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Fixed points of almost Geraghty contraction type maps/generalized contraction maps with rational expressions in b-metric spaces

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Abstract

In this paper, we introduce almost Geraghty contraction type maps for a single selfmap and prove the existence and uniqueness of fixed points. We extend it to a pair of selfmaps by defining almost Geraghty contraction type pair of maps in which one of the maps is b-continuous in a complete b-metric space. Further, we prove the existence of common fixed points for a pair of selfmaps satisfying a generalized contraction condition with rational expression in which one of the maps is b-continuous. Our results extend and generalize some of the known results that are available in the literature. We draw some corollaries from our results and provide examples in support of our results.

Keywords: Common fixed points, b-metric space, b-continuous, almost Geraghty contraction type maps.
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1. Introduction and preliminaries

The development of fixed point theory is based on the generalization of contraction conditions in one direction or/and generalization of ambient spaces of the operator under consideration on the other. Banach contraction principle plays an important role in solving nonlinear equations, and it is one of the most useful result in fixed point theory. In the direction of generalization of contraction conditions, in 1973, Geraghty [19] proved a fixed point theorem, generalizing Banach contraction principle. Several authors proved later various results using Geraghty-type conditions. In continuation to the extensions of contraction maps, Berinde [7] introduced 'weak contractions' as a generalization of contraction maps. Berinde renamed 'weak contractions' as 'almost contractions' in his later work [8]. For more works on almost contractions and its generalizations, we refer Babu, Sandhya and Kameswari [4], Abbas, Babu and Alemayehu [1] and the related references cited in these papers. In 1975, Dass and Gupta [14] established fixed point results using contraction condition involving rational expressions.

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The main idea of b-metric was initiated from the works of Bourbaki [11] and Bakhtin [6]. The concept of b-metric space or metric type space was introduced by Czerwik [12] as a generalization of metric space. Afterwards, many authors studied fixed point theorems for single-valued and multi-valued mappings in b-metric spaces, for more information we refer [3, 5, 9, 10, 13, 20, 21, 22, 23, 24, 25].

Definition 1.1. [12] Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied: for any $x, y, z \in X$

- (i) $0 \leq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
- (ii) $d(x, y) = d(y, x)$,
- (iii) there exists $s \geq 1$ such that $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space with coefficient s.

Every metric space is a b-metric space with $s = 1$. In general, every b-metric space is not a metric space.

Definition 1.2. [10] Let (X, d) be a b-metric space and let $\{x_n\}$ be a sequence in X.

- (i) A sequence $\{x_n\}$ in X is called b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$.
- (ii) A sequence $\{x_n\}$ in X is called b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.
- (iii) A b-metric space (X, d) is said to be a complete b-metric space if every b-Cauchy sequence in X is b-convergent in X.
- (iv) A set $B \subset X$ is said to be b-closed if for any sequence $\{x_n\}$ in B such that $\{x_n\}$ is b-convergent to $z \in X$ then $z \in B$.

In general, a b-metric is not necessarily continuous.

Example 1.3. [18] Let $X = \mathbb{N} \cup \{\infty\}$. We define a mapping $d : X \times X \to [0, \infty)$ as follows:

$$d(m, n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5 & \text{if one of } m, n \text{ is odd and the other is odd or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

Then (X, d) is a b-metric space with coefficient $s = \frac{5}{2}$.

Definition 1.4. [10] Let (X, d_X) and (Y, d_Y) be two b-metric spaces. A function $f : X \to Y$ is a b-continuous at a point $x \in X$, if it is b-sequentially continuous at x. i.e., whenever $\{x_n\}$ is b-convergent to x, fx_n is b-convergent to fx.

In 1973, Geraghty [19] introduced a class of functions

$$\mathfrak{S} = \{\beta : [0, \infty) \to [0, 1) / \lim_{n\to\infty} \beta(t_n) = 1 \implies \lim_{n\to\infty} t_n = 0\}.$$

Theorem 1.5. [19] *Let (X, d) be a complete metric space. Let $T : X \to X$ be a selfmap satisfying the following: there exists $\beta \in \mathfrak{S}$ such that*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y) \text{ for all } x, y \in X.$$

Then T has a unique fixed point.

We denote

$$\mathfrak{B} = \{\alpha : [0, \infty) \to [0, \frac{1}{s}) / \lim_{n\to\infty} \alpha(t_n) = \frac{1}{s} \implies \lim_{n\to\infty} t_n = 0\}.$$

In 2011, Dukic, Kadelburg and Radenović [15] extended Theorem 1.5 to the case of b-metric spaces as follows.

Theorem 1.6. [15] *Let (X,d) ba a complete b-metric space with coefficient $s \geq 1$ and let $T : X \to X$ be a selfmap of X. Suppose that there exists $\alpha \in \mathfrak{B}$ such that*

$$d(Tx,Ty) \leq \alpha(d(x,y))d(x,y) \text{ for all } x,y \in X.$$

Then T has a unique fixed point in X.

Throughout this paper, we denote

$$\mathfrak{F} = \{\beta : [0,\infty) \to [0,\frac{1}{s})/\limsup_{n\to\infty} \beta(t_n) = \frac{1}{s} \implies \lim_{n\to\infty} t_n = 0\}$$

and $\mathbb{N}$, the set of all natural numbers.

The following lemmas are useful in proving our main results.

Lemma 1.7. [17] *Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a selfmap. Suppose that $\{x_n\}$ is a sequence in X induced by $x_{n+1} = Tx_n$ such that $d(x_n,x_{n+1}) \leq \lambda d(x_{n-1},x_n)$ for all $n \in \mathbb{N}$, where $\lambda \in [0,1)$ is a constant. Then $\{x_n\}$ is a b-Cauchy sequence in X.*

Lemma 1.8. [2] *Let (X,d) be a b-metric space with coefficient $s \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y respectively, then we have*

$$\frac{1}{s^2}d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y)$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n,y_n) = 0$. Moreover for each $z \in X$ we have

$$\frac{1}{s}d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq sd(x,z).$$

In 2019, Faraji, Savić and Radenović [16] proved the following two theorems.

Theorem 1.9. [16] *Let (X,d) be a complete b-metric space with parameter $s \geq 1$. Let $T : X \to X$ be a selfmap satisfying: there exists $\beta \in \mathfrak{F}$ such that*

$$d(Tx,Ty) \leq \beta(M(x,y))M(x,y) \text{ for all } x,y \in X,$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2s}(d(x,Ty) + d(y,Tx))\}$$

Then T has a unique fixed point.

Theorem 1.10. [16] *Let (X,d) be a complete b-metric space with parameter $s \geq 1$. Let $T,S : X \to X$ be selfmaps on X which satisfy: there exists $\beta \in \mathfrak{F}$ such that*

$$sd(Tx,Sy) \leq \beta(M(x,y))M(x,y) \text{ for all } x,y \in X,$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Sy)\}$. If T or S are continuous, then T and S have a unique common fixed point.

The following theorem is due to Haung, Deng and Radenović [17].

Theorem 1.11. [17] *Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a selfmap such that*

$$d(Tx,Ty) \leq \lambda_1 d(x,y) + \lambda_2 \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} + \lambda_3 \frac{d(x,Ty)d(y,Tx)}{1+d(x,y)} + \lambda_4 \frac{d(x,Tx)d(x,Ty)}{1+d(x,y)} + \lambda_5 \frac{d(y,Tx)d(y,Ty)}{1+d(x,y)}$$

where $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ and λ_5 are nonnegative constants with $\lambda_1+\lambda_2+\lambda_3+2s\lambda_4+2s\lambda_5 < 1$. Then T has a unique fixed point in X. Moreover, for any $x \in X$, the iterative sequence $\{T^n x\}$ is b-convergent to the fixed point.

In Section 2, we introduce almost Geraghty contraction type maps for a single selfmap and prove the existence and uniqueness of fixed points. We extend it to a pair of selfmaps by defining almost Geraghty contraction type pair of maps in which one of the maps is b-continuous in a complete b-metric space. In Section 3, we prove the existence of common fixed points for a pair of selfmaps satisfying a generalized contraction condition with rational expressions in which one of the maps is b-continuous. Our results extend and generalize some of the known results that are available in the literature. We draw some corollaries from our results and provide examples in support of our results.

2. Fixed points of almost Geraghty contraction type maps

The following we introduce almost Geraghty contraction type maps for a single and a pair of maps in b-metric spaces as follows:

Definition 2.1. Let (X, d) be a b-metric space with coefficient $s \geq 1$, and let f be a selfmap of X. If there exist $\beta \in \mathfrak{F}$ and $L \geq 0$ such that

$$d(fx, fy) \leq \beta(M(x, y))M(x, y) + LN(x, y) \tag{2.1}$$

for all $x, y \in X$, where

$$M(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2s}[d(x, fy) + d(y, fx)]\}$$

and

$$N(x, y) = \min\{d(x, fx), d(x, fy), d(y, fx)\}$$

then we say that f is an almost Geraghty contraction type map.

The importance of the class of almost Geraghty contraction type maps is that this class properly includes the class of Geraghty contraction type maps studied by Faraji, Savić and Radenović [16] so that the class of almost Geraghty contraction type maps is lager than the class of Geraghty contraction type maps, which is illustrated in the following example (also in Example 2.8 and Example 2.9).

Example 2.2. Let $X = [0, \infty)$ and let $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ (x + y)^2 & \text{if } x \neq y. \end{cases}$$

Then clearly (X, d) is a complete b-metric space with coefficient $s = 2$.
We define $f : X \to X$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1) \\ 2x - 1 & \text{if } x \in [1, \infty). \end{cases}$$

We define $\beta : [0, \infty) \to [0, \frac{1}{s})$ by $\beta(t) = \frac{1}{3+t}$ for all $t > 0$. Then $\beta \in \mathfrak{F}$. Without loss of generality we assume that $x \geq y$.

Case (i). Let $x, y \in [0, 1)$.
$d(fx, fy) = 0$ and clearly the inequality (2.1) holds in this case.

