Available online at http://www.cna-journal.com
Commun. Nonlinear Anal. 6(1) (2019), 23–31
Research Article

Communications in Nonlinear Analysis

Journal Homepage: www.cna-journal.com

Fixed Point Theorems For Dislocated Quasi G-Fuzzy Metric Spaces

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Abstract

The aim of this paper is to introduce the new concept of ordered complete dislocated quasi G-fuzzy metric space. The notion of dominated mappings is applied to approximate the unique solution of nonlinear functional equations. In this paper, we find the fixed point results for mappings satisfying the locally contractive conditions on a closed ball in an ordered complete dislocated quasi G-fuzzy metric space.

Keywords: Fixed Point, G-Fuzzy Metric Spaces, Closed Ball, Dislocated Quasi Metric Spaces.
2010 MSC: 54A40, 47H10.

1. Introduction

Mustafa and Sims [10] introduced a G- metric space and obtained some fixed point theorems in it. Some interesting references in G- metric spaces are [2,8,9,12,13,14,16]. In this paper, we find the fixed point results for mappings satisfying the locally contractive conditions on a closed ball in an ordered complete dislocated quasi G-fuzzy metric space. Our results improve several well known classical results.

Definition 1.1. Let X be a nonempty set and let $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all$x, y \in X$ with $x \neq y$,

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Received 2019-03-12

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$, symmetry in all three variables,

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ is called a G- metric space.

Definition 1.2. The G- metric space (X, G) is called symmetric if $G(x, x, y) = G(x, y, y)$ for all $x, y \in X$.

Definition 1.3. A 3-tuple $(X, G, *)$ is called a G- fuzzy metric space if X is an arbitrary nonempty set, $*$ is a continuous t-norm, and G is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions for each $t, s > 0$:

(i) $G(x, x, y, t) > 0$ for all $x, y \in X$ with $x \neq y$,

(ii) $G(x, x, y, t) \geq G(x, y, z, t)$ for all $x, y, z \in X$ with $y \neq z$,

(iii) $G(x, y, z, t) = 1$ if and only if $x = y = z$,

(iv) $G(x, y, z, t) = G(p(x, y, z), t)$, where p is a permutation function,

(v) $G(x, y, z, t + s) \geq G(a, y, z, t) * G(x, a, a, s)$ for all $x, y, z, a \in X$,

(vi) $G(x, y, z, .) : (0, \infty) \to [0, 1]$ is continuous.

Definition 1.4. A G- fuzzy metric space $(X, G, *)$ is said to be symmetric if $G(x, x, y, t) = G(x, y, y, t)$ for all $x, y \in X$ and for each $t > 0$.

Definition 1.5. Let X be a nonempty set and let $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following axioms:

(i) If $G(x, y, z, t) = G(y, z, x, t) = G(z, x, y, t) = 1$, then $x = y = z$,

(ii) $G(x, y, z, t) \geq G(x, a, a, t) + G(a, y, z, t)$ for all $x, y, z, a \in X$. (Rectangle inequality).
Then the pair $(X, G, *)$ is called the dislocated quasi G-metric space.

It is clear that if $G(x, y, z, t) = G(y, z, x, t) = G(z, x, y, t) = 1$, then from (i) $x = y = z$. But if $x = y = z$ then $G(x, y, z, t)$ may not be 0. It is observed that if $G(x, y, z, t) = G(y, z, x, t) = G(z, x, y, t)$

Example 1.6. If $X = R^+ \cup \{0\}$ then $G(x, y, z, t) = x + max\{x, y, z, t\}$ defines a dislocated quasi metric on X.

Definition 1.7. Let $(X, G, *)$ be a G-fuzzy metric space and let $\{x_n\}$ be a sequence of points in X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $lim_{m,n\to\infty} G(x, x_n, x_m, t) = 1$, and one says that sequence $\{x_n\}$ is G-convergent to x. Thus, if $x_n \to x$ in a dislocated quasi G-fuzzy metric space $(X, G, *)$, then for any $\epsilon > 0$, there exists $n, m \in N$ such that $G(x, x_n, x_m, t) > \epsilon$, for all $n, m \geq N$.

