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A Predictor-Corrector Method for Fractional Delay-Differential System with Multiple Lags

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Abstract

The purpose of this work is to present numerical solutions of variable-order fractional delay differential equations with multiple lags based on the Adams-Bashforth-Moulton method, where the derivative is defined in the Caputo variable-order fractional sense. Since the variable-order fractional derivatives contain classical and fractional derivatives as special cases and also single delay is a special case of multiple delays, several results of references are significantly generalized. The error analysis for this method is given and the effectiveness of the algorithm is highlighted with numerical examples.

Keywords: Variable-order fractional calculus, delay differential equations, Adams Bashforth-Moulton method.
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1. Introduction

Due to its effectiveness in real world fractional delay differential equations is receiving importance in various branches of science [3, 5, 6], the last one is dynamical systems involving non-integer order as well as time delays. The delay introduces information from the past and introduction it in the model enriches its dynamics and allows a precise description of the real life phenomena, the applications of delay differential equations is clearly observed in many practical systems such as neuroscience, automatic control, traffic models, lasers, and so on [16, 4].

On the other hand, a fractional calculus has been acknowledged as a promising mathematical tool to efficiently describe the historical memory and hereditary properties of complex dynamic systems [19]. However, various literature indicated that the memory and hereditary properties of the system may change with time

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or other conditions [17, 18]. Hence, the variable-order fractional derivatives provide an excellent approach for the modeling of memory and hereditary properties [14, 11].
The variable-order fractional calculus are an extension of the classical fractional calculus, namely the order of fractional derivatives or integrals depends on the time and/or another variable. In 1993, Samko and Ross [9] firstly proposed the variable-order integral and differential as well as some basic properties. After that, the variable-order differential operators have been discussed by several authors [1, 10, 12]. Since the kernel of the variable order derivatives which appear in differential equations has a variable-exponent, analytical solutions of variable order fractional differential equations are more difficult to obtain, thus the effective and applicable numerical techniques for solving such equations are always needed. But numerical techniques to solve variable order fractional differential equations are at the early stage of growth. Several recent researches concerned with the exictence, uniqueness and numerical solutions of variable order fractional delay differential equations [13, 20, 21].
The main goal of this paper is to derive a numerical method to solve variable order fractional delay differential systems with multiple lags. The remainder of the work is organized as follows. In Section 2, we introduce the basic definitions of variable order Fractional Calculus. In section 3, we give the modified algorithm of Adams-Bashforth-Moulton method. In section 4, the error analysis is discussed. In Section 5, we tested the validity of algorithm by numerical examples.

2. Variable-Order Fractional Calculus

In this section, we present definitions of variable-order fractional integral and derivatives [1, 9, 10, 12].

Definition 2.1. Let $f \in C([0,T])$, $\alpha(t) \geqslant 0$, and $T>0$, the Riemann-Liouville variable order integral is defined as follows:

$$ {}_0I_t^{\alpha(t)} f(t) = \int_0^t \frac{1}{\Gamma(\alpha(t))}(t-\eta)^{\alpha(t)-1} f(\eta)d\eta, \quad t>0, \tag{2.1}$$

where $\Gamma(.)$ denotes the Gamma function.

Definition 2.2. Let $f \in C^1([0,T])$, $T>0$, $0<\alpha(t)\leqslant 1$ and be continuous, the Caputo variable order derivative is defined as follows:

$$ {}_0^C D_t^{\alpha(t)} f(t) = \begin{cases} {}_0I_t^{1-\alpha(t)} \frac{\mathrm{d}}{\mathrm{d}t} f(t), & 0<\alpha(t)<1, \\ \frac{\mathrm{d}}{\mathrm{d}t} f(t), & \alpha(t)=1. \end{cases} \tag{2.2}$$