Case (ii). Let $x, y \in [1, \infty)$.

$$d(fx, fy) = 4(x + y - 1)^2, \quad d(x, y) = (x + y)^2, \quad d(x, fx) = (3x - 1)^2, \quad d(y, fy) = (3y - 1)^2,$$
$$d(x, fy) = (x + 2y - 1)^2, \quad d(y, fx) = (y + 2x - 1)^2$$

$$\begin{aligned} M(x, y) &= \max\{d(x, y), d(x, fx), d(y, fy), \tfrac{1}{2s}[d(x, fy) + d(y, fx)]\} \\ &= \max\{(x + y)^2, (3x - 1)^2, (3y - 1)^2, \tfrac{1}{4}[(x + 2y - 1)^2 + (y + 2x - 1)^2]\} \\ &= (3x - 1)^2 \end{aligned}$$

and

$$N(x,y) = \min\{(3x-1)^2, (x+2y-1)^2, (y+2x-1)^2\} = (x+2y-1)^2.$$

We consider

$$d(fx,fy) = 4(x+y-1)^2 \leq \frac{1}{3+(3x-1)^2}(3x-1)^2 + \frac{11}{3}(x+2y-1)^2 \leq \beta(M(x,y))M(x,y) + LN(x,y).$$

Case (iii). Let $x \in [1,\infty), y \in [0,1)$.

$$d(fx,fy) = 4x^2, \quad d(x,y) = (x+y)^2, \quad d(x,fx) = (3x-1)^2, \quad d(y,fy) = (y+1)^2,$$
$$d(x,fy) = (x+1)^2, \quad d(y,fx) = (y+2x-1)^2.$$

$$\begin{aligned} M(x,y) &= \max\{(x+y)^2, (3x-1)^2, (y+1)^2, \tfrac{1}{4}[(x+1)^2 + (y+2x-1)^2]\} \\ &= (3x-1)^2 \end{aligned}$$

and

$$\begin{aligned} N(x,y) &= \min\{d(x,fx), d(x,fy), d(y,fx)\} \\ &= \min\{(3x-1)^2, (x+1)^2, (y+2x-1)^2\}. \end{aligned}$$

We consider

$$\begin{aligned} d(fx,fy) = 4x^2 &\leq \tfrac{1}{3+(3x-1)^2}(3x-1)^2 + \tfrac{11}{3}\min\{(x+1)^2, (y+2x-1)^2\} \\ &\leq \beta(M(x,y))M(x,y) + LN(x,y). \end{aligned}$$

From all above cases f is an almost Geraghty contraction type map with $L = \frac{11}{3}$.

Here we observe that if $L = 0$ then the inequality (2.1) fails to hold.
For, we choose $x = 3$ and $y = 2$, we have

$$d(fx,fy) = 64, d(x,fx) = 64, d(y,fy) = 25, d(x,y) = 25, d(x,fy) = 36, d(y,fx) = 49.$$

Thus,

$$M(x,y) = \max\{25, 64, 25, \frac{1}{4}[36+49]\} = 64.$$

Here we note that

$$d(fx,fy) = 64 \nleq \beta(64)64 = \beta(M(x,y))M(x,y) \ for\ any\ \beta \in \mathfrak{F}.$$

Definition 2.3. Let (X,d) be a b-metric space with coefficient $s \geq 1$, and let f and g be selfmaps of X. If there exist $\beta \in \mathfrak{F}$ and $L \geq 0$ such that

$$sd(fx,gy) \leq \beta(M(x,y))M(x,y) + LN(x,y) \tag{2.2}$$

for all $x, y \in X$, where

$$M(x,y) = \max\{d(x,y), d(x,fx), d(y,gy)\}$$

and

$$N(x,y) = \min\{d(x,fx), d(x,gy), d(y,fx)\}$$

then we call (f,g) is an almost Geraghty contraction type pair of maps.

Example 2.4. Let $X = [0,\infty)$ and let $d : X \times X \to [0,\infty)$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ (x+y)^2 & \text{if } x \neq y. \end{cases}$$

Then clearly (X,d) is a complete b-metric space with coefficient $s = 2$. We define $f, g : X \to X$ by
$f(x) = \begin{cases} x+1 & \text{if } x \in [0,1) \\ 2x-1 & \text{if } x \in [1,\infty) \end{cases}$ and $g(x) = 1$ for $x \in [0,\infty)$.
We define $\beta : [0,\infty) \to [0,\frac{1}{s})$ by $\beta(t) = \frac{1}{3+t}$ for all $t > 0$. Then $\beta \in \mathfrak{F}$.
Without loss of generality we assume that $x \geq y$.

Case (i). Let $x, y \in [0,1)$.

$$d(fx,gy)=(x+2)^2,\quad d(x,y)=(x+y)^2,\qquad d(x,fx)=(2x+1)^2,\quad d(y,gy)=(y+1)^2,$$
$$d(x,gy)=(x+1)^2,\qquad d(y,fx)=(x+y+1)^2$$

$M(x,y)=\max\{d(x,y),d(x,fx),d(y,gy)\}=\max\{(x+y)^2,(2x+1)^2,(y+1)^2\}=(2x+1)^2$
and $N(x,y)=\min\{d(x,fx),d(x,gy),d(y,fx)\}=\min\{(2x+1)^2,(x+1)^2,(x+y+1)^2\}=(x+1)^2$.
We consider
$sd(fx,gy)=2(x+2)^2\leq\frac{1}{3+(2x+1)^2}(2x+1)^2+6(x+1)^2$
$=\beta(M(x,y))M(x,y)+LN(x,y)$.

Case (ii). Let $x, y \in [1,\infty)$.
$d(fx,gy)=(2x)^2, d(x,y)=(x+y)^2, d(x,fx)=(3x-1)^2, d(y,gy)=(y+1)^2$,
$d(x,gy)=(x+1)^2, d(y,fx)=(y+2x-1)^2$.
$M(x,y)=\max\{d(x,y),d(x,fx),d(y,gy)\}=\max\{(x+y)^2,(3x-1)^2,(y+1)^2\}=(3x-1)^2$ and
$N(x,y)=\min\{d(x,fx),d(x,gy),d(y,fx)\}$
$=\min\{(3x-1)^2,(x+1)^2,(y+2x-1)^2\}=(x+1)^2$.
We consider
$sd(fx,gy)=2(2x)^2\leq\frac{1}{3+(3x-1)^2}(3x-1)^2+6(x+1)^2$
$=\beta(M(x,y))M(x,y)+LN(x,y)$.

Case (iii). Let $x\in[1,\infty), y\in[0,1)$.
$d(fx,gy)=(2x)^2, d(x,y)=(x+y)^2, d(x,fx)=(3x-1)^2, d(y,gy)=(y+1)^2$,
$d(x,gy)=(x+1)^2, d(y,fx)=(y+2x-1)^2$.
$M(x,y)=\max\{d(x,y),d(x,fx),d(y,gy)\}$
$=\max\{(x+y)^2,(3x-1)^2,(y+1)^2\}=(3x-1)^2$ and
$N(x,y)=\min\{d(x,fx),d(x,gy),d(y,fx)\}$
$=\min\{(3x-1)^2,(x+1)^2,(y+2x-1)^2\}=\min\{(x+1)^2,(y+2x-1)^2\}$.
We consider
$sd(fx,gy)=2(2x)^2\leq\frac{1}{3+(3x-1)^2}(3x-1)^2+6\min\{(x+1)^2,(y+2x-1)^2\}$
$=\beta(M(x,y))M(x,y)+LN(x,y)$.
From all above cases, (f,g) is an almost Geraghty contraction type pair of maps with $L=6$.

Here we observe that if $L=0$ then the inequality (2.2) fails to hold.
For, take $x=\frac{1}{2}$ and $y=0$. Then $d(fx,gy)=\frac{25}{4}, d(x,fx)=4, d(y,gy)=1, d(x,y)=\frac{1}{4}$.
$M(x,y)=\max\{d(x,y),d(x,fx),d(y,gy)\}=4$.
$sd(fx,gy)=\frac{25}{2}\not\leq\beta(4)4=\beta(M(x,y))M(x,y)$ for any $\beta\in\mathfrak{F}$.

Theorem 2.5. *Let (X,d) be a complete b-metric space with coefficient $s\geq 1$ and let $f: X\to X$ be an almost Geraghty contraction type map. Then f has a unique fixed point in X.*

Proof. Let x_0 be arbitrary. We define the sequence $\{x_n\}$ in X by $x_n=fx_{n-1}=f^nx_0, n\in\mathbb{N}$. If $x_n=x_{n+1}$ for some $n\in\mathbb{N}$, then x_n is a fixed point of T. Suppose that $x_n\neq x_{n+1}$ for all $n\in\mathbb{N}$. From the inequality (2.1), we have

$$d(x_n,x_{n+1})=d(fx_{n-1},fx_n)\leq\beta(M(x_{n-1},x_n))M(x_{n-1},x_n)+LM(x_{n-1},x_n)\tag{2.3}$$

in which

$$\begin{aligned} M(x_{n-1},x_n) &= \max\{d(x_{n-1},x_n), d(x_{n-1},fx_{n-1}), d(x_n,fx_n), \\ &\quad \tfrac{1}{2s}[d(x_{n-1},fx_n)+d(x_n,fx_{n-1})]\} \\ &= \max\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), \\ &\quad \tfrac{1}{2s}[d(x_{n-1},x_{n+1})+d(x_n,x_n)]\} \\ &= \max\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \tfrac{1}{2s}d(x_{n-1},x_{n+1})\} \\ &\leq \max\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \\ &\quad \tfrac{1}{2s}s[d(x_{n-1},x_n)+d(x_n,x_{n+1})]\} \\ &= \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} \end{aligned}$$

and

$$\begin{aligned} N(x_{n-1},x_n) &= \min\{d(x_{n-1},fx_{n-1}), d(x_{n-1},fx_n), d(x_n,fx_{n-1})\} \\ &= \min\{d(x_{n-1},x_n), d(x_{n-1},x_{n+1}), d(x_n,x_n)\} \\ &= \min\{d(x_{n-1},x_n), d(x_{n-1},x_{n+1}), 0\} = 0. \end{aligned}$$

If $M(x_{n-1},x_n) = d(x_n,x_{n+1})$ then from the inequality (2.3), we have

$$d(x_n,x_{n+1}) \leq \beta(d(x_n,x_{n+1}))d(x_n,x_{n+1}) < \frac{1}{s}d(x_n,x_{n+1})$$

which is a contradiction.