Definition 1.8. Let $(X, G, *)$ be a dislocated quasi G-fuzzy metric space. A sequence $\{x_n\}$ is called G-Cauchy sequence if, for each $\epsilon > 0$ there exists a positive integer $n \in N$ such that $G(x_n, x_m, x_l, t) > \epsilon$ for all $n, m, l \geq N$; i.e. if, $G(x_n, x_m, x_l, t) \to 1$ as $n, m, l \to \infty$.

Definition 1.9. A dislocated quasi G-fuzzy metric space $(X, G, *)$ is said to be G-complete if every G-Cauchy sequence in $(X, G, *)$ is G-convergent in X.

Proposition 1.10. *Let (X, G) be a dislocated quasi G-fuzzy metric space, then the following are equivalent:*

(i) $\{x_n\}$ is G convergent to x,

(ii) $G(x_n, x_n, x, t) \to 1$ *as* $n \to \infty$,

(iii) $G(x_n, x, x, t) \to 1$ *as* $n \to \infty$,

(iv) $G(x_n, x_m, x, t) \to 1$ *as* $m, n \to \infty$.

Definition 1.11. Let $(X, G, *)$ be a G-fuzzy metric space. For $x_0 \in X, r > 0$, the G-ball with centre x_0 and radius r is, $\overline{(B(x_0, r, t))} = \{y \in X : G(x_0, y, y, t) \geq 1 - r\}$.

Definition 1.12. Let $(X, \preceq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 1.13. Let $(X, \preceq)$ be a partially ordered set. A self mapping f on X is called dominated if $fx \preceq x$ for each $x \in X$.

Example 1.14. Let $X = [0, 1]$ be endowed with usual ordering and $f : X \to X$ be defined by $fx = x^n$ for some $n \in N$. Since $fx = x^n \preceq x$ for all $x \in X$, therefore f is a dominated map.

2. Main Result

Theorem 2.1. *Let $(X, \preceq)$ be partially ordered set and let $(X, G, *)$ be an complete dislocated quasi G-fuzzy metric space, $S : X \to X$ be a dominated mapping and x_0 be any arbitrary point in X. Suppose there exists $k \in [0, 1)$ with,*
(2.1.1) $G(Sx, Sy, Sz, t) \geq kG(x, y, z, t)$, for all $x, y, z \in \overline{B(x_0, r, t)}$ and
(2.1.2) $G(x_0, Sx_0, Sx_0, t) \geq (1 - k)r$.
If for a non increasing sequence $\{x_n\} \to u$ implies that $u \preceq x_n$. Then there exists a point $x* \in \overline{(B(x_0, r, t))}$ such that $x = Sx*$ and $G(x*, x*, x*, t) = 1$. Moreover, if for any three points $x, y, z \in \overline{B(x_0, r, t)}$ such that there exists a point $v \in \overline{B(x_0, r, t)}$ such that $v \preceq x, v \preceq y$ and $v \preceq z$, that is, every three of elements in $\overline{B(x_0, r, t)}$ has a lower bound, then the point $x*$ is unique.*

Proof. Consider a Picard sequence $x_{n+1} = Sx_n$ with initial guess x_0. As $x_{n+1} = Sx_n \preceq x_n$ for all $n \in \{0\} \cup N$. Now by inequality (2.1.2) we have
$G(x_0, x_1, x_1, t) \geq r$, which implies that $x_1 \in \overline{B(x_0, r, t)}$. By rectangular inequality

$$G(x_0, x_2, x_2, t) \geq G(x_0, x_1, x_1, t) + G(x_1, x_2, x_2, t)$$

then we get,

$$\begin{aligned} G(x_0, x_2, x_2, t) &\geq G(x_0, Sx_0, Sx_0, t) + G(Sx_0, Sx_1, Sx_1, t) \\ &\geq (1 - k)r + k(1 - k)r \\ &\geq (1 - k^2)r \geq r. \end{aligned}$$