3. Formulation of Numerical Method

Throughout this paper we denote by VOFDDEs for variable-order fractional delay-differential equations with multiple lags.
In this part, we modified the Adams-Bashforth-Moulton predictor-corrector method described in [2] to solve VOFDDEs. We consider VOFDDEs defined by

$$ {}_0^C D_t^{\alpha(t)} y(t) = \Psi\left(t, y(t), y(t-\tau_1), \ldots, y(t-\tau_k)\right), \quad t \in [0,T], \quad k \in \mathbb{N}^*, \tag{3.1}$$

$$ y(t) = \phi(t), \quad t \in [-\tau, 0], \quad \tau = \max\{\tau_1 \ldots, \tau_k\}, \tag{3.2}$$

where $y(t) = (y_1, \ldots, y_N)$, $N \in \mathbb{N}^*$, $T>0$, $0<\alpha(t)\leqslant 1$ and $\tau_j \geqslant 0$, $j=1,\ldots,k$; denotes the delay coefficients. Furthermore, we assume that $\Psi \in C\left([0,T]\times\mathbb{R}^{kN+N}, \mathbb{R}^N\right)$. Consider a uniform grid $\left\{t_i = ih: \; i=-m_j, -m_j+1, \ldots, -1, 0, 1, \ldots, n\right\}$, where m_j and n are integers such that $n=\left[\frac{T}{h}\right]$ and $m_j = \frac{\tau_j}{h}$, for $j=1,\ldots,k$. Note that

$$ y(t_i-\tau_j) = y(ih-m_jh) = y(t_{i-m_j}), \quad i=0,\ldots,n; \quad j=1,\ldots,k. \tag{3.3}$$

Now, the approximation to the delayed term $y(t_i-\tau_j)$ which consist of the following two types.

- **When τ_j is constant.** Susppose that $(m_j-\delta_j)h=\tau_j$ with $0\leqslant\delta_j<1$. When $\delta_j=0$, $y(t_i-\tau_j)$ can be approximated by

$$y(t_i-\tau_j)\approx\begin{cases}y_{i-m_j} & if \quad i>m_j\\ \phi_i & if \quad i\leqslant m_j\end{cases},\quad j=1,\dots,k. \tag{3.4}$$

When $0<\delta_j<1$, $j=1,\dots,k$ cannot be calculated directly. Let $\omega_{i+1,j}$ be the approximation to $y(t_{i+1}-\tau_j)$ for the case $(m_j-1)h<\tau_j<m_jh$, $j=1,\dots,k$. On interpolating it by the two nearest points, that is

$$\omega_{i+1,j}=\delta_j y_{i-m_j+2}+(1-\delta_j)y_{i-m_j+1}, \tag{3.5}$$

the last equality implies the implicit of the numerical equation if $m_j>1$ which can be directly determined. However, when $m_j=1$ and $\delta_j\neq 0$, that is $\tau_j<h$ the first term in the right-hand side of (3.5) is $\delta_j y_{i+1}$. Further prediction is required in this case, that is

$$\omega_{i+1,j}=\delta_j y^P_{i+1}+(1-\delta_j)y_i. \tag{3.6}$$

- **When τ_j is time varying.** If $\tau_j=\tau_j(t)$ the approximation seems to be intricate. Let $\omega_{i+1,j}\approx y(t_{i+1}-\tau_j)$, the linear interpolation of y_l at point $t=t_{i+1}-\tau_j(t_{i+1})$ is used to approximate the delay term. Let $\tau_j(t_{i+1})=(m_{i+1,j}-\delta_{i+1,j})h$ where $m_{i+1,j}\in\mathbb{Z}_+$ and $\delta_{i+1,j}\in[0,1)$, then

$$\omega_{i+1,j}=\delta_{i+1,j}y_{i-m_{i+1,j}+2}+(1-\delta_{i+1,j})y_{i-m_{i+1,j}+1}. \tag{3.7}$$

Further prediction is required if $m_{i+1,j}=1$ in the first term in the right-hand side of (3.7) and it is not needed if $m_{i+1,j}>1$. Hence in each step of the computational procedure, a condition $m_{i,j}=1$ or not is initially checked for further prediction or not.