Therefore $M(x_{n-1},x_n) = d(x_{n-1},x_n)$. Hence from the inequality (2.3), we have

$$d(x_n,x_{n+1}) \leq \beta(d(x_{n-1},x_n))d(x_{n-1},x_n) < \frac{1}{s}d(x_{n-1},x_n). \tag{2.4}$$

Therefore, $d(x_n,x_{n+1}) \leq d(x_{n-1},x_n)$ for all $n \in \mathbb{N}$. Thus, $\{d(x_n,x_{n+1})\}$ is a decreasing sequence of non-negative reals and bounded below by 0. Hence, there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} d(x_n,x_{n+1}) = r$. Now on taking limit superior as $n \to \infty$ in (2.4), we get

$$r \leq \limsup_{n\to\infty} \beta(d(x_n,x_{n+1}))r \quad \text{ implies } \quad 1 \leq \limsup_{n\to\infty} \beta(d(x_n,x_{n+1})) \leq \frac{1}{s}$$

which implies that $\frac{1}{s} \leq 1 \leq \limsup\limits_{n\to\infty} \beta(d(x_n,x_{n+1})) \leq \frac{1}{s}$ and so $\limsup\limits_{n\to\infty} \beta(d(x_n,x_{n+1})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, we have $\lim\limits_{n\to\infty} d(x_n,x_{n+1}) = 0$. So we have $r = 0$.

We now prove that $\{x_n\}$ is a b-Cauchy sequence in X. On the contrary suppose that $\{x_n\}$ is not a b-Cauchy sequence.Thus, there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$d(x_{m_k},x_{n_k}) \geq \epsilon \quad \textit{and} \quad d(x_{m_k},x_{n_k-1}) < \epsilon \tag{2.5}$$

From the inequality (2.1), (2.5) and by the b-triangular inequality, we have

$$\epsilon \leq d(x_{m_k},x_{n_k}) \leq s[d(x_{m_k},x_{m_k+1}) + d(x_{m_k+1},x_{n_k})].$$

Taking limit superior as $n \to \infty$, we get

$$\frac{\epsilon}{s} \leq \limsup_{n\to\infty} d(x_{m_k+1},x_{n_k}).$$

We now consider

$$d(x_{m_k+1},x_{n_k}) = d(fx_{m_k},fx_{n_k-1}) \leq \beta(M(x_{m_k},x_{n_k-1}))M(x_{m_k},x_{n_k-1}) + LN(x_{m_k},x_{n_k-1}) \tag{2.6}$$

where

$$\begin{aligned} M(x_{m_k},x_{n_k-1}) &= \max\{d(x_{m_k},x_{n_k-1}), d(x_{m_k},fx_{m_k}), d(x_{n_k-1},fx_{n_k-1}), \\ &\quad \tfrac{1}{2s}[d(x_{m_k},fx_{n_k-1}) + d(x_{n_k-1},fx_{m_k})]\} \\ &= \max\{d(x_{m_k},x_{n_k-1}), d(x_{m_k},x_{m_k+1}), d(x_{n_k-1},x_{n_k}), \\ &\quad \tfrac{1}{2s}[d(x_{m_k},x_{n_k}) + d(x_{n_k-1},x_{m_k+1})]\} \\ &\leq \max\{d(x_{m_k},x_{n_k-1}), d(x_{m_k},x_{m_k+1}), d(x_{n_k-1},x_{n_k}), \\ &\quad \tfrac{1}{2}[d(x_{m_k},x_{n_k-1}) + d(x_{n_k-1},x_{n_k}) + d(x_{n_k-1},x_{m_k}) + d(x_{m_k},x_{m_k+1})]\} \end{aligned}$$

and

$$\begin{aligned} N(x_{m_k}, x_{n_k-1}) &= \min\{d(x_{m_k}, fx_{m_k}), d(x_{m_k}, fx_{n_k-1}), d(x_{n_k-1}, fx_{m_k})\} \\ &= \min\{d(x_{m_k}, x_{m_k+1}), d(x_{m_k}, x_{n_k}), d(x_{n_k-1}, x_{m_k+1})\}. \end{aligned}$$

Taking limit superior as $n \to \infty$ on $M(x_{m_k}, x_{n_k-1})$ and $N(x_{m_k}, x_{n_k-1})$, we get

$$\limsup_{n\to\infty} M(x_{m_k}, x_{n_k-1}) = \limsup_{n\to\infty} d(x_{m_k}, x_{n_k-1}) \leq \epsilon$$

and $\limsup_{n\to\infty} N(x_{m_k}, x_{n_k-1}) = 0$. Taking limit superior as $n \to \infty$ in (2.6), we get

$$\begin{aligned} \tfrac{\epsilon}{s} &\leq \limsup_{n\to\infty} d(x_{m_k+1}, x_{n_k}) \\ &\leq \limsup_{n\to\infty} [\beta(M(x_{m_k}, x_{n_k-1}))M(x_{m_k}, x_{n_k-1}) + LN(x_{m_k}, x_{n_k-1})] \\ &\leq \limsup_{n\to\infty} \beta(M(x_{m_k}, x_{n_k-1}))\epsilon \end{aligned}$$

which implies that

$$\frac{1}{s} \leq \limsup_{n\to\infty} \beta(M(x_{m_k}, x_{n_k-1})) \leq \frac{1}{s}.$$

Therefore,

$$\limsup_{n\to\infty} \beta(M(x_{m_k}, x_{n_k-1})) = \frac{1}{s}.$$

Since $\beta \in \mathfrak{F}$, we have $\limsup_{n\to\infty} M(x_{m_k}, x_{n_k-1}) = 0$. i.e., $\limsup_{n\to\infty} d(x_{m_k}, x_{n_k-1}) = 0$.
Therefore $\lim_{n\to\infty} d(x_{m_k}, x_{n_k-1}) = 0$.
From (2.5) and using b-triangular inequality, we have

$$\epsilon \leq d(x_{m_k}, x_{n_k}) \leq s[d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})].$$

Taking limit superior as $k \to \infty$, we get

$$\epsilon \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_k}) = 0,$$

it is a contradiction. Therefore, the sequence $\{x_n\}$ is a b-Cauchy sequence in X.
Since X is b-complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.
From the inequality (2.1) and the b-triangular inequality, we have

$$\begin{aligned} d(u, fu) &\leq s[d(u, fx_n) + d(fx_x, fu)] \\ &\leq sd(u, fx_n) + s[\beta(M(u, x_n))M(u, x_n) + LN(u, x_n)] \end{aligned} \tag{2.7}$$

where

$$M(u, x_n) = \max\{d(u, x_n), d(u, fu), d(x_n, fx_n), \frac{1}{2s}[d(u, fx_n) + d(x_n, fu)]\}$$

and

$$N(u, x_n) = \min\{d(u, fu), d(u, fx_n), d(x_n, fu)\}.$$

Taking limit superior as $n \to \infty$ on $M(u, x_n)$ and $N(u, x_n)$ and using Lemma 1.8, we get

$$\limsup_{n\to\infty} M(u, x_n) = d(u, fu) \text{ and } \limsup_{n\to\infty} N(u, x_n) = 0.$$

Taking limit superior as $n \to \infty$ in (2.7), we get

$$d(u, fu) \leq s \limsup_{n\to\infty} \beta(M(u, x_n)) \limsup_{n\to\infty} M(u, x_n) + sL \limsup_{n\to\infty} N(u, x_n)$$

implies that
$$d(u, fu) \leq s \limsup_{n\to\infty} \beta(M(u, x_n))d(u, fu)$$
which implies that
$$\frac{1}{s} \leq \limsup_{n\to\infty} \beta(M(u, x_n)) \leq \frac{1}{s}.$$
Therefore $\limsup_{n\to\infty} \beta(M(u, x_n)) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, we have $\lim_{n\to\infty} M(u, x_n) = 0$. i.e. $\lim_{n\to\infty} d(u, fu) = 0$.
Therefore $fu = u$. i.e., u is a fixed point of f.

Uniqueness of fixed point follows from the inequality (2.1). □

Theorem 2.6. *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$ and let (f, g) be an almost Geraghty contraction type pair of maps. If either f or g is b-continuous then f and g have a unique common fixed point in X.*

