{rs}\right)$.
Conversely, if we take $\mu \in \overline{\alpha\left(Ab^{rs}\right)}$, then $\mu \in \alpha\left(Ab^{rs}\right)$. Since α is closed. Then $\overline{\alpha\left(Ab^{rs}\right)} = \bar{\alpha}\left(Ab^{rs}\right)$. Hence $\alpha\left(Ab^{rs}\right)$ is a closed subset of $P_{\chi^3}\left(Ab^{rs}\right)$.

4.2. Corollary

If P_{χ^3} is a separable space, then $P_{\chi^3}\left(Ab^{rs}\right)$ is also a separable space.

4.3. Definition

A Cheney and Sharma operators of Pascal triple sequence $\mu = \left(\mu_{mnk}\right)$ is said to be Cheney and Sharma operators of Pascal Fibonacci binomial matrix on rough statistically convergence if there is a number l such that for every $\epsilon > 0$ and r be a positive number the set
$K_{r+\epsilon}\left(Ab^{rs}\right) := \left\{(m \leq u, n \leq v, k \leq w) : \left|Ab^{rs}\mu_{mnk} - l\right| \geq r + \epsilon\right\}$ has natural density zero,
i.e; $d\left(K_{r+\epsilon}\left(Ab^{rs}\right)\right) = 0$. That is

$$lim_{u,v,w\to\infty}\frac{1}{uvw}\left|\left\{(m \leq u, n \leq v, k \leq w) : \left|Ab^{rs}\mu_{mnk} - l\right| \geq r + \epsilon\right\}\right| = 0.$$

In this case we write $d\left(Ab^{rs}\right) - lim\mu_{mnk} = l$ or $\mu_{mnk} \to l\left(rs\left(Ab^{rs}\right)\right)$. The set of $Ab^{rs}-$ rough statistically convergent of Cheney and Sharma operators of Pascal triple sequence space will be denoted by $rs\left(Ab^{rs}\right)$. In this case $l = 0$, we will write $rs_0\left(Ab^{rs}\right)$.

4.4. Definition

A Cheney and Sharma operators of Pascal triple sequence $\mu = \left(\mu_{mnk}\right)$ is said to be Cheney and Sharma operators of Pascal Fibonacci binomial matrix on rough statistically Cauchy if there exists a number $N = N\left(r + \epsilon\right)$ such that for every $\epsilon > 0$ and r be a positive number the set

$$lim_{u,v,w\to\infty}\frac{1}{uvw}\left|\left\{(m \leq u, n \leq v, k \leq w) : \left|Ab^{rs}\mu_{mnk} - Ab^{rs}\mu_N\right| \geq r + \epsilon\right\}\right| = 0.$$

4.5. Theorem

If a Cheney and Sharma operators of Pascal triple sequence space μ is a Cheney and Sharma operators of Pascal Fibonacci binomial matrix on rough statistically convergent sequence then μ is a Cheney and Sharma operators of Pascal Fibonacci binomial matrix on rough statistically Cauchy sequence.
Proof: Let $\epsilon > 0$ and r be a positive real number. Assume that $\left(\mu_{mnk}\right) \to l\left(rs\left(Ab^{rs}\right)\right)$. Then

$$\left|Ab^{rs}\mu_{mnk} - l\right| < \tfrac{r+\epsilon}{2} \text{ for almost all } m, n, k.$$

If N is chosen so that

$$\left|Ab^{rs}\mu_N - l\right| < \tfrac{r+\epsilon}{2}$$

then we have

$$|Ab^{rs}\mu_{mnk} - Ab^{rs}\mu_N| < |Ab^{rs}\mu_{mnk} - l| + |Ab^{rs}\mu_N - l| < \left(\frac{r+\epsilon}{2}\right) + \left(\frac{r+\epsilon}{2}\right) = r + \epsilon$$

for almost all m, n, k.
$\Longrightarrow \mu$ is Cheney and Sharma operators of Pascal Fibonacci binomial matrix on rough statistically Cauchy sequence.

4.6. Theorem

If μ is Cheney and Sharma operators of Pascal triple sequence for which there is a Cheney and Sharma operators of Pascal Fibonacci binomial matrix on rough statistically convergent sequence $\gamma = (\gamma_{mnk})$ such that $Ab^{rs}\mu_{mnk} = Ab^{rs}\gamma_{mnk}$ for almost all m, n, k, then μ is Cheney and Sharma operators of Pascal Fibonacci binomial matrix on rough statistically convergent sequence.