Without loss of generality, we restrict to the first case study, we display the numerical algorithm for VOFD-DEs (3.1)-(3.2).
By applying ${}_0I^{\alpha(t_{i+1})}_{t_{i+1}}$ on both sides of (3.1) and using (3.2), we get to:

$$y(t_{i+1})=\phi(0)+\frac{1}{\Gamma(\alpha(t_{i+1}))}\int_0^{t_{i+1}}(t_{i+1}-\sigma)^{\alpha(t_{i+1}-1)}\Psi(\sigma,y(\sigma-\tau_1),\dots,y(\sigma-\tau_k))d\sigma. \tag{3.8}$$

Further, the integral in equation (3.8) is evaluated using product trapezoidal quadrature formula. Then by using (3.3) the corrector formula is thus (we denote the numerical calculation of y by $\overline{y}$)

$$\begin{aligned}\overline{y}(t_{i+1})=\phi(0)&+\frac{h^{\alpha_{i+1}}}{\Gamma(\alpha_{i+1}+2)}\Psi(t_{i+1},\overline{y}(t_{i+1}),\overline{y}(t_{i+1-m_1}),\dots,\overline{y}(t_{i+1-m_k}))\\ &+\frac{h^{\alpha_{i+1}}}{\Gamma(\alpha_{i+1}+2)}\sum_{j=0}^{i}a_{j,i+1}\Psi(t_j,\overline{y}(t_j),\overline{y}(t_{j-m_1}),\dots,\overline{y}(t_{j-m_k})),\end{aligned} \tag{3.9}$$

where $\alpha_{i+1}=\alpha(t_{i+1})$ and

$$a_{j,i+1}=\begin{cases}i^{\alpha_{i+1}+1}-(i-\alpha_{i+1})(i+1)^{\alpha_{i+1}}, & j=0,\\ (i-j+2)^{\alpha_{i+1}+1}-2(i-j+1)^{\alpha_{i+1}+1}+(i-j)^{\alpha_{i+1}+1}, & 1\leqslant j\leqslant i-1,\\ 2(2^{\alpha_{i+1}}-1), & j=i,\\ 1, & j=i+1.\end{cases} \tag{3.10}$$

Now, we replace $\overline{y}(t_{i+1})$ on the right hand side of (3.9) by an approximation $\overline{y}^P(t_{i+1})$, called predictor. Product rectangle rule is used in (3.8) to derive predictor term:

$$\overline{y}^P(t_{i+1})=\phi(0)+\frac{h^{\alpha_{i+1}}}{\Gamma(\alpha_{i+1}+1)}\sum_{j=0}^{i}b_{j,i+1}\Psi(t_j,\overline{y}(t_j),\overline{y}(t_{j-m_1}),\dots,\overline{y}(t_{j-m_k})), \tag{3.11}$$

where

$$b_{j,i+1} = \begin{cases} (i-j+1)^{\alpha_{i+1}} - (i-j)^{\alpha_{i+1}}, & 0 \leqslant j \leqslant i-1, \\ 1, & j = i. \end{cases} \tag{3.12}$$

The algorithm to solve (3.1)-(3.2) is as follows

Algorithm 1 Solving fractional delay-differential system with multiple lags.

Input: $\Psi = [\Psi_1, \dots, \Psi_N], \alpha(t), \tau = [\tau_1, \dots, \tau_k], T, \phi(t) = [\phi_1(t), \dots, \phi_N(t)], h.$

1. Compute values $n, m_j, \delta_j, \alpha_i$.
2. Set, $M = \max\{m_1, \dots, m_k\}$.
3. For $i = 0 : -1 : -M$, $\overline{y}(ih) = \phi(ih)$.
4. For $i = 1 : n$ do
   - Compute $\overline{y}(t_i - \tau_j)$;
   - Compute $\overline{y}^P(t_{i+1})$;
   - Evaluate $\omega_{i+1,j}$;
   - Compute $\overline{y}(t_{i+1})$.

Output: $\phi(-Mh), \phi(-Mh+h), \dots, \phi(-h), \phi(0), \overline{y}(t_1), \dots, \overline{y}(t_n)$.

4. Detailed Error Analysis of the Algorithm

Under the following condition on Ψ,

$$\|\Psi(t, u_1, u_2 \dots, u_{k+1}) - \Psi(t, w_1, w_2 \dots, w_{k+1})\|_{\mathbb{R}^N} \leq \sum_{j=1}^{k+1} L_j \|u_j - w_j\|_{\mathbb{R}^N}, \tag{4.1}$$

for all $t \in [0, T]$, $u_j, w_j \in \mathbb{R}^N$ and $L_j > 0$, for $j = 1, \dots, k+1$. we can get