Thus, $x_2 \in \overline{(B(x_0, r, t))}$. We suppose that $x_3, ..., x_j \in \overline{(B(x_0, r, t))}$, for some $j \in N$.
Now using (2.1.1) we get,

$$\begin{aligned} G(x_j, x_{j+1}, x_{j+1}, t) &= G(Sx_{j-1}, Sx_j, Sx_j, t) \\ &\geq k[G(x_{j-1}, x_j, x_j, t)] \\ &\geq k^2[G(x_{j-2}, x_{j-1}, x_{j-1}, t)] \\ &\vdots \\ &\geq k^j[G(x_0, x_1, x_1, t)] \end{aligned} \tag{2.1.3}$$

By using inequalities (2.1.1) and (2.1.3) we have,

$$\begin{aligned} G(x_0, x_{j+1}, x_{j+1}, t) &\geq G(x_0, x_1, x_1, t) + G(x_1, x_2, x_2, t) + ... + G(x_j, x_{j+1}, x_{j+1}, t) \\ &\geq (1 - k)r + rk(1 - k) + ... + rk^j(1 - k) \\ &= r(1 - k)[1 + k + k^2 + ... + k^j] \\ &\geq r(1 - k)(1 - k^{j+1})/((1 - k)) \geq r. \end{aligned}$$

Thus, $x_{j+1} \in \overline{(B(x_0, r, t))}$. Hence $x_n \in \overline{(B(x_0, r, t))}$, for all $n \in N$. Now inequality (2.1.3) can be written as,

$$G(x_n, x_{n+1}, x_{n+1}, t) \geq k^n[G(x_0, x_1, x_1, t)]. \tag{2.1.4}$$

Using inequality (2.1.4) we get,

$$\begin{aligned} G(x_n, x_{n+i}, x_{n+i}, t) &\geq G(x_n, x_{n+1}, x_{n+1}, t) + ... + G(x_{n+i-1}, x_{n+i}, x_{n+i}, t) \\ &\geq k^n(1-k^i)/((1-k))G(x_0, x_1, x_1, t) \to 1 \end{aligned}$$

as $n \to \infty$.

This proves that the sequence $\{x_n\}$ is a G-Cauchy sequence in $\overline{((B(x_0, r, t)), G, *)}$. Therefore there exists a point $x* \in \overline{(B(x_0, r, t))}$, with, $lim_{n\to\infty} G(x_n, x*, x*, t) = 1$.
Similarly, it can be proved that $lim_{n\to\infty} G(x*, x*, x_n, t) = 1$.
Therefore $lim_{n\to\infty} G(x_n, x*, x*, t) = lim_{n\to\infty} G(x*, x*, x_n, t) = 1$. (2.1.5)
Now, $G(x*, Sx*, Sx*, t) \geq G(x*, x_n, x_n, t) + G(x_n, Sx*, Sx*, t)$.
By assumption $x* \preceq x_n \preceq x_{n-1}$, therefore,

$$\begin{aligned} G(x*, Sx*, Sx*, t) &\geq G(x*, x_n, x_n, t) + G(Sx_{n-1}, Sx*, Sx*, t) \\ &\geq G(x*, x_n, x_n, t) + kG(x_{n-1}, x*, x*, t) \\ &\geq \lim_{n\to\infty}[G(x*, x_n, x_n, t) + kG(x_{n-1}, x*, x*, t)] \geq 1. \end{aligned}$$

This implies $G(x*, Sx*, Sx*, t) = 1$. Therefore, $x* = Sx*$. Similarly,$G(Sx*, Sx*, x*, t) \geq 1$, and hence $x* = Sx*$. Now,

$$G(x*, x*, x*, t) = G(Sx*, Sx*, Sx*, t) \geq kG(x*, x*, x*, t).$$

Since, $k \in [0, 1)$, then $G(x*, x*, x*, t) = 1$. Uniqueness: Let $y*$ be another point in $\overline{(B(x_0, r, t))}$, such that $y* = Sy*$. If $x*$ and $y*$ are comparable then, $G(x*, x*, y*, t) = G(Sx*, Sx*, Sy*, t) \geq kG(x*, x*, y*, t)$. Therefore, $G(y*, x*, x*, t) \geq 1$. This shows that $x* = y*$.