Proof. Let x_0 be arbitrary.
We define the sequence $\{x_n\}$ in X by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n = 0, 1, 2, \ldots$.
From the inequality (2.2), we have
$$\begin{aligned} sd(x_{2n+1}, x_{2n+2}) &= sd(fx_{2n}, gx_{2n+1}) \\ &\leq \beta(M(x_{2n}, x_{2n+1}))M(x_{2n}, x_{2n+1}) + LN(x_{2n}, x_{2n+1}) \end{aligned} \tag{2.8}$$
where
$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1})\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \end{aligned}$$
and
$$\begin{aligned} N(x_{2n}, x_{2n+1}) &= \min\{d(x_{2n}, fx_{2n}), d(x_{2n}, gx_{2n+1}), d(x_{2n+1}, fx_{2n})\} \\ &= \min\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})\} \\ &= \min\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+2}), 0\} = 0. \end{aligned}$$
If $M(x_{2n}, x_{2n+1}) = d(x_{2n+1}, x_{2n+2})$ then from the inequality (2.8), we have
$$sd(x_{2n+1}, x_{2n+2}) \leq \beta(d(x_{2n+1}, x_{2n+2}))d(x_{2n+1}, x_{2n+2}) < \frac{1}{s}d(x_{2n+1}, x_{2n+2}),$$
which is a contradiction. Therefore $M(x_{2n}, x_{2n+1}) = d(x_{2n}, x_{2n+1})$.
Hence from the inequality (2.8), we have
$$sd(x_{2n+1}, x_{2n+2}) \leq \beta(d(x_{2n}, x_{2n+1}))d(x_{2n}, x_{2n+1}) < \frac{1}{s}d(x_{2n}, x_{2n+1}). \tag{2.9}$$
Therefore $d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1})$. Similarly, we obtain $d(x_{2n+2}, x_{2n+3}) \leq d(x_{2n+1}, x_{2n+2})$.
Hence $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$.
Thus $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of nonnegative reals and bounded below by 0.
Hence there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$.
Now on taking limit superior as $n \to \infty$ in (2.9), we get
$$sr \leq \limsup_{n\to\infty} \beta(d(x_{2n}, x_{2n+1}))r$$
implies that
$$s \leq \limsup_{n\to\infty} \beta(d(x_{2n}, x_{2n+1})) \leq \frac{1}{s}$$

which implies that

$$\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta(d(x_{2n}, x_{2n+1})) \leq \frac{1}{s^2} \leq \frac{1}{s}$$

which implies that

$$\limsup_{n\to\infty} \beta(d(x_{2n}, x_{2n+1})) = \frac{1}{s}.$$

Since $\beta \in \mathfrak{F}$, we have $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$ so that $r = 0$.
We now prove that $\{x_n\}$ is a b-Cauchy sequence in X. It is sufficient to show that $\{x_{2n}\}$ is b-Cauchy. Suppose that $\{x_{2n}\}$ is not a b-Cauchy sequence. Then there exists an $\epsilon > 0$ for which we can find subsequences $\{x_{2m_k}\}$ and $\{x_{2n_k}\}$ of $\{x_{2n}\}$ with
$n_k > m_k > k$ such that

$$d(x_{2m_k}, x_{2n_k}) \geq \epsilon \quad \text{and} \quad d(x_{2m_k}, x_{2n_k-2}) < \epsilon \tag{2.10}$$

From the inequality (2.2), (2.10) and by the b-triangular inequality, we have

$$\begin{aligned} \epsilon &\leq d(x_{2n_k}, x_{2m_k}) \\ &\leq s[d(x_{2n_k}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2m_k})] \\ &= sd(x_{2n_k}, x_{2n_k-1}) + sd(fx_{2n_k-2}, gx_{2m_k-1}) \\ &\leq sd(x_{2n_k}, x_{2n_k-1}) + \beta(M(x_{2n_k-2}, x_{2m_k-1}))M(x_{2n_k-2}, x_{2m_k-1}) + L(x_{2n_k-2}, x_{2m_k-1}) \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} M(x_{2n_k-2}, x_{2m_k-1}) &= \max\{d(x_{2n_k-2}, x_{2m_k-1}), d(x_{2n_k-2}, fx_{2n_k-2}), d(x_{2m_k-1}, gx_{2m_k-1})\} \\ &= \max\{d(x_{2n_k-2}, x_{2m_k-1}), d(x_{2n_k-2}, x_{2n_k-1}), d(x_{2m_k-1}, x_{2m_k})\} \end{aligned}$$

and

$$\begin{aligned} N(x_{2n_k-2}, x_{2m_k-1}) &= \min\{d(x_{2n_k-2}, fx_{2n_k-2}), d(x_{2n_k-2}, gx_{2m_k-1}), d(x_{2m_k-1}, fx_{2n_k-2})\} \\ &= \min\{d(x_{2n_k-2}, x_{2n_k-1}), d(x_{2n_k-2}, x_{m_k}), d(x_{2m_k-1}, x_{2n_k-1})\}. \end{aligned}$$

Taking limit superior as $n \to \infty$ on $M(x_{2n_k-2}, x_{2m_k-1})$ and $N(x_{2n_k-2}, x_{2m_k-1})$, we get

$$\limsup_{n\to\infty} M(x_{2n_k-2}, x_{2m_k-1}) = \limsup_{n\to\infty} d(x_{2n_k-2}, x_{2m_k-1})$$

and $\limsup\limits_{n\to\infty} N(x_{2n_k-2}, x_{2m_k-1}) = 0$. From the b-triangular inequality, we have

$$d(x_{2n_k-2}, x_{2m_k-1}) \leq s[d(x_{2n_k-2}, x_{2m_k}) + d(x_{2m_k}, x_{2m_k-1})]$$

Taking limit superior as $n \to \infty$ and using (2.10) in the above inequality, we get

$$\limsup_{n\to\infty} d(x_{2n_k-2}, x_{2m_k-1}) \leq s\epsilon.$$

Taking limit superior as $n \to \infty$ in (2.11), we get

$$\begin{aligned} \epsilon &\leq \limsup_{n\to\infty} [sd(x_{2n_k}, x_{2n_k-1}) + \beta(M(x_{2n_k-2}, x_{2m_k-1}))M(x_{2n_k-2}, x_{2m_k-1}) + LN(x_{2n_k-2}, x_{2m_k-1})] \\ &= \limsup_{n\to\infty} \beta(M(x_{2n_k-2}, x_{2m_k-1})) \limsup_{n\to\infty} M(x_{2n_k-2}, x_{2m_k-1}) + L \limsup_{n\to\infty} N(x_{2n_k-2}, x_{2m_k-1}) \\ &\leq \epsilon s \limsup_{n\to\infty} \beta(M(x_{2n_k-2}, x_{2m_k-1})). \end{aligned}$$

Therefore

$$\tfrac{1}{s} \leq \limsup_{n\to\infty} \beta(M(x_{2n_k-2}, x_{2m_k-1})) \leq \tfrac{1}{s}$$

implies that $\limsup\limits_{n\to\infty} \beta(M(x_{2n_k-2}, x_{2m_k-1})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, it follows that $\limsup\limits_{n\to\infty} M(x_{2n_k-2}, x_{2m_k-1})$. i.e., $\limsup\limits_{n\to\infty} d(x_{2n_k-2}, x_{2m_k-1}) = 0$.

From the inequality (2.10) and by using b-triangular inequality, we get

$$\begin{aligned}\epsilon &\leq d(x_{2m_k}, x_{2n_k}) \\ &\leq s[d(x_{2m_k}, x_{2m_k-1}) + d(x_{2m_k-1}, x_{2n_k})] \\ &\leq sd(x_{2m_k}, x_{2m_k-1}) + s^2[d(x_{2m_k-1}, x_{2n_k-2}) + d(x_{2n_k-2}, x_{2n_k})] \\ &\leq sd(x_{2m_k}, x_{2m_k-1}) + s^2 d(x_{2m_k-1}, x_{2n_k-2}) + s^3 d(x_{2n_k-2}, x_{2n_k-1}) + s^3 d(x_{2n_k-1}, x_{2n_k})\end{aligned}$$

Taking limit superior as $n \to \infty$, we get

$$0 < \epsilon \leq \limsup_{n\to\infty} d(x_{2m_k}, x_{2n_k}) \leq 0.$$

Therefore $\lim\limits_{n\to\infty} d(x_{2m_k}, x_{2n_k}) = 0$,
it is a contradiction.
Therefore, the sequence $\{x_n\}$ is a b-Cauchy sequence in X.
Since X is b-complete, there exists $x \in X$ such that $\lim\limits_{n\to\infty} x_n = x$. Suppose f is b-continuous, we have

$$fx = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} x_{2n+1} = x.$$

Therefore x is a fixed point of f.
We now prove that x is also a fixed point of g.
Suppose that $d(x, gx) > 0$.
From the inequality (2.2), we have

$$sd(x, gx) = sd(fx, gx) \leq \beta(M(x, x))M(x, x) + LN(x, x)$$

where

$$M(x, x) = \max\{d(x, x), d(x, fx), d(x, gx)\} = d(x, gx)$$

and

$$N(x, x) = \min\{d(x, fx), d(x, gx), d(x, fx)\} = 0.$$

Therefore

$$sd(x, gx) \leq \beta(d(x, gx))d(x, gx) < \frac{1}{s}d(x, gx),$$

which is a contradiction. Therefore x is a common fixed point of f and g.

On the similar lines, we can obtain x is a common fixed point of f and g, when g is b-continuous.

Uniqueness of common fixed point follows from the inequality (2.2). □

Remark 2.7. Theorem 1.9 and Theorem 1.10 follow as corollaries of Theorem 2.5 and Theorem 2.6 respectively by choosing $L = 0$.

The following is an example in support of Theorem 2.5, in which we show the importance of L.

Example 2.8. Let $X = [0, \infty)$ and let $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 4 & \text{if } x, y \in (0, 1), \\ \frac{9}{2} + \frac{1}{x+y} & \text{if } x, y \in [1, \infty), \\ \frac{12}{5} & \text{otherwise.} \end{cases}$$

Then clearly (X, d) is a complete b-metric space with coefficient $s = \frac{25}{24}$.
Here we observe that when $x = \frac{10}{9}, z = 1 \in [1, \infty)$ and $y \in (0, 1)$, we have $d(x, z) = \frac{9}{2} + \frac{1}{x+z} = \frac{189}{38}$ and $d(x, y) + d(y, z) = \frac{12}{5} + \frac{12}{5} = \frac{24}{5}$ so that $d(x, z) \neq d(x, y) + d(y, z)$.
Hence d is a b-metric with $s = \frac{25}{24}$ but not a metric.
We define $f : X \to X$ by

$$f(x) = \begin{cases} 2 & \text{if } x \in [0,1) \\ \frac{1+x}{2} & \text{if } x \in [1,\infty). \end{cases}$$

We define $\beta : [0,\infty) \to [0, \frac{1}{s})$ by $\beta(t) = \frac{24}{25}e^{-t}$. Then $\beta \in \mathfrak{F}$.

Case (i). Let $x, y \in [0,1)$.
$d(fx, fy) = 0$ and clearly the inequality (2.1) holds in this case.