Theorem 4.1. *Suppose the solution $y \in C^2([0, T])$ of (3.1)-(3.2) satisfies the following two conditions:*

$$\left\| \int_0^{t_{i+1}} (t_{i+1} - \sigma)_0^{\alpha_{i+1}-1\,C} D_\sigma^{\alpha_{i+1}} y(\sigma) d\sigma - \frac{h^{\alpha_{i+1}}}{\alpha_{i+1}} \sum_{j=0}^{i} b_{j,i+1\,0}{}^C D_t^{\alpha_{i+1}} y(t_j) \right\|_{\mathbb{R}^N} \leq C t_{i+1}^{\gamma_1} h^{\theta_1}, \tag{4.2}$$

$$\left\| \int_0^{t_{i+1}} (t_{i+1} - \sigma)^{\alpha_{i+1}-1}{}_0{}^C D_\sigma^{\alpha_{i+1}} y(\sigma) d\sigma - \frac{h^{\alpha_{i+1}}}{\alpha_{i+1}(\alpha_{i+1}+1)} \sum_{j=0}^{i} a_{j,i+1\,0}{}^C D_t^{\alpha_{i+1}} y(t_j) \right\|_{\mathbb{R}^N} \leq C t_{i+1}^{\gamma_2} h^{\theta_2}, \tag{4.3}$$

with some $\gamma_1, \gamma_2 \geqslant 0$, and $\theta_1, \theta_2 > 0$, then for some suitable $T > 0$, we have

$$\max_{0 \leqslant j \leqslant n} \|y(t_j) - \overline{y}(t_j)\|_{\mathbb{R}^N} \leq \mathcal{K} h^q, \tag{4.4}$$

where $n = [\frac{T}{h}]$, $q = \min\{\theta_1 + \alpha(t), \theta_2\}$, and $C, \mathcal{K}$ are a positive constants.

Proof. We prove the result by using the mathematical induction. Suppose that the conclusion is true for $j = 0, \dots, i$. From the assumptions (4.1), note that

$$\|\Psi(t, u_1, u_2 \dots, u_{k+1}) - \Psi(t, w_1, w_2 \dots, w_{k+1})\|_{\mathbb{R}^N} \leq C_N h^q \sum_{j=1}^{k+1} L_j, \tag{4.5}$$

where $C_N > 0$, from (3.1), (4.5) and the following inequality

$$\frac{h^{\alpha_{i+1}}}{\alpha_{i+1}} \sum_{j=0}^{i} b_{j,i+1} \leqslant \frac{T^{\alpha_{i+1}}}{\alpha_{i+1}}, \tag{4.6}$$

we have

$$\left\|y(t_{i+1}) - \overline{y}^P(t_{i+1})\right\|_{\mathbb{R}^N} \leq \frac{CT^{\gamma_1}}{\Gamma(\alpha_{i+1})} h^{\theta_1} + \frac{C_1 T^{\alpha_{i+1}}}{\Gamma(\alpha_{i+1}+1)} h^q, \tag{4.7}$$

where $C_1 > 0$. Since

$$\frac{h^{\alpha_{i+1}}}{\alpha_{i+1}(\alpha_{i+1}+1)} \sum_{j=0}^{i} a_{j,i+1} \leq T^{\alpha_{i+1}} \tag{4.8}$$

and thanks to (4.3), (4.5), (4.7), we have

$$\|y(t_{i+1}) - \overline{y}(t_{i+1})\|_{\mathbb{R}^N} \leq \mathcal{K} h^q. \tag{4.9}$$

For the detailed, see [13]. □

5. Numerical Examples

The computer code of Algorithm 1, was written in Matlab, and the time step used in the simulation was $h = \frac{1}{10}$.

Example 5.1. Consider a VOFDDEs version of delay differential equation given in [15]

$$\begin{cases} {}_0^C D_t^{\alpha(t)} y(t) = \frac{2y(t-2)}{1+y(t-2.6)^{9.65}} - y(t), \\ \\ \qquad y(t) = 0.5, \quad t \leqslant 0. \end{cases} \tag{5.1}$$