Now, if $x*$ and $y*$ are not comparable then there exists a point $v \in \overline{(B(x_0, r, t))}$ which is the lower bound of both $x*$ and $y*$ that is $v \preceq x*$ and $v \preceq y*$. Moreover, by assumption $x* \preceq x_n$ as $x_n \to x*$. Therefore, by (2.1.1) and (2.1.2) and the fact that $v \preceq x* \preceq x_n \preceq ... \preceq x_0$

$$\begin{aligned} G(x_0, Sv, Sv, t) &\geq G(x_0, x_1, x_1, t) + G(x_1, Sv, Sv, t) \\ &\geq G(x_0, Sx_0, Sx_0, t) + G(Sx_0, Sv, Sv, t) \\ &\geq (1-k)r + kG(x_0, v, v, t) \\ &\geq (1-k)r + kr. \end{aligned}$$

But, $x_0, v \in \overline{(B(x_0, r, t))}$ then $G(x_0, Sv, Sv, t) \geq r - rk + rk \geq r$.
This implies $G(x_0, Sv, Sv, t) \geq r$.
It follows that $Sv \in \overline{(B(x_0, r, t))}$. Now, we will prove that $S^n v \in \overline{(B(x_0, r, t))}$, by using mathematical induction. Let $S^2 v, S^3 v, ..., S^j v \in \overline{(B(x_0, r, t))}$ for some $j \in N$.
As $S^j v \preceq S^(j-1)v \preceq ... \preceq v \preceq x* \preceq x_n \preceq ... \preceq x_0$, then,
$G(x_{j+1}, S^{j+1}v, S^{j+1}v, t) = G(Sx_j, S(S^j v), S(S^j v), t)$. Thus, by (2.1.1),

$$G(x_{j+1}, S^{j+1}v, S^{j+1}v, t) \geq kG(x_j, S^j v, S^j v, t) \geq ...k^{j+1}G(x_0, v, v, t). \tag{2.1.6}$$

Now,

$$\begin{aligned} G(x_0, S^{j+1}v, S^{j+1}v, t) &\geq G(x_0, x_1, x_1, t) + ... + G(x_j, x_{j+1}, x_{j+1}, t) + G(x_{j+1}, S^{j+1}v, S^{j+1}v, t) \\ &\geq G(x_0, x_1, x_1, t) + ... + k^j G(x_0, x_1, x_1, t) + k^{j+1}G(x_0, v, v, t) \\ &\geq G(x_0, x_1, x_1, t)[1 + k + k^2 + ... + k^j] + rk^{j+1} \\ &\geq (1-k)r(1-k^{j+1})/((1-k)) + rk^{j+1} = r. \end{aligned}$$

This implies $G(x_0, S^{j+1}v, S^{j+1}v, t) \geq r$.

It follows that $S^{j+1} \in \overline{(B(x_0, r, t))}$and hence $S^n v \in \overline{(B(x_0, r, t))}$ for all n. Now

$$G(x*, y*, y*, t) \geq G(S^n x*, S^{n-1}v, S^{n-1}v, t) + G(S^{n-1}v, S^n y*, S^n y*, t).$$

As $S^{n-1}v \preceq S^{n-2}v \preceq ... \preceq v \preceq x*$ and $S^{n-1}v \preceq y*$ for all $n \in N$as $S^n x* = x*$ and $S^n y* = y*$ for all $n \in N$. Then by (2.1.1)

$$\begin{aligned} G(x*, y*, y*, t) &\geq kG(S^{n-1}x*, S^{n-2}v, S^{n-2}v, t) + kG(S^{n-2}v, S^{n-1}y*, S^{n-1}y*, t) \\ &\vdots \\ &\geq k^n G(x*, Sv, Sv, t) + k^n G(Sv, y*, y*, t) \to 1 \ as \ n \to \infty. \end{aligned}$$

So $G(x*, y*, y*, t) \geq 1$, hence $x* = y*$.