Case (ii). $x, y \in [1,\infty)$.

$$\begin{array}{lcl} d(fx,fy) & = & \frac{9}{2} + \frac{1}{x+y}, d(x,fx) = \frac{9}{2} + \frac{1}{(x+y)}, d(y,fy) = \frac{9}{2} + \frac{1}{x+y}, d(x,y) = \frac{9}{2} + \frac{1}{(x+y)}, \\ d(x,fy) = \frac{9}{2} + \frac{1}{x+y}, & & \\ d(y,fx) = \frac{9}{2} + \frac{1}{(x+y)} & & \\ M(x,y) & = & \max\{d(x,y), d(x,fx), d(y,fy, \frac{1}{2s}[d(x,fy) + d(y,fx)])\} \\ & = & \max\{\frac{9}{2} + \frac{1}{(x+y)}, \frac{9}{2} + \frac{1}{(x+y)}, \frac{9}{2} + \frac{1}{(x+y)}, \frac{12}{25}[\frac{9}{2} + \frac{1}{(x+y)} + \frac{9}{2} + \frac{1}{(x+y)}]\} = \frac{9}{2} + \frac{1}{(x+y)} \end{array}$$

and

$$N(x,y) = \min\{d(x,fx), d(x,fy), d(y,fx)\} = \frac{9}{2} + \frac{1}{x+y}.$$

We consider

$$\begin{array}{lcl} d(fx,fy) = \frac{9}{2} + \frac{1}{(x+y)} & \leq & \frac{24}{25}e^{-(\frac{9}{2} + \frac{1}{(x+y)})}(\frac{9}{2} + \frac{1}{(x+y)}) + 3 \times (\frac{9}{2} + \frac{1}{(x+y)}) \\ & \leq & \beta(M(x,y))M(x,y) + LN(x,y). \end{array}$$

Case (iii). $x \in [0,1), y \in [1,\infty)$.

$$\begin{array}{lcl} d(fx,fy) & = & \frac{9}{2} + \frac{1}{(x+y)}, d(x,fx) = \frac{12}{5}, d(y,fy) = \frac{9}{2} + \frac{1}{(x+y)}, d(x,y) = \frac{12}{5}, d(x,fy) = \frac{12}{5}, \\ d(y,fx) & = & \frac{9}{2} + \frac{1}{(x+y)}. \\ M(x,y) & = & \max\{d(x,y), d(x,fx), d(y,fy), \frac{1}{2s}[d(x,fy) + d(y,fx)]\} \\ & = & \max\{\frac{12}{5}, \frac{12}{5}, \frac{9}{2} + \frac{1}{(x+y)}, \frac{12}{25}[\frac{12}{5} + \frac{9}{2} + \frac{1}{(x+y)}]\} = \frac{9}{2} + \frac{1}{(x+y)} \end{array}$$

and

$$N(x,y) = \min\{d(x,fx), d(x,gy), d(y,fx)\} = \min\{\frac{12}{5}, \frac{12}{5}, \frac{9}{2} + \frac{1}{(x+y)}\} = \frac{12}{5}.$$

We consider

$$\begin{array}{lcl} d(fx,fy) = \frac{9}{2} + \frac{1}{(x+y)} & \leq & \frac{24}{25}e^{-(\frac{9}{2} + \frac{1}{(x+y)})}(\frac{9}{2} + \frac{1}{(x+y)}) + 3 \times \frac{12}{5} \\ & \leq & \beta(M(x,y))M(x,y) + LN(x,y). \end{array}$$

Case (iv). $x \in [1,\infty), y \in [0,1)$.

$$\begin{array}{lcl} d(fx,fy) & = & \frac{9}{2} + \frac{1}{(x+y)}, d(x,fx) = \frac{9}{2} + \frac{1}{(x+y)}, d(y,fy) = \frac{12}{5}, d(x,y) = \frac{12}{5}, \\ d(x,fy) & = & \frac{9}{2} + \frac{1}{(x+y)}, d(y,fx) = \frac{12}{5}. \\ M(x,y) & = & \max\{d(x,y), d(x,fx), d(y,fy), \frac{1}{2s}[d(x,fy) + d(y,fx)]\} \\ & = & \max\{\frac{12}{5}, \frac{9}{2} + \frac{1}{(x+y)}, \frac{12}{5}, \frac{12}{25}[\frac{9}{2} + \frac{1}{(x+y)} + \frac{12}{5}]\} = \frac{9}{2} + \frac{1}{(x+y)} \end{array}$$

and

$$N(x,y) = \min\{d(x,fx), d(x,fy), d(y,fx)\} = \min\{\frac{9}{2} + \frac{1}{(x+y)}, \frac{9}{2} + \frac{1}{(x+y)}, \frac{12}{5}\} = \frac{12}{5}.$$

We consider

$$\begin{aligned} d(fx,fy)=\tfrac{9}{2}+\tfrac{1}{(x+y)} &\leq \tfrac{24}{25}e^{-(\frac{9}{2}+\frac{1}{(x+y)})}(\tfrac{9}{2}+\tfrac{1}{(x+y)})+3\times\tfrac{12}{5} \\ &\leq \beta(M(x,y))M(x,y)+LN(x,y) \end{aligned}$$

From all the above cases, f is an almost Geraghty contraction type map with $L=3$.
Threfore f satisfies all the hypotheses of Theorem 2.5 and 1 is the unique fixed point of f.

Here we observe that if $L=0$ then the inequality (2.1) fails to hold.
For, by choosing $x=0$ and $y=2$ we have

$$\begin{aligned} d(fx,fy) &= 5, d(x,fx)=\tfrac{12}{5}, d(y,fy)=5, d(x,y)=\tfrac{12}{5}, d(x,fy)=\tfrac{12}{5}, d(y,fx)=5. \\ M(x,y) &= \max\{d(x,y),d(x,fx),d(y,fy),\tfrac{1}{2s}[d(x,fy)+d(y,fx)]\} \\ &= \max\{\tfrac{12}{5},\tfrac{12}{5},5,\tfrac{12}{25}[\tfrac{12}{5}+5]\}=5. \end{aligned}$$

Here we note that

$$d(fx,fy)=5\not\leq\beta(5)5=\beta(M(x,y))M(x,y)$$

for any $\beta\in\mathfrak{F}$. Hence Theorem 1.9 is not applicable.

The following is an example in support of Theorem 2.6, in which we show the importance of L.

Example 2.9. Let $X=[0,\infty)$ and let $d:X\times X\to[0,\infty)$ defined by

$$d(x,y)=\begin{cases} 0 & \text{if } x=y, \\ 4 & \text{if } x,y\in(0,1), \\ \frac{9}{2}+\frac{1}{x+y} & \text{if } x,y\in[1,\infty), \\ \frac{12}{5} & \text{otherwise.} \end{cases}$$

Then clearly (X,d) is a complete b-metric space with coefficient $s=2$.
Here we observe that when $x=\frac{10}{9}, z=1\in[1,\infty)$ and $y\in(0,1)$, we have $d(x,z)=\frac{9}{2}+\frac{1}{x+z}=\frac{189}{38}$ and $d(x,y)+d(y,z)=\frac{12}{5}+\frac{12}{5}=\frac{24}{5}$ so that $d(x,z)\neq d(x,y)+d(y,z)$.
Hence d is a b-metric with $s=2$ but not a metric.
We define $f,g:X\to X$ by

$$f(x)=\begin{cases} \frac{x(5-x)}{4} & \text{if } x\in[0,1) \\ \frac{1+2x}{2} & \text{if } x\in[1,\infty) \end{cases} \quad rm\ and\ g(x)=\begin{cases} x & \text{if } x\in[0,1) \\ \frac{1}{x} & \text{if } x\in[1,\infty), \end{cases}$$

Clearly g is b-continuous.
We define $\beta:[0,\infty)\to[0,\frac{1}{s})$ by $\beta(t)=\frac{e^{-t}}{3}$. Then $\beta\in\mathfrak{F}$.

Case (i). $x,y\in[0,1)$.

$$\begin{aligned} d(fx,gy) &= 4, d(x,fx)=4, d(y,gy)=4, d(x,y)=4, d(x,gy)=4, d(y,fx)=4. \\ M(x,y) &= \max\{d(x,y),d(x,fx),d(y,gy)\}=4, \\ N(x,y) &= \min\{d(x,fx),d(x,gy),d(y,fx)\}=4. \end{aligned}$$

We consider

$$sd(fx,gy)=8\leq\frac{e^{-4}}{3}4+2\times4=\beta(M(x,y))M(x,y)+LN(x,y).$$

Case (ii). $x,y\in[1,\infty)$.

$$\begin{aligned} d(fx,gy) &= \tfrac{12}{5}, d(x,fx)=\tfrac{9}{2}+\tfrac{1}{(x+y)}, d(y,gy)=\tfrac{12}{5}, \\ d(x,y) &= \tfrac{9}{2}+\tfrac{1}{(x+y)}, d(x,gy)=\tfrac{12}{5}, \\ d(y,fx) &= \tfrac{9}{2}+\tfrac{1}{(x+y)}. \\ M(x,y) &= \max\{d(x,y),d(x,fx),d(y,gy)\}=\tfrac{9}{2}+\tfrac{1}{(x+y)} \\ N(x,y) &= \min\{d(x,fx),d(x,gy),d(y,fx)\}=\tfrac{12}{5}. \end{aligned}$$

We consider

$$sd(fx,gy) = \tfrac{24}{5} \leq \frac{e^{-(\frac{9}{2}+\frac{1}{(x+y)})}}{3}(\tfrac{9}{2}+\tfrac{1}{(x+y)}) + 2\times\tfrac{12}{5} = \beta(M(x,y))M(x,y)+LN(x,y).$$

Case (iii). $x\in[0,1), y\in[1,\infty)$.

$$\begin{aligned} d(fx,gy) &= \tfrac{12}{5}, d(x,fx)=4, d(y,gy)=\tfrac{12}{5}, d(x,y)=\tfrac{12}{5}, d(x,gy)=4, d(y,fx)=\tfrac{12}{5}.\\ M(x,y) &= \max\{d(x,y),d(x,fx),d(y,gy)\}=4 \text{ and}\\ N(x,y) &= \min\{d(x,fx),d(x,gy),d(y,fx)\}=\tfrac{12}{5}.\end{aligned}$$