The approximate solution of (5.1) for fractional derivative $\alpha = 0.95$ is shown in Fig. 1, whereas Fig. 2 shows phase portrait of the system i.e. plots of $y(t)$ versus $y(t-2)$ and $y(t)$ versus $y(t-2.6)$ for the same value of α. It be analysed from this figures that the system (5.1) shows chaotic behaviour almost different to those generated by a single delay (see [13]).
Fig. 3 shows the numerical solution of (5.1) for the variable order fractional derivative $\alpha(t) = 0.93 - \frac{1}{e^t+1}$, and phase portrait of the system (5.1) for $\alpha(t) = 0.93 - \frac{1}{e^t+1}$ is shown in Fig. 4. In the following experiment, we have changed the variable order fractional derivative by $\alpha(t) = \frac{2.8+\sin(t)}{4}$, and the approximate solution is shown in Fig. 5, Fig. 6 shows the phase portrait of the system (5.1) for the same value of derivative.
The portrait phase of the variable order fractional derivative (Fig. 4 and Fig. 6) shows us that the chaotic behaviour is more complicated than fractional derivative and thus the general behavior of the solution of system (5.1) depends on kind of derivative.

Example 5.2. Consider a VOFDDEs version of four dimensional enzyme kinetics with an inhibitor molecule given in [7]

$$\begin{cases} {}_0^C D_t^{\alpha(t)} y_1(t) &=& 10.5 - \frac{y_1(t)}{1+0.0005 y_4^3(t-4)}, \\ {}_0^C D_t^{\alpha(t)} y_2(t) &=& \frac{y_1(t)}{1+0.0005 y_4^3(t-4.8)} - y_2(t), \\ {}_0^C D_t^{\alpha(t)} y_3(t) &=& y_2(t) - y_3(t), \\ {}_0^C D_t^{\alpha(t)} y_4(t) &=& y_3(t) - y_4(t), \\ y(t) &=& [60, 10, 10, 20]^T, \quad t \leqslant 0. \end{cases} \tag{5.2}$$

where $y(t) = [y_1(t), y_2(t), y_3(t), y_4(t)]^T$.

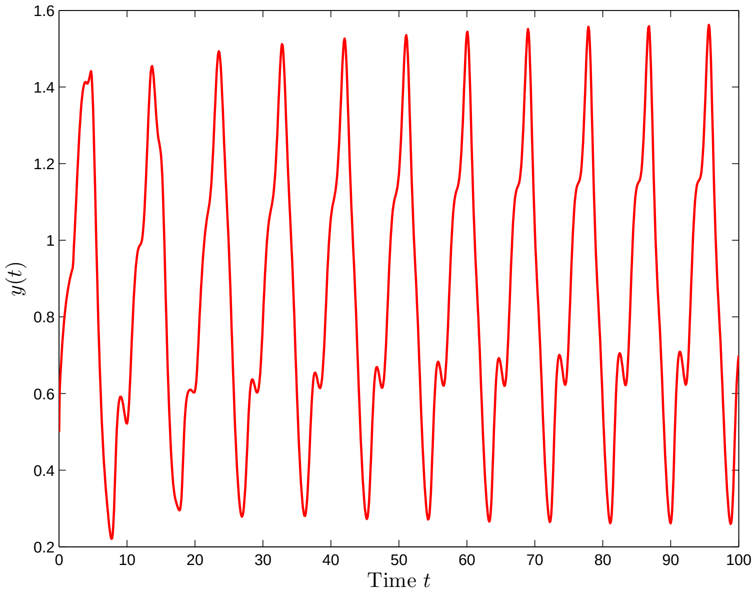

Figure 1: The numerical solution of system (5.1) with fractional derivative.