Similarly, $G(y*, x*, x*, t) \geq 1$, hence $y* = x*$. This proves the uniqueness of the fixed point. □

Theorem 2.2. *Let $(X, \preceq)$ be partially ordered set and let $(X, G, *)$ be an complete dislocated quasi G-fuzzy metric space, $S : X \to R$ be a mapping and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, 1/2)$ with*

$$G(Sx, Sy, Sz, t) \geq k(G(x, Sx, Sx, t) + G(y, Sy, Sy, t) + G(z, Sz, Sz, t)) \tag{2.2.1}$$

for all comparable elements $x, y, z \in \overline{(B(x_0, r, t))}$ and

$$G(x_0, Sx_0, Sx_0, t) \geq (1 - \theta)r \tag{2.2.2}$$

where $\theta = k/(1 - 2k)$. If for non increasing sequence $\{x_n\} \to u$ implies that $u \preceq x_n$. Then there exists a point $x* \in \overline{(B(x_0, r, t))}$, such that $x = Sx*$ and $G(x*, x*, x*, t) = 1$. Moreover, if for any three points $x, y, z \in \overline{(B(x_0, r, t))}$, there exists a point $v \in \overline{(B(x_0, r, t))}$ such that $v \preceq x$ and $v \preceq y, v \preceq z$, where*

$$\begin{aligned} G(x_0, Sx_0, Sx_0, t) + G(v, Sv, Sv, t) + G(v, Sv, Sv, t) \geq\ & G(x_0, v, v, t) + \\ & G(Sx_0, Sv, Sv, t) + G(Sx_0, Sv, Sv, t) \end{aligned} \tag{2.2.3}$$

then the point $x*$ is unique.

Proof. Proof Consider a Picard sequence $x_{n+1} = Sx_n$ with initial guess x_0. Then $x_{n+1} = Sx_n \preceq x_n$ for all $n \in \{0\} \cup N$ and by using inequality (2.2.2), we have,

$$G(x_0, Sx_0, Sx_0, t) \geq (1 - \theta)r \geq r.$$

Therefore,$x_1 \in \overline{(B(x_0, r, t))}$. Let $x_1, ..., x_j \in \overline{(B(x_0, r, t))}$ for some $j \in N$. Thus, by using inequality (2.2.1) we have

$$\begin{aligned} G(x_j, x_{j+1}, x_{j+1}, t) &= G(Sx_{j-1}, Sx_j, Sx_j, t) \\ &\geq k[G(x_{j-1}, Sx_{j-1}, Sx_{j-1}, t) + G(x_j, Sx_j, Sx_j, t) + G(x_j, Sx_j, Sx_j, t)] \end{aligned}$$

which implies that,

$$\begin{aligned} G(x_j, x_{j+1}, x_{j+1}, t) &\geq \theta G(x_{j-1}, x_j, x_j, t) \geq \theta^2 G((x_{j-2}, x_{j-1}, x_{j-1}, t) \\ &\geq \theta^j G(x_0, x_1, x_1, t) \end{aligned}$$

then,

$$G(x_j, x_{j+1}, x_{j+1}, t) \geq \theta^j G(x_0, x_1, x_1, t) \tag{2.2.4}$$

Now, by using the inequality (2.2.2) and (2.2.4) we have,

$$\begin{aligned} G(x_0, x_{j+1}, x_{j+1}, t) &\geq G(x_0, x_1, x_1, t) + G(x_1, x_2, x_2, t) + ... + G(x_j, x_{j+1}, x_{j+1}, t) \\ &\geq (1 - \theta)r[1 + \theta + \theta^2 + ... + \theta^j] \\ &\geq (1 - \theta)r(1 - \theta^{j+1})/((1 - \theta)) \geq r, \end{aligned}$$

which gives, $x_{j+1} \in \overline{(B(x_0, r, t))}$. Hence $x_n \in \overline{(B(x_0, r, t))}$, for all $n \in N$. It implies that inequality (2.2.4) can be written as,

$$G(x_n, x_{n+1}, x_{n+1}, t) \geq \theta^n G(x_0, x_1, x_1, t). \tag{2.2.5}$$

Now, by using inequality (2.2.5) we have,

$$\begin{aligned} G(x_n, x_{n+i}, x_{n+i}, t) &\geq G(x_n, x_{n+1}, x_{n+1}, t) + G(x_{n+1}, x_{n+2}, x_{n+2}, t) + ... + G(x_{n+1-i}, x_(n+i), x_{n+i}, t) \\ &\geq \theta^n (1-\theta^i)/(1-\theta) G(x_0, x_1, x_1, t) \to 1 \ as \ m, n \ \to \infty. \end{aligned}$$