We consider

$$sd(fx,gy)=\frac{24}{5}\leq\frac{e^{-4}}{3}4+2\times\frac{12}{5}=\beta(M(x,y))M(x,y)+LN(x,y).$$

Case (iv). $x\in[1,\infty), y\in[0,1)$.

$$\begin{aligned} d(fx,gy) &= \tfrac{12}{5}, d(x,fx)=\tfrac{9}{2}+\tfrac{1}{(x+y)}, d(y,gy)=4, d(x,y)=\tfrac{12}{5}, d(x,gy)=\tfrac{12}{5}, d(y,fx)=\tfrac{12}{5}.\\ M(x,y) &= \max\{d(x,y),d(x,fx),d(y,gy)\}=\tfrac{9}{2}+\tfrac{1}{(x+y)} \text{ and}\\ N(x,y) &= \min\{d(x,fx),d(x,gy),d(y,fx)\}=\tfrac{12}{5}.\end{aligned}$$

We consider

$$\begin{aligned} sd(fx,gy)=\tfrac{24}{5} &\leq \frac{e^{-(\frac{9}{2}+\frac{1}{(x+y)})}}{3}(\tfrac{9}{2}+\tfrac{1}{(x+y)})+2\times\tfrac{12}{5}\\ &= \beta(M(x,y))M(x,y)+LN(x,y).\end{aligned}$$

From all the above cases, (f,g) is an almost Geraghty contraction type pair of maps with $L=2$.
Threfore f and g satisfy all the hypotheses of Theorem 2.6 and 0 is the unique common fixed point of f and g.

Here we observe that if $L=0$ then the inequality (2.2) fails to hold.
For, we choose $x=0$ and $y=2$ we have

$$d(fx,gy)=\frac{12}{5}, d(x,fx)=4, d(y,gy)=\frac{12}{5}, d(x,y)=\frac{12}{5}.$$

$$M(x,y)=\max\{d(x,y),d(x,fx),d(y,gy)\}=\max\{\frac{12}{5},4,\frac{12}{5}\}=4.$$

Here we note that

$$sd(fx,gy)=\frac{24}{5}\nleq\beta(4)4=\beta(M(x,y))M(x,y)$$

for any $\beta\in\mathfrak{F}$.
Hence Theorem 1.10 is not applicable.

Remark 2.10. Remark 2.7, Example 2.8 and Example 2.9 suggest that Theorem 2.5 and Theorem 2.6 are generalizations of Theorem 1.9 and Theorem 1.10 respectively.

3. Common fixed points of genaralized contraction pair of maps with rational expressions

Definition 3.1. Let (X,d) be a b-metric space with coefficient $s\geq 1$ and $f,g: X\to X$ be two selfmaps. Assume that there exist non-negative reals $\lambda_i, i=1,2,3,4,5$ with $\lambda_1+\lambda_2+\lambda_3+2s\lambda_4+2s\lambda_5<1$ satisfying:

$$d(fx,gy)\leq\lambda_1 d(x,y)+\lambda_2\frac{d(x,fx)d(y,gy)}{1+d(x,y)}+\lambda_3\frac{d(x,gy)d(y,fx)}{1+d(x,y)}+\lambda_4\frac{d(x,fx)d(x,gy)}{1+d(x,y)}+\lambda_5\frac{d(y,fx)d(y,gy)}{1+d(x,y)} \quad (3.1)$$

for all $x,y\in X$, then we say that the (f,g) is a generalized contraction pair of maps with rational expressions.

Example 3.2. Let $X = [0,1]$ and let $d : X \times X \to [0,\infty)$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ (x+y)^2 & \text{if } x \neq y. \end{cases}$$

Then clearly (X,d) is a complete b-metric space with coefficient $s = 2$.
We define $f, g : X \to X$ by

$$f(x) = \tfrac{x}{64} \text{ and } g(x) = \begin{cases} 0 & \text{if } x \in [0,1] \setminus \{\frac{1}{2}\} \\ \frac{1}{128} & \text{if } x = \frac{1}{2}. \end{cases}$$

Take $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{3}, \lambda_3 = \lambda_4 = \lambda_5 = 0$.

Case (i). $x = \frac{1}{2}, y = \frac{1}{2}$.
$d(fx, gy) = 0$. Clearly the inequality (3.1) holds in this case.

Case (ii). $x \neq \frac{1}{2}, y = \frac{1}{2}$.

$$\begin{aligned}
d(fx,gy) &= (\tfrac{x}{64} + \tfrac{1}{128})^2, d(x,fx) = (x + \tfrac{x}{64})^2, d(y,gy) = (\tfrac{65}{128})^2, d(x,y) = (x + \tfrac{1}{2})^2, \\
d(x,gy) &= (x + \tfrac{1}{128})^2, d(y,fx) = (\tfrac{1}{2} + \tfrac{x}{64})^2, \\
\tfrac{d(x,fx)d(y,gy)}{1+d(x,y)} &= \tfrac{(x+\frac{x}{64})^2(\frac{65}{128})^2}{1+(x+\frac{1}{2})^2}, \tfrac{d(x,gy)d(y,fx)}{1+d(x,y)} = \tfrac{(x+\frac{1}{128})^2(\frac{1}{2}+\frac{x}{64})^2}{1+(x+\frac{1}{2})^2}, \tfrac{d(x,fx)d(x,gy)}{1+d(x,y)} = \tfrac{(x+\frac{x}{64})^2(x+\frac{1}{128})^2}{1+(x+\frac{1}{2})^2}, \\
\tfrac{d(y,fx)d(y,gy)}{1+d(x,y)} &= (\tfrac{66}{91})(5 + \tfrac{1}{(x+y)}).
\end{aligned}$$

We consider

$$\begin{aligned}
d(fx,gy) &= (\tfrac{x}{64} + \tfrac{1}{128})^2 \\
&\leq \tfrac{1}{2}(x + \tfrac{1}{2})^2 + \tfrac{1}{3}\tfrac{(x+\frac{x}{64})^2(\frac{65}{128})^2}{1+(x+\frac{1}{2})^2} \\
&= \lambda_1 d(x,y) + \lambda_2 \tfrac{d(x,fx)d(y,gy)}{1+d(x,y)} + \lambda_3 \tfrac{d(x,gy)d(y,fx)}{1+d(x,y)} + \lambda_4 \tfrac{d(x,fx)d(x,gy)}{1+d(x,y)} + \lambda_5 \tfrac{d(y,fx)d(y,gy)}{1+d(x,y)}.
\end{aligned}$$

Case (iii). $x = \frac{1}{2}, y \neq \frac{1}{2}$.

$$d(fx,gy) = (\frac{1}{128})^2, d(x,fx) = (x + \frac{1}{128})^2, d(y,gy) = y^2,$$

$$d(x,y) = (\frac{1}{2} + y)^2, d(x,gy) = x^2,$$

$$d(y,fx) = (y + \frac{x}{64})^2, \frac{d(x,fx)d(y,gy)}{1+d(x,y)} = \frac{(x+\frac{1}{128})^2 y^2}{1+(\frac{1}{2}+y)^2}, \frac{d(x,gy)d(y,fx)}{1+d(x,y)} = \frac{x^2(y+\frac{x}{64})^2}{1+(\frac{1}{2}+y)^2},$$

$$\frac{d(x,fx)d(x,gy)}{1+d(x,y)} = \frac{(x+\frac{1}{128})^2 x^2}{1+(\frac{1}{2}+y)^2}, \frac{d(y,fx)d(y,gy)}{1+d(x,y)} = \frac{((y+\frac{x}{64})^2)y^2}{(1+(\frac{1}{2}+y)^2)}.$$

We consider

$$\begin{aligned}
d(fx,gy) &= (\tfrac{1}{128})^2 \\
&\leq \tfrac{1}{2}(\tfrac{1}{2} + y)^2 + \tfrac{1}{3}\tfrac{(x+\frac{1}{128})^2 y^2}{1+(\frac{1}{2}+y)^2} \\
&= \lambda_1 d(x,y) + \lambda_2 \tfrac{d(x,fx)d(y,gy)}{1+d(x,y)} + \lambda_3 \tfrac{d(x,gy)d(y,fx)}{1+d(x,y)} + \lambda_4 \tfrac{d(x,fx)d(x,gy)}{1+d(x,y)} + \lambda_5 \tfrac{d(y,fx)d(y,gy)}{1+d(x,y)}.
\end{aligned}$$

Case (iv). $x \neq \frac{1}{2}, y \neq \frac{1}{2}$.

$$d(fx,gy) = (\frac{x}{64})^2, d(x,fx) = (x + \frac{x}{64})^2, d(y,gy) = y^2,$$

$$d(x,y) = (x+y)^2, d(x,gy) = x^2,$$

$$d(y,fx) = (y + \frac{x}{64})^2, \frac{d(x,fx)d(y,gy)}{1+d(x,y)} = \frac{(x+\frac{x}{64})^2 y^2}{1+(x+y)^2}, \frac{d(x,gy)d(y,fx)}{1+d(x,y)} = \frac{x^2(y+\frac{x}{64})^2}{1+(x+y)^2},$$

$$\frac{d(x,fx)d(x,gy)}{1+d(x,y)} = \frac{(x+\frac{x}{64})^2x^2}{1+(x+y)^2}, \frac{d(y,fx)d(y,gy)}{1+d(x,y)} = \frac{((y+\frac{x}{64})^2)y^2}{(1+(x+y)^2)}.$$

We consider

$$\begin{aligned} d(fx,gy) &= (\tfrac{x}{64})^2 \\ &\leq \tfrac{1}{2}(x+y)^2 + \tfrac{1}{3}\tfrac{(x+\frac{x}{64})^2y^2}{1+(x+y)^2} \\ &= \lambda_1 d(x,y) + \lambda_2\tfrac{d(x,fx)d(y,gy)}{1+d(x,y)} + \lambda_3\tfrac{d(x,gy)d(y,fx)}{1+d(x,y)} + \lambda_4\tfrac{d(x,fx)d(x,gy)}{1+d(x,y)} + \lambda_5\tfrac{d(y,fx)d(y,gy)}{1+d(x,y)}. \end{aligned}$$

From all the above cases, (f,g) is a generalized contraction pair of maps with rational expressions.