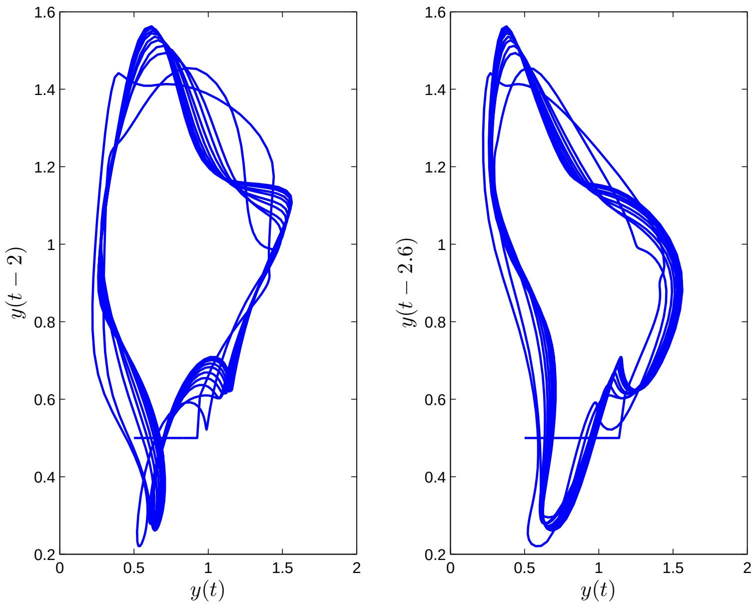

Figure 2: Phase plot of system (5.1) with fractional derivative.

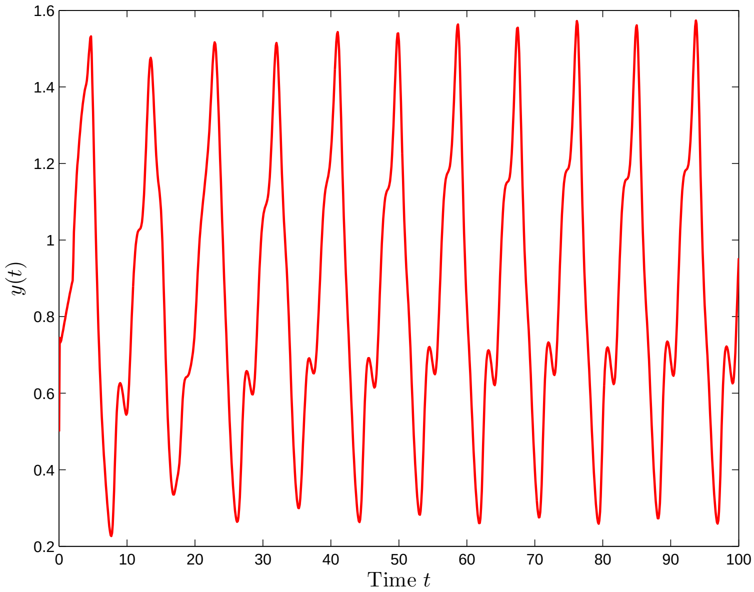

Figure 3: The numerical solution of system (5.1) with VO fractional derivative.

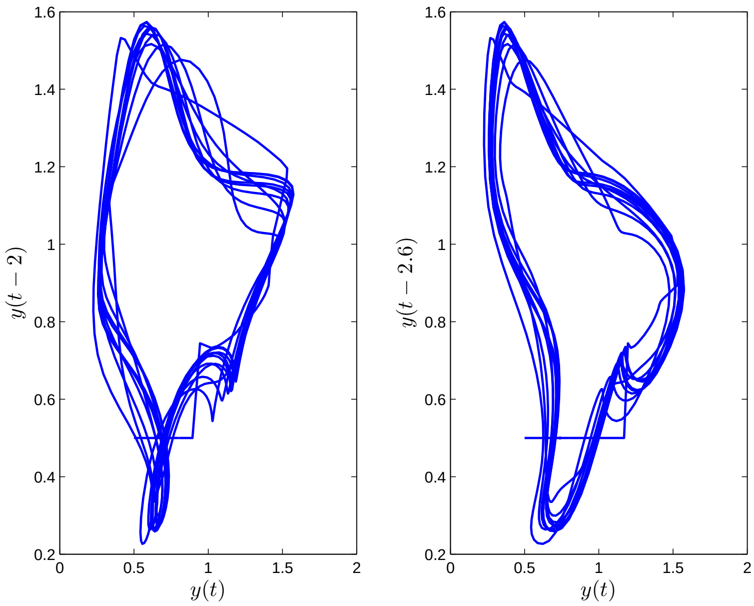

Figure 4: Phase plot of system (5.1) with VO fractional derivative.