Notice that the sequence $\{x_n\}$ is G-Cauchy sequence in $(\overline{(B(x_0, r, t))}, G, *)$.
Therefore, there exists a point $x* \in \overline{(B(x_0, r, t))}$, with $lim_{n\to\infty} x_n = x*$. Also,

$$\lim_{n\to\infty} G(x_n, x*, x*, t) = lim_{n\to\infty} G(x*, x*, x_n, t) = 1. \tag{2.2.6}$$

Now, $G(x*, Sx*, Sx*, t) \geq G(x*, x_n, x_n, t) + G(x_n, Sx*, Sx*, t)$.
By assumption $x* \preceq x_n \preceq x_(n-1)$, therefore,

$$\begin{aligned} G(x*, Sx*, Sx*, t) &\geq \lim_{n\to\infty}[G(x*, x_n, x_n, t) + k(G(x_{n-1}, Sx_{n-1}, Sx_{n-1}, t) \\ &\quad + G(x*, Sx*, Sx*, t) + G(x*, Sx*, Sx*, t))]. \end{aligned}$$

Thus, $(1-2k)G(x*, Sx*, Sx*, t) \geq 1$ and this implies $G(x*, Sx*, Sx*, t) = 1$. Similarly, $G(Sx*, Sx*, x*, t) \geq 1$ and hence $x* = Sx*$. Now,

$$G(x*, x*, x*, t) = G(Sx*, Sx*, Sx*, t) \geq k[G(x*, Sx*, Sx*, t) + G(x*, Sx*, Sx*, t) + G(x*, Sx*, Sx*, t)],$$

which implies that, $(1-3k)G(x*, x*, x*, t) \geq 1$.
This implies that,

$$G(x*, x*, x*, t) = 1 \tag{2.2.7}$$

Uniqueness: Now we show that $x*$ is unique. Let $y*$ be another point in$\overline{(B(x_0, r, t))}$, such that $y* = Sy*$. By following similar arguments as in inequality (2.2.6) we obtain,

$$G(y*, y*, y*, t) = 1. \tag{2.2.8}$$

Now, if $x* \preceq y*$, then,

$$G(x*, y*, y*, t) = G(Sx*, Sy*, Sy*, t) \geq k[G(x*, Sx*, Sx*, t) + G(y*, Sy*, Sy*, t) + G(y*, Sy*, Sy*, t)]$$

then, $G(x*, y*, y*, t) = 1$ by using (2.2.7) and (2.2.8). Similarly, $G(y*, y*, x*, t) = 1$.
Hence, we have $x* = y*$. Now if $x*$ and $y*$ are not comparable then there exists a point $v \in \overline{(B(x_0, r, t))}$,which is a lower bound of both $x*$ and $y*$. Now we will prove that $S^n v \in \overline{(B(x_0, r, t))}$. Moreover, by assumptions $v \preceq x * x_n \preceq ... \preceq x_0$.
Now, by using inequality (2.2.1) and (2.2.3), we have,

$$\begin{aligned} G(Sx_0, Sv, Sv, t) &\geq k[G(x_0, Sx_0, Sx_0, t) + G(v, Sv, Sv, t) + G(v, Sv, Sv, t)] \\ &\geq k[G(x_0, x_1, x_1, t) + G(v, Sv, Sv, t) + G(v, Sv, Sv, t)] \\ &\geq k[G(x_0, v, v, t) + G(Sx_0, Sv, Sv, t) + G(Sx_0, Sv, Sv, t)] \end{aligned}$$

Hence,

$$G(Sx_0, Sv, Sv, t) \geq k[G(x_0, v, v, t) + G(x_1, Sv, Sv, t) + G(x_1, Sv, Sv, t)].$$

Thus,

$$G(x_1, Sv, Sv, t) \geq \theta G(x_0, v, v, t). \tag{2.2.9}$$

Now,

$$\begin{aligned} G(x_0, Sv, Sv, t) &\geq G(x_0, x_1, x_1, t) + G(x_1, Sv, Sv, t) \\ &\geq G(x_0, x_1, x_1, t) + \theta G(x_0, v, v, t), \\ &\geq (1-\theta)r + \theta r. \end{aligned}$$