Proposition 3.3. *Let (X,d) be a b-metric space with coefficient $s \geq 1$ and (f,g) be a generalized contraction pair of maps with rational expressions. Then u is a fixed point of f if and only if u is a fixed point of g. Moreover, in that case u is unique.*

Proof. Let u be a fixed point of f. i.e., $fu = u$.
Suppose that $gu \neq u$.
We now consider

$$\begin{aligned} d(u,gu) &= d(fu,gu) \\ &\leq \lambda_1 d(u,u) + \lambda_2\tfrac{d(u,fu)d(u,gu)}{1+d(u,u)} + \lambda_3\tfrac{d(u,gu)d(u,fu)}{1+d(u,u)} + \lambda_4\tfrac{d(u,fu)d(u,gu)}{1+d(u,u)} + \lambda_5\tfrac{d(u,fu)d(u,gu)}{1+d(u,u)} = 0. \end{aligned}$$

Therefore $d(u,gu) = 0$ and hence $u = gu$.
Therefore u is a fixed point of g. Hence u is a common fixed point of f and g.
Similarly, it is easy to see that if u is a fixed point of g then u is a fixed point of f also.
Suppose u and v are two common fixed points of f and g with $u \neq v$.
From the inequality (3.1), we have

$$\begin{aligned} d(u,v) &= d(fu,gv) \\ &\leq \lambda_1 d(u,v) + \lambda_2\tfrac{d(u,fu)d(v,gv)}{1+d(u,v)} + \lambda_3\tfrac{d(u,gv)d(v,fu)}{1+d(u,v)} + \lambda_4\tfrac{d(u,fu)d(u,gv)}{1+d(u,v)} + \lambda_5\tfrac{d(v,fu)d(v,gv)}{1+d(u,v)} \\ &\leq \lambda_1 d(u,v) + \lambda_2\tfrac{d(u,u)d(v,v)}{1+d(u,v)} + \lambda_3\tfrac{d(u,v)d(v,u)}{1+d(u,v)} + \lambda_4\tfrac{d(u,u)d(u,v)}{1+d(u,v)} + \lambda_5\tfrac{d(v,u)d(v,v)}{1+d(u,v)} \\ &= (\lambda_1+\lambda_3)d(u,v) \\ &< d(u,v), \end{aligned}$$

which is a contradiction.
Therefore $d(u,v) = 0$ and hence $u = v$. □

Theorem 3.4. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and (f,g) be a generalized contraction pair of maps with rational expressions. Then f and g have a unique common fixed point in X, provided either f or g is b-continuous.*

Proof. Let $x_0 \in X$ be arbitrary. We define the sequence $\{x_n\}$ in X by
$x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n = 0,1,2,\ldots$. If $x_{2n} = x_{2n+1}$ for some n, then $x_{2n} = x_{2n+1} = fx_{2n}$ so that x_{2n} is a fixed point of f.
By Proposition 3.3, we have x_{2n} is a fixed point of g so that x_{2n} is a common fixed point of f and g.
Similarly, if $x_{2n+1} = x_{2n+2}$ for some n then also we have x_{2n+1} is a common fixed point of g and f.
Hence without loss of generality we assume that $x_n \neq x_{n+1}$ for all n.
Suppose n is even. Then $n = 2m$, $m \in \mathbb{N}$.
We now consider

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) &= d(x_{2m+1}, x_{2m+2}) \\
&= d(fx_{2m}, gx_{2m+1}) \\
&\leq \lambda_1 d(x_{2m}, x_{2m+1}) + \lambda_2 \tfrac{d(x_{2m}, fx_{2m}) d(x_{2m+1}, gx_{2m+1})}{1+d(x_{2m}, x_{2m+1})} \\
&\quad + \lambda_3 \tfrac{d(x_{2m}, gx_{2m+1}) d(x_{2m+1}, fx_{2m})}{1+d(x_{2m}, x_{2m+1})} + \lambda_4 \tfrac{d(x_{2m}, fx_{2m}) d(x_{2m}, gx_{2m+1})}{1+d(x_{2m}, x_{2m+1})} \\
&\quad + \lambda_5 \tfrac{d(x_{2m+1}, gx_{2m}) d(x_{2m+1}, gx_{2m+1})}{1+d(x_{2m}, x_{2m+1})} \\
&= \lambda_1 d(x_{2m}, x_{2m+1}) + \lambda_2 \tfrac{d(x_{2m}, x_{2m+1}) d(x_{2m+1}, x_{2m+2})}{1+d(x_{2m}, x_{2m+1})} \\
&\quad + \lambda_3 \tfrac{d(x_{2m}, x_{2m+2}) d(x_{2m+1}, x_{2m+1})}{1+d(x_{2m}, x_{2m+1})} + \lambda_4 \tfrac{d(x_{2m}, x_{2m+1}) d(x_{2m}, x_{2m+2})}{1+d(x_{2m}, x_{2m+1})} \\
&\quad + \lambda_5 \tfrac{d(x_{2m+1}, x_{2m+1}) d(x_{2m+1}, x_{2m+2})}{1+d(x_{2m}, x_{2m+1})} \\
&\leq \lambda_1 d(x_{2m}, x_{2m+1}) + \lambda_2 \tfrac{d(x_{2m}, x_{2m+1}) d(x_{2m+1}, x_{2m+2})}{1+d(x_{2m}, x_{2m+1})} + \lambda_4 d(x_{2m}, x_{2m+2}) \\
&\leq \lambda_1 d(x_{2m}, x_{2m+1}) + \lambda_2 \tfrac{d(x_{2m}, x_{2m+1}) d(x_{2m+1}, x_{2m+2})}{1+d(x_{2m}, x_{2m+1})} + \lambda_4 s[d(x_{2m}, x_{2m+1}) + d(x_{2m+1}, x_{2m+2})].
\end{aligned}$$

Therefore

$$d(x_{2m+1}, x_{2m+2}) \leq \frac{\lambda_1 + s\lambda_4}{1 - \lambda_2 - s\lambda_4} d(x_{2m}, x_{2m+1}) = h_1 d(x_{2m}, x_{2m+1}),$$

where $0 \leq h_1 = \frac{\lambda_1 + s\lambda_4}{1-\lambda_2 - s\lambda_4} < 1$.
Similarly, we can prove that

$$d(x_{2m+2}, x_{2m+3}) \leq \frac{\lambda_1 + s\lambda_5}{1 - \lambda_2 - s\lambda_5} d(x_{2m+1}, x_{2m+2}) = h_2 d(x_{2m}, x_{2m+1}),$$

where $0 \leq h_2 = \frac{\lambda_1 + s\lambda_5}{1-\lambda_2 - s\lambda_5} < 1$.
Now, if n is odd, then $n = 2m + 1$, $m \in \mathbb{N}$. We now consider

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) &= d(x_{2m+2}, x_{2m+3}) \\
&= d(x_{2m+3}, x_{2m+2}) \\
&= d(fx_{2m+2}, gx_{2m+1}) \\
&\leq \lambda_1 d(x_{2m+2}, x_{2m+1}) + \lambda_2 \tfrac{d(x_{2m+2}, fx_{2m+2}) d(x_{2m+1}, gx_{2m+1})}{1+d(x_{2m+2}, x_{2m+1})} \\
&\quad + \lambda_3 \tfrac{d(x_{2m+2}, gx_{2m+1}) d(x_{2m+1}, fx_{2m+2})}{1+d(x_{2m+2}, x_{2m+1})} + \lambda_4 \tfrac{d(x_{2m+2}, fx_{2m+2}) d(x_{2m+2}, gx_{2m+1})}{1+d(x_{2m+2}, x_{2m+1})} \\
&\quad + \lambda_5 \tfrac{d(x_{2m+1}, fx_{2m+2}) d(x_{2m+1}, gx_{2m+1})}{1+d(x_{2m+2}, x_{2m+1})} \\
&= \lambda_1 d(x_{2m+2}, x_{2m+1}) + \lambda_2 \tfrac{d(x_{2m+2}, x_{2m+3}) d(x_{2m+1}, x_{2m+2})}{1+d(x_{2m+2}, x_{2m+1})} \\
&\quad + \lambda_3 \tfrac{d(x_{2m+2}, x_{2m+2}) d(x_{2m+1}, x_{2m+3})}{1+d(x_{2m+2}, x_{2m+1})} + \lambda_4 \tfrac{d(x_{2m+2}, x_{2m+3}) d(x_{2m+2}, x_{2m+2})}{1+d(x_{2m+2}, x_{2m+1})} \\
&\quad + \lambda_5 \tfrac{d(x_{2m+1}, x_{2m+3}) d(x_{2m+1}, x_{2m+2})}{1+d(x_{2m+2}, x_{2m+1})} \\
&\leq \lambda_1 d(x_{2m+2}, x_{2m+1}) + \lambda_2 d(x_{2m+3}, x_{2m+2}) + \lambda_5 d(x_{2m+1}, x_{2m+3}) \\
&\leq \lambda_1 d(x_{2m+1}, x_{2m+2}) + \lambda_2 d(x_{2m+3}, x_{2m+2}) + \lambda_5 s[d(x_{2m+1}, x_{2m+2}) + d(x_{2m+2}, x_{2m+3})].
\end{aligned}$$

Therefore $d(x_{2m+3}, x_{2m+2}) \leq \frac{\lambda_1 + s\lambda_5}{1-\lambda_2 - s\lambda_5} d(x_{2m+2}, x_{2m+1}) = h_2 d(x_{2m+2}, x_{2m+1})$,
where $0 \leq h_2 = \frac{\lambda_1 + s\lambda_5}{1-\lambda_2 - s\lambda_5} < 1$.
Similarly, we can prove that $d(x_{2m+3}, x_{2m+4}) \leq \frac{\lambda_1 + s\lambda_4}{1-\lambda_2 - s\lambda_4} d(x_{2m+2}, x_{2m+3}) = h_1 d(x_{2m}, x_{2m+1})$,
where $0 \leq h_1 = \frac{\lambda_1 + s\lambda_4}{1-\lambda_2 - s\lambda_4} < 1$.