Proof: Suppose that $Ab^{rs}\mu_{mnk} = Ab^{rs}\gamma_{mnk}$ for almost all m, n, k, and $(\gamma_{mnk}) \to l\,(rs\,(Ab^{rs}))$. Then, $\epsilon > 0$ and r be a positive real number and for each u, v, w,
$\{(m \le u, n \le v, k \le w) : |Ab^{rs}\mu_{mnk} - l| \ge r + \epsilon\} \subseteq$
$\{(m \le u, n \le v, k \le w) : |Ab^{rs}\mu_{mnk} \ne Ab^{rs}\gamma_{mnk}| \ge r + \epsilon\} \bigcup$
$\{(m \le u, n \le v, k \le w) : |Ab^{rs}\mu_{mnk} - l| \le r + \epsilon\}$.
Since $(\gamma_{mnk}) \to l\,(rs\,(Ab^{rs}))$, the latter set contains a fixed number of integers, say $g = g\,(r+\epsilon)$. Therefore, for $Ab^{rs}\mu_{mnk} = Ab^{rs}\gamma_{mnk}$ for almost all m, n, k,
$lim_{u,v,w\to\infty} \frac{1}{uvw} |\{(m \le u, n \le v, k \le w) : |Ab^{rs}\mu_{mnk} - l| \ge r + \epsilon\}| \le$
$lim_{u,v,w\to\infty} \frac{1}{uvw} |\{(m \le u, n \le v, k \le w) : |Ab^{rs}\mu_{mnk} \ne Ab^{rs}\gamma_{mnk}|\}| + lim_{u,v,w} \frac{g(r+\epsilon)}{uvw} = 0.$
Hence $(\mu_{mnk}) \to l\,(rs\,(Ab^{rs}))$.

4.7. Definition

A Cheney and Sharma operators of Pascal triple sequence $\mu = (\mu_{mnk})$ is said to be rough statistically analytic if there exists some $l \ge 0$ such that

$$d\left(\left\{(m, n, k) : |\mu_{mnk}|^{1/m+n+k} > l\right\}\right) = 0,$$

i.e., $|\mu_{mnk}|^{1/m+n+k} \le l\ a.a.k$. Analytic sequences are obviously rough statistically analytic as the empty set has zero natural density. However the converse is not true.

For example, we consider the Cheney and Sharma operators of Pascal triple sequence

$$\mu = (\mu_{uvw}) = \begin{cases} (uvw)^{u+v+w}, & \text{if } m, n, k \textit{ is a square} \\ 0, & \text{if } m, n, k \textit{ is not a square} \end{cases}$$

clearly the Cheney and Sharma operators of Pascal triple sequence (μ_{mnk}) is not a analytic sequence. However,

$$d\left(\left\{(m, n, k) : |\mu_{mnk}|^{1/m+n+k} > \frac{1}{6}\right\}\right) = 0,$$

as the of squares has zero natural density and hence the Cheney and Sharma operators of Pascal triple sequence (μ_{mnk}) is rough statistically analytic.

4.8. Corollary

Every convergent sequence is rough statistically triple Cheney and Sharma operators of Pascal analytic.

4.9. Corollary

Every rough statistical convergent sequence is rough statistically triple Cheney and Sharma operators of Pascal analytic.

4.10. Corollary

Every Cheney and Sharma operators of Pascal Fibonacci binomial matrix of rough statistical convergent sequence is Cheney and Sharma operators of Pascal Fibonacci binomial matrix of rough statistically triple Cheney and Sharma operators of Pascal analytic.

5. Rate of Cheney and Sharma operators of Pascal Fibonaci binomial matrix on rough statistical convergence

Let $F(\mathbb{R})$ denote the linear space of real value function on $\mathbb{R}$. Let $C(\mathbb{R})$ be space of all real-valued continuous functions f on $\mathbb{R}$. $C(\mathbb{R})$ with the metric given as follows:

$$d\left((f,\mu),(f,\gamma)\right) = sup_{\mu\in\mathbb{R}} \left|(f,\mu)-(f,\gamma)\right|^{1/m+n+k}, f \in C(\mathbb{R})$$

and we denote $C_{2\pi}(\mathbb{R})$ the space of all $2\pi-$ periodic functions $f \in C(\mathbb{R})$ with the metric is given by

$$d\left((f,\mu),(f,\gamma)\right)_{2\pi} = sup_{t\in\mathbb{R}} \left|(f,\mu(t))-(f,\gamma(t))\right|^{1/m+n+k}, f \in C(\mathbb{R}).$$

We estimate rate of Cheney and Sharma operators of Pascal Fibonaci binomial matrix on rough statistical convergence of a triple Cheney and Sharma operators of Pascal sequence of positive linear operators defined $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$.

5.1. Definition

Let (a_{uvw}) be a positive non-increasing sequence. The triple Cheney and Sharma operators of Pascal sequence $\mu = (\mu_{mnk})$ is rate of Cheney and Sharma operators of Pascal Fibonaci binomial matrix on rough statistical convergence to l with the rate $o(a_{uvw})$ if for every $\epsilon > 0$ and r be a real number such that

$$lim_{u,v,w\to\infty} \frac{1}{h_{uvw}} \left|\left\{(m \le u, n \le v, k \le w) : \left|Ab^{rs}\mu_{mnk} - l\right| \ge r+\epsilon\right\}\right| = 0.$$

We can write $(\mu_{mnk}) - l = d(Ab^{rs}) - o(a_{uvw})$.