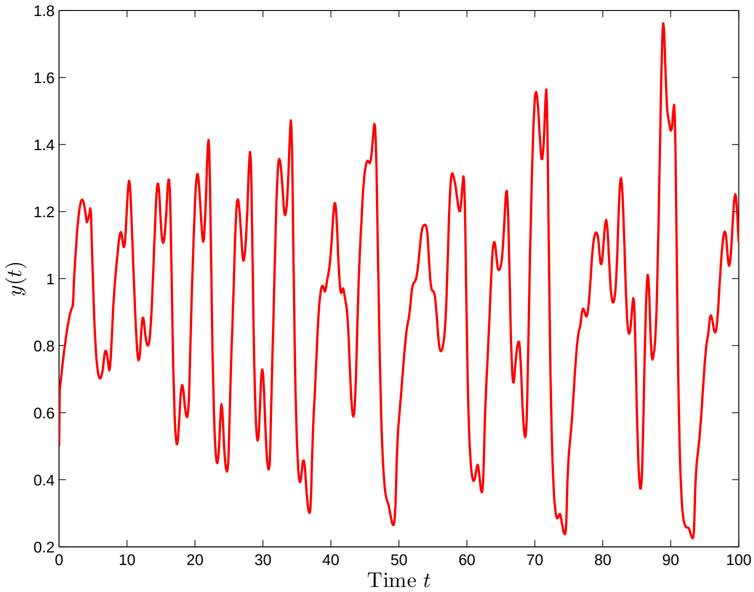

Figure 5: The numerical solution of system (5.1) with VO fractional derivative.

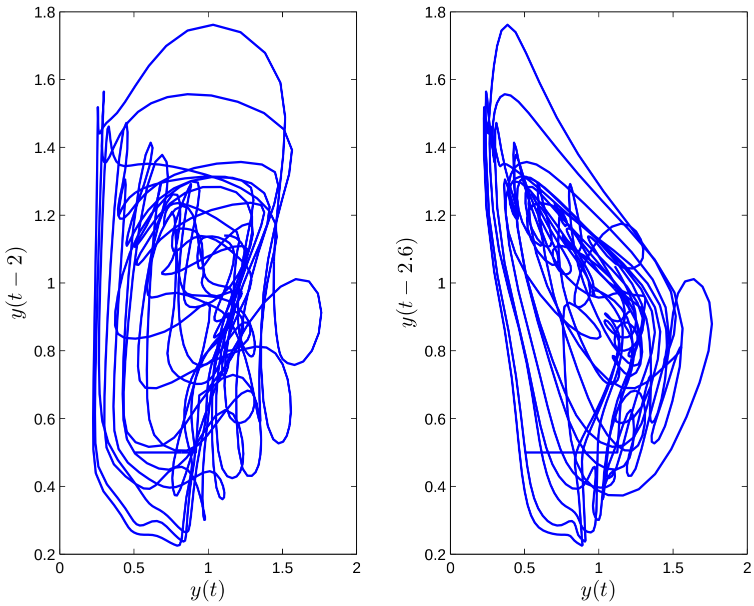

Figure 6: Phase plot of system (5.1) with VO fractional derivative.

The approximate solution of (5.2) for fractional derivative $\alpha = 0.98$ is shown in Fig. 7, which looks a bit different to the system (5.2) generated by a single delay (see [13]). In Fig. 8, we depict approximate solutions of system (5.2) for variable order fractional derivative $\alpha(t) = 0.97 - \frac{1}{e^t+1}$, Fig. 9 shows numerical solution of system (5.2) for variable order fractional derivative $\alpha(t) = 0.93 - \frac{1}{e^t+1}$. In the following, we have selected the variable order fractional derivative $\alpha(t) = \frac{2.8+\sin(t)}{4}$, it has many variation on $[0, 160]$, Fig. 10, showed a completely different behavior of system (5.2).

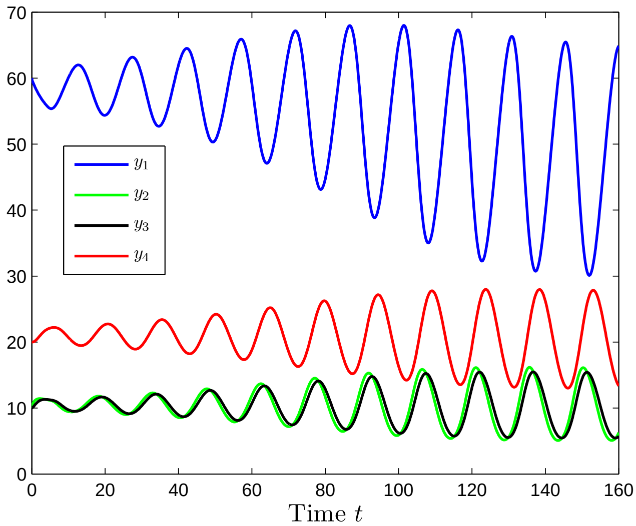

Figure 7: The numerical solution of system (5.2) with fractional derivative.