Let $0 \leq h = \max\{h_1, h_2\} < 1$.
Therefore $d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. Now by Lemma 1.7, $\{x_n\}$ is a b-Cauchy sequence in X.
Since (X, d) is b-complete, we have $\{x_n\}$ is b-convergent to some point x(say) in X.
Therefore $x = \lim\limits_{n\to\infty} x_{2n+1} = \lim\limits_{n\to\infty} fx_{2n}$ and $x = \lim\limits_{n\to\infty} x_{2n+2} = \lim\limits_{n\to\infty} gx_{2n+1}$
so that $\lim\limits_{n\to\infty} fx_{2n} = x = \lim\limits_{n\to\infty} gx_{2n+1}$.
We assume that f is b-continuous. Since $x_{2n} \to x$ as $n \to \infty$, we have $fx_{2n} \to fx$ as $n \to \infty$.
By b-triangular inequality, we have $0 \leq d(x, fx) \leq s[d(x, fx_{2n}) + d(fx_{2n}, fx)]$.

Taking limit superior as $n \to \infty$, we get
$0 \leq d(x, fx) \leq 0$. Therefore x is a fixed point of f.
By Proposition 3.3, we have x is a unique common fixed point of f and g.

On the similar lines, we can prove that x is a unique common fixed point of f and g, whenever g is b-continuous. □

The following is an example in support of Theorem 3.4.

Example 3.5. Let $X = [0, \infty)$ and let $d : X \times X \to [0, \infty)$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 4 & \text{if } x, y \in (0,1), \\ 5 + \frac{1}{x+y} & \text{if } x, y \in [1, \infty), \\ \frac{66}{25} & \text{otherwise.} \end{cases}$$

Then clearly (X, d) is a complete b-metric space with coefficient $s = \frac{25}{24}$.
We define $f, g : X \to X$ by

$$f(x) = \begin{cases} \frac{x^2}{4} + 2 & \text{if } x \in [0,1) \\ 3x^2 - 2 & \text{if } x \in [1, \infty) \end{cases} \text{ and } g(x) = \begin{cases} x^2 & \text{if } x \in [0,1) \\ \frac{1}{x^2} & \text{if } x \in [1, \infty), \end{cases}$$

Clearly g is b-continuous. We take $\lambda_1 = \frac{2}{3}, \lambda_2 = \frac{1}{4}, \lambda_3 = \frac{1}{80}, \lambda_4 = \frac{1}{320}, \lambda_5 = \frac{1}{320}$.
Then $\lambda_1 + \lambda_2 + \lambda_3 + 2s\lambda_4 + 2s\lambda_5 < 1$. Without loss of generality we assume that $x \geq y$.

Case (i). $x, y \in (0, 1)$.

$$d(fx, gy) = \frac{66}{25}, d(x, fx) = \frac{66}{25}, d(y, gy) = 4, d(x, y) = 4, d(x, gy) = 4, d(y, fx) = \frac{66}{25},$$

$$\frac{d(x, fx)d(y, gy)}{1 + d(x, y)} = \frac{264}{125}, \frac{d(x, gy)d(y, fx)}{1 + d(x, y)} = \frac{264}{125}, \frac{d(x, fx)d(x, gy)}{1 + d(x, y)} = \frac{264}{125}, \frac{d(y, fx)d(y, gy)}{1 + d(x, y)} = \frac{264}{125}.$$

We consider

$$\begin{aligned} d(fx, gy) = \frac{66}{25} &\leq \frac{2}{3} \times 4 + \frac{1}{4} \times \frac{264}{125} + \frac{1}{80} \times \frac{264}{125} + \frac{1}{320} \times \frac{264}{125} + \frac{1}{320} \times \frac{264}{125} \\ &= \lambda_1 d(x, y) + \lambda_2 \frac{d(x,fx)d(y,gy)}{1+d(x,y)} + \lambda_3 \frac{d(x,gy)d(y,fx)}{1+d(x,y)} + \lambda_4 \frac{d(x,fx)d(x,gy)}{1+d(x,y)} + \lambda_5 \frac{d(y,fx)d(y,gy)}{1+d(x,y)}. \end{aligned}$$

Case (ii). $x, y \in [1, \infty)$.

$$\begin{aligned} d(fx, gy) &= \frac{66}{25}, d(x, fx) = 5 + \frac{1}{(x+y)}, d(y, gy) = \frac{66}{25}, \\ d(x, y) &= 5 + \frac{1}{(x+y)}, d(x, gy) = \frac{66}{25}, \\ d(y, fx) &= 5 + \frac{1}{(x+y)}, \\ \frac{d(x,fx)d(y,gy)}{1+d(x,y)} &= \frac{(\frac{22}{25})(5+\frac{1}{(x+y)})}{6+\frac{1}{(x+y)}}, \frac{d(x,gy)d(y,fx)}{1+d(x,y)} = \frac{(\frac{33}{125})(5+\frac{1}{(x+y)})}{6+\frac{1}{(x+y)}}, \frac{d(x,fx)d(x,gy)}{1+d(x,y)} = \frac{(\frac{33}{625})(5+\frac{1}{(x+y)})}{6+\frac{1}{(x+y)}}, \\ \frac{d(y,fx)d(y,gy)}{1+d(x,y)} &= \frac{(\frac{33}{1250})(5+\frac{1}{(x+y)})}{6+\frac{1}{(x+y)}}. \end{aligned}$$

We consider

$$\begin{aligned} d(fx, gy) = \frac{66}{25} &\leq \frac{2}{3} \times (5 + \frac{1}{(x+y)}) + \frac{1}{4} \times \frac{(\frac{22}{25})(5+\frac{1}{(x+y)})}{6+\frac{1}{(x+y)}} + \frac{1}{80} \times \frac{(\frac{33}{125})(5+\frac{1}{(x+y)})}{6+\frac{1}{(x+y)}} + \frac{1}{320} \times \frac{(\frac{33}{625})(5+\frac{1}{(x+y)})}{6+\frac{1}{(x+y)}} \\ &\quad + \frac{1}{320} \times \frac{(\frac{33}{1250})(5+\frac{1}{(x+y)})}{6+\frac{1}{(x+y)}} \\ &= \lambda_1 d(x, y) + \lambda_2 \frac{d(x,fx)d(y,gy)}{1+d(x,y)} + \lambda_3 \frac{d(x,gy)d(y,fx)}{1+d(x,y)} + \lambda_4 \frac{d(x,fx)d(x,gy)}{1+d(x,y)} + \lambda_5 \frac{d(y,fx)d(y,gy)}{1+d(x,y)}. \end{aligned}$$

Case (iii). $x \in [1, \infty), y \in [0, 1)$.

$$\begin{aligned} d(fx, gy) &= \frac{66}{25}, d(x, fx) = 5 + \frac{1}{(x+y)}, d(y, gy) = 4, d(x, y) = \frac{66}{25}, d(x, gy) = \frac{66}{25}, d(y, fx) = \frac{66}{25}, \\ \frac{d(x,fx)d(y,gy)}{1+d(x,y)} &= \frac{(100)(5+\frac{1}{(x+y)})}{91}, \frac{d(x,gy)d(y,fx)}{1+d(x,y)} = \frac{4356}{2275}, \frac{d(x,fx)d(x,gy)}{1+d(x,y)} = (\frac{66}{91})(5 + \frac{1}{(x+y)}), \\ \frac{d(y,fx)d(y,gy)}{1+d(x,y)} &= \frac{264}{91}. \end{aligned}$$

We consider

$$\begin{aligned} d(fx,gy)=\tfrac{66}{25} &\leq \tfrac{2}{3}\times\tfrac{66}{25}+\tfrac{1}{4}\times\frac{(100)(5+\frac{1}{(x+y)})}{91}+\tfrac{1}{80}\times\tfrac{4356}{2275}+\tfrac{1}{320}\times(\tfrac{66}{91})(5+\tfrac{1}{(x+y)}) \\ &\quad +\tfrac{1}{320}\times\tfrac{264}{91} \\ &= \lambda_1 d(x,y)+\lambda_2\tfrac{d(x,fx)d(y,gy)}{1+d(x,y)}+\lambda_3\tfrac{d(x,gy)d(y,fx)}{1+d(x,y)}+\lambda_4\tfrac{d(x,fx)d(x,gy)}{1+d(x,y)} \\ &\quad +\lambda_5\tfrac{d(y,fx)d(y,gy)}{1+d(x,y)}. \end{aligned}$$

From all the above cases, f and g satisfy all hypotheses of Theorem 3.4 and 1 is the unique common fixed point of f and g.

Corollary 3.6. *Let (X,d) be a b-metric space with coefficient $s\geq 1$ and $T:X\to X$ be a mapping such that*

$$d(Tx,Ty)\leq\lambda_1 d(x,y)+\lambda_2\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}+\lambda_3\frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}+\lambda_4\frac{d(x,Tx)d(x,Ty)}{1+d(x,y)}+\lambda_5\frac{d(y,Tx)d(y,Ty)}{1+d(x,y)}$$

where $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ and λ_5 are nonnegative constants with $\lambda_1+\lambda_2+\lambda_3+2s\lambda_4+2s\lambda_5<1$. Then T has a unique fixed point in X. Moreover, for any $x\in X$, the iterative sequence $\{T^n x\}$ is b-convergent to the fixed point.

Proof. By choosing $f=g=T$ in Theorem 3.4, the conclusion of this corollary follows. □

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