5.2. Lemma

Let (a_{uvw}) and (b_{uvw}) be two positive non-increasing sequences. Let $\mu = (\mu_{mnk})$ and $\gamma = (\gamma_{mnk})$ be two triple Cheney and Sharma operators of Pascal sequences such that $(\mu_{mnk}) - l_1 = d(Ab^{rs}) - o(a_{uvw})$ and $(\gamma_{mnk}) - l_2 = d(Ab^{rs}) - o(b_{uvw})$. Then we have
(i)$\alpha(\mu_{mnk} - l_1) = d(Ab^{rs}) - o(a_{uvw})$ for any scalar α,
(ii) $(\mu_{mnk} - l_1) \pm (\gamma_{mnk} - l_2) = d(Ab^{rs}) - o(c_{uvw})$,
(iii) $(\mu_{mnk} - l_1) \cdot (\gamma_{mnk} - l_2) = d(Ab^{rs}) - o(a_{uvw}b_{uvw})$, where $c_{uvw} = max\{a_{uvw}, b_{uvw}\}$.
For any $\delta > 0$, the modulus of continuity of $f, w(f,\delta)$ is defined by

$$w(f,\delta) = sup_{|\mu-\gamma|<\delta} \left|(f,\mu)-(f,\gamma)\right|.$$

A function $f \in C[a,b], lim_{u,v,w\to 0^+} w(f,\delta) = 0$. For any $\delta > 0$

$$\left|(f,\mu)-(f,\gamma)\right| \le w(f,\delta)\left(\frac{|\mu-\gamma|}{\delta}+1\right). \tag{5.1}$$

5.3. Theorem

Let (l_{mnk}) be triple Cheney and Sharma operators of Pascal sequence of positive linear operator from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$. Assume that
(i) $d(l_{mnk}((1,\mu)-\mu),0)_{2\pi} = d(Ab^{rs}) - o(h_{uvw})$
(ii) $w(f,\theta_{mnk}) = d(Ab^{rs}) - o(g_{uvw})$ where $\theta_{mnk} = \sqrt{L_{mnk}\left[sin^2\left(\frac{t-\mu}{2}\right),\mu\right]}$. Then for all $f \in C_{2\pi}(\mathbb{R})$, we get $d(l_{mnk}((f,\mu)-f(\mu)),0)_{2\pi} = d(Ab^{rs}) - o(e_{uvw})$ where $e_{uvw} = max\{h_{uvw}, g_{uvw}\}$.
Proof: Let $f \in C_{2\pi}(\mathbb{R})$ and $\mu \in [-\pi,\pi]$, we can write
$|l_{mnk}((f,\mu)-f(\mu))| \leq l_{mnk}((f,t)-f(\mu),\mu) + |f(\mu)|\,|l_{mnk}((1,\mu)-f(1))|$
$\leq l_{mnk}\left(\frac{|\mu-\gamma|}{\delta}+1,\mu\right) w(f,\delta) + |f(\mu)|\, l_{mnk}((1,\mu)-f(1))$
$\leq l_{mnk}\left(\frac{\pi^2}{\delta^2} sin^2\left(\frac{\gamma-\mu}{2}\right)+1,\mu\right) w(f,\delta) + |f(\mu)|\, l_{mnk}((1,\mu)-f(1))$
$\leq \left\{l_{mnk}(1,\mu) + \frac{\pi^2}{\delta^2} l_{mnk}\left(sin^2\left(\frac{\gamma-\mu}{2}\right),x\right)\right\} w(f,\delta) + |f(\mu)|\, l_{mnk}((1,\mu)-f(1))$
$= \left\{l_{mnk}(1,\mu) + \frac{\pi^2}{\delta^2} l_{mnk}\left(sin^2\left(\frac{\gamma-\mu}{2}\right),\mu\right)\right\} w(f,\delta) + |f(\mu)|\, l_{mnk}((1,\mu)-f(1))$.
By choosing $\sqrt{\theta_{mnk}} = \delta$, we get
$d(l_{mnk}((f,\mu)-f(\mu)),0)_{2\pi} \leq d((f,\mu),(f,\gamma))_{2\pi}$
$d(l_{mnk}((f,\mu)-f(\mu)) + 2w(f,\theta_{mnk}) + w(f,\theta_{mnk}))\, d(l_{mnk}((1,\mu)-f(\mu)),0)_{2\pi}$
$\leq K\{d(l_{mnk}((1,\mu)-f(\mu)),0)_{2\pi} + w(f,\theta_{mnk}) + w(f,\theta_{mnk})\, l_{mnk}((1,\mu)-f(\mu))_{2\pi}\}$, where

$$K = max\{2, d((f,\mu),(f,\gamma))_{2\pi}\}.$$

6. Competing Interests:

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

7. Acknowledgement

The authors are extremely grateful to the anonymous referees for their valuable suggestion and constructive comments for the improvement of the manuscript.

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