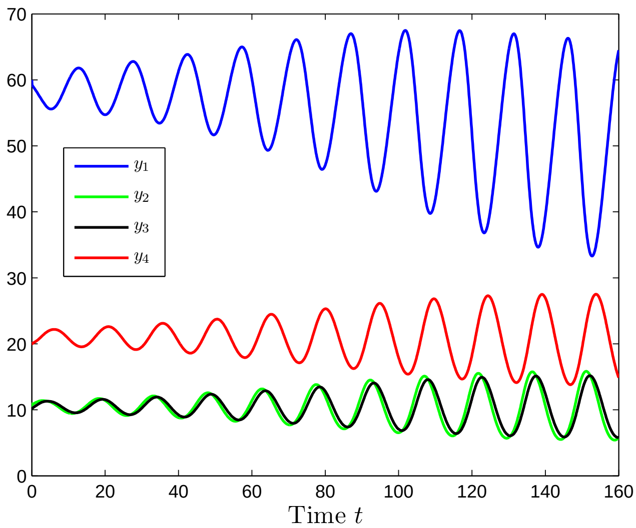

Figure 8: The numerical solution of system (5.2) with VO fractional derivative.

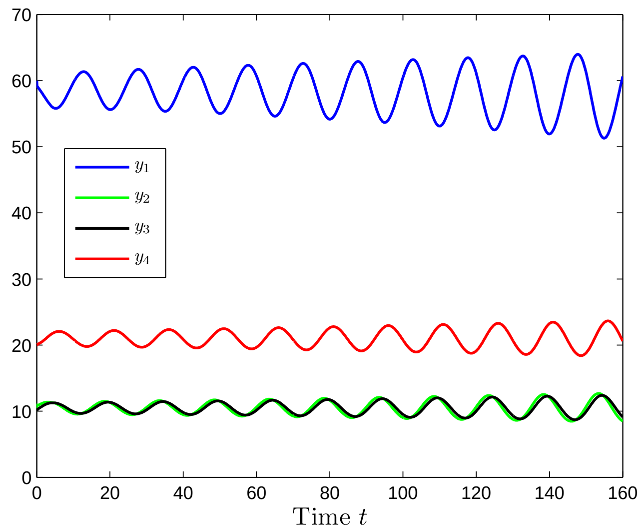

Figure 9: The numerical solution of system (5.2) with VO fractional derivative.

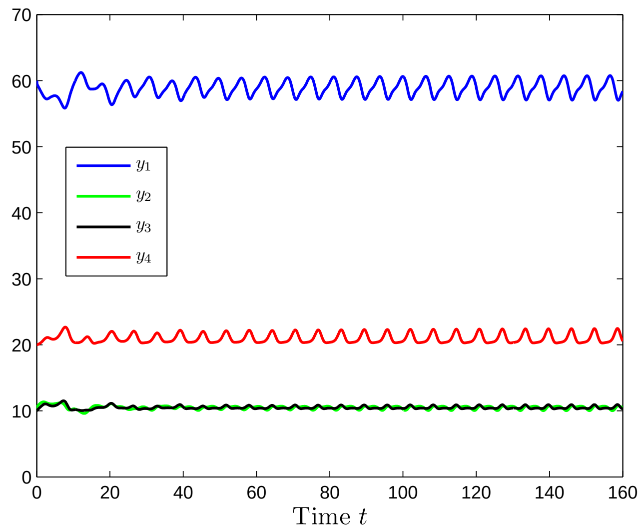

Figure 10: The numerical solution of system (5.2) with VO fractional derivative.

6. Conclusion

In this paper, we derived a numerical approximation to the variable-order fractional delay differential equations with multiple lags, based on Adams-Bashforth-Moulton method. The detailed error analysis of the numerical method was studied under a specific condition on the non-linear term. Numerical examples showed the efficiency of the suggested algorithm. Another new approach can be applied to solve the variable-order fractional delay differential equations with multiple lags as a future work.

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