Thus, $G(x_0, Sv, Sv, t) \geq r$, then it follows that $Sv \in \overline{(B(x_0, r, t))}$.
Now, we will prove that $S^n v \in \overline{(B(x_0, r, t))}$. By using the mathematical induction to apply inequality (2.2.1). Let $S^2 v, ..., S^j v \in \overline{(B(x_0, r, t))}$, for some $j \in N$.
As

$$S^j v \preceq S^{j-1} v \preceq ... \preceq v \preceq x* \preceq x_n \preceq ... \preceq x_0,$$

then,

$$\begin{aligned} G(S^j v, S^{j+1} v, S^{j+1} v, t) &= G(S(S^{j-1})v, S(S^j v), S(S^j v, t)) \\ &\geq k[G(S^{j-1}, S^j v, S^j v, t) + G(S^j v, S^{j+1}, S^{j+1} v, t) \\ &\quad +G(S^j v, S^{j+1}, S^{j+1}, t)], \end{aligned}$$

which implies that,

$$\begin{aligned} G(S^j v, S^{j+1} v, S^{j+1} v, t) &\geq \theta G(S^{j-1}, S^j v, S^j v, t) \\ &\geq \theta^2 G(S^{j-2}, S^{j-1} v, S^{j-1} v, t) \\ &\vdots \\ &\geq \theta^j G(v, Sv, Sv, t). \end{aligned} \tag{2.2.10}$$

Now,

$$\begin{aligned} G(x_{j+1}, S^j v, S^{j+1} v, t) &= G(Sx_j, S(S^j v), S(S^j v), t) \\ &\geq k(G(x_j, Sx_j, Sx_j, t) + G(S^j v, S^{j+1} v, S^{j+1} v, t) \\ &\quad +G(S^j v, S^{j+1} v, S^{j+1} v, t)). \end{aligned}$$

By (2.2.4) and (2.2.10), we get

$$\begin{aligned} G(x_{j+1}, S^{j+1} v, S^{j+1} v, t) &\geq k[\theta^j G(x_0, x_1, x_1, t) + \theta^j G(v, Sv, Sv, t) + \theta^j G(v, Sv, Sv, t)] \\ &\geq k\theta^j [G(x_0, x_1, x_1, t) + G(v, Sv, Sv, t) + G(v, Sv, Sv, t)] \\ &\geq k\theta^j [G(x_0, v, v, t) + G(x_1, Sv, Sv, t) + G(x_1, Sv, Sv, t)] \\ &\geq k\theta^j [G(x_0, v, v, t) + \theta G(x_0, v, v, t) + \theta G(x_0, v, v, t)] \\ &\geq \theta^{j+1} G(x_0, v, v, t). \end{aligned} \tag{2.2.11}$$

Now,

$$\begin{aligned} G(x_0, S^{j+1} v, S^{j+1} v, t) &\geq G(x_0, x_1, x_1, t) + ... + G(x_j, x_{j+1}, x_{j+1}, t) + G(x_{j+1}, S^{j+1} v, S^{j+1} v, t) \\ &\geq G(x_0, x_1, x_1, t) + \theta G(x_0, x_1, x_1, t) + ... + \theta^{j+1} G(x_0, v, v, t) \\ &\geq G(x_0, x_1, x_1, t)[1 + \theta + \theta^2 + ... + \theta^j] + \theta^{j+1} r \\ &\geq (1-\theta)r((1-\theta^{j+1})/(1-\theta)) + \theta^{j+1} r = r. \end{aligned}$$

It follows that $S^{j+1} v \in \overline{(B(x_0, r, t))}$, and hence $S^n v \in \overline{(B(x_0, r, t))}$. Now inequality (2.2.10) can be written as, $G(S^n v, S^{n+1} v, S^{n+1} v, t) \geq \theta^n G(v, Sv, Sv, t) \to 1$ as $n \to \infty$. (2.2.12)
Now, by (2.2.7), (2.2.8) and (2.2.12)

$$\begin{aligned} G(x*, y*, y*, t) &= G(Sx*, Sy*, Sy, t) \\ &\geq G(Sx*, S^{n+1} v, S^{n+1} v, t) + G(S^{n+1} v, Sy*, Sy*, t) \\ &\geq k[G(x*, Sx*, Sx*, t) + G(S^n v, S^{n+1} v, S^{n+1} v, t) \\ &\quad +G(S^n v, S^{n+1} v, S^{n+1} v, t)] + kG(S^n v, S^{n+1} v, S^{n+1} v, t) + 2G(y*, Sy*, Sy*, t) \\ &\geq kG(x*, x*, x*, t) + 3kG(S^n v, S^{n+1} v, S^{n+1}, t) + 2kG(y*, y*, y*, t) G(x*, y*, y*, t) \geq 1 \end{aligned}$$

Similarly, $G(y*, x*, x*, t) = 1$. Thus, $x* = y*$. The following example exhibits the superiority of our Theorem (2.2). The mapping is contractive on the closed ball instead on the whole space. □

Example 2.3. Let $X = R^+ \cup \{0\}$ be endowed with usual order and $G : X \times X \times X \to X$ be an ordered complete dislocated quasi G-fuzzy metric space defined by, $G(x, y, z, t) = x/2 + y + z$. Let $S : X \to X$ be defined by, $Sx = x/8$ if $x \in [0, 1/2]$, $Sx = x - 1/2$ if $x \in [0, \infty)$. Clearly, S is a dominated mappings. Then for $x_0 = 1/2, r = 3/2, = 3/8, \overline{(B(x_0, r, t))} = [0, 1/2]$ and for $k = 3/10$.
$(1-\theta)r = (1-3/8)3/2 = 15/16$ and $G(x_0, Sx_0, Sx_0, t) = G(1/2, S1/2, S1/2, t) = 1/4+1/16+(1)/16 = 3/8$.
This implies $G(x_0, Sx_0, Sx_0, t) \geq (1 - \theta)r = 3/8 \leq 15/16$.
Implies $48 \leq 120$..
Also, if $x, y, z \in (1, \infty)$ we assume that $x < y$ and $y < z$, then

$$5x + 10y + 10z \leq 15/2x + 15/2y + 15/2z + 7/2.$$

Thus,

$$5x + 10y + 10z - 5 - 5 - 5/2 \leq 15/2x + 15/2y + 15/2z - 9$$

It means that

$$\begin{aligned} 10[(x/2 - 1/4) + (y - 1/2) + (z - 1/2)] \quad &\leq \quad 3[((x/2 + x + x - 1) + (y/2 + y + y - 1) \\ &\quad +(z/2 + z + z - 1))] \end{aligned}$$

Hence,

$$\begin{aligned} G(Sx, Sy, Sz, t) \quad &\leq \quad k[((x/2 + x - 1/2 + x - 1/2) + (y/2 + y - 1/2 + y - 1/2) \\ &\quad +(z/2 + z - 1/2 + z - 1/2))] \\ &\leq \quad k[G(x, Sx, Sx, t) + G(y, Sy, Sy, t) + G(z, Sz, Sz, t)]. \end{aligned}$$

So, the contractive condition does not hold in X. Now if $x, y, z \in \overline{(B(x_0, r, t))}$, then

$$\begin{aligned} G(Sx, Sy, Sz, t) \quad &= \quad x/16 + y/8 + z/8 = 1/8\{x/2 + y + z\} \\ &\geq \quad 3/10x/2 + y/2 + z/2 \\ &\geq \quad 3/10\{(x/2 + x/8 + x/8) + (y/2 + y/8 + y/8) + (z/2 + z/8 + z/8)\} \\ &\geq \quad 3/10\{G(x, Sx, Sx, t) + G(y, Sy, Sy, t) + G(z, Sz, Sz, t)\} \\ &\geq \quad k\{G(x, Sx, Sx, t) + G(y, Sy, Sy, t) + G(z, Sz, Sz, t)\}. \end{aligned}$$

Hence it satisfies all the requirements of Theorem (2.2).

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