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On the zeros of the Polar Derivative of a polynomial

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Abstract

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that $a_n \geq a_{n-1} \geq ... \geq a_0 > 0$, then according to the Enström-Kakeya Theorem all the zeros of $P(z)$ lie in $|z| \leq 1$. Aziz and Mohammad have shown that under the same condition on coefficients the zeros of $P(z)$ whose modulus is greater than or equal to $\frac{n}{n+1}$ are simple.In this paper, we extend the above result to the polar derivative.

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1. Introduction and Preliminaries

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, then concerning the distribution of zeros of $P(z)$, Eneström and Kakeya proved the following result:

Theorem 1.1. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq ... \geq a_0 > 0.$$

Then $P(z)$ has all its zeros in $|z| \leq 1$.

Regarding the multiplicity of zeros of a polynomial, A.Aziz and Mohammad[1] proved the following result:

Theorem 1.2. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq ... \geq a_0 > 0.$$

Then all the zeros of $P(z)$ of modulus greater than or equal to $\frac{n}{n+1}$ are simple.

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In [2] A.Aziz and Mohammad gave a generalisation as well as a refinement of Theorem (1.2) as follows:

Theorem 1.3. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* $n > 1$ *such that for some* $t > 0$,

$$ta_j \geq a_{j-1}, j = 2, 3, ..., n$$

a_0 *may be real or complex number. Then all the zeros of* $P(z)$ *of modulus greater than or equal to* $\frac{t(n-1)}{n}$ *are simple.*

In literature, there exist generalizations and extensions of Theorem (1.2) and Theorem (1.3)(see [3], [6]). Let α be a complex number. If $P(z)$ is a polynomial of degree n, then the polar derivative of $P(z)$ with respect to α, denoted by $D_\alpha P(z)$ is defined by

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$$

Clearly $D_\alpha P(z)$ is a polynomial of degree at most $n - 1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

Ramulu and Reddy [8] found the bounds for the zeros of $D_\alpha P(z)$ under certain conditions on its coefficients. In fact they proved:

Theorem 1.4. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that*

$$na_0 \leq (n-1)a_1 \leq (n-2)a_2 \leq ... \leq 3a_{n-3} \leq 2a_{n-2} \leq a_{n-1}.$$

Then all the zeros of $D_0 P(z)$ *lie in*

$$|z| \leq \frac{a_{n-1} - na_0 + |na_0|}{a_{n-1}}.$$

Theorem 1.5. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that*

$$na_0 \geq (n-1)a_1 \geq (n-2)a_2 \geq ... \geq 3a_{n-3} \geq 2a_{n-2} \geq a_{n-1}.$$

Then all the zeros of $D_0 P(z)$ *lie in*

$$|z| \leq \frac{|na_0| + na_0 - a_{n-1}}{a_{n-1}}.$$

In literature, there exist generalizations and extensions of Theorem (1.4) and Theorem (1.5)(see [5], [9]).

2. Main Result

In this paper, we obtain a region in which the zeros of polar derivative are simple. In fact we prove the following:

Theorem 2.1. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n*. Let* $\alpha \in \mathbb{R}$ *such that*

$$(n-1)\left[n\alpha a_n + a_{n-1}\right] \geq (n-2)\left[(n-1)\alpha a_{n-1} + 2a_{n-2}\right] \geq ... \geq 3\left[4\alpha a_4 + (n-3)a_3\right]$$
$$\geq 2\left[3\alpha a_3 + (n-2)a_2\right] \geq \left[2\alpha a_2 + (n-1)a_1\right].$$

Then all the zeros of $D_\alpha P(z)$ whose modulus is greater than or equal to

$$\frac{(n-1)\left[n\alpha a_n + a_{n-1}\right] - \left[2\alpha a_2 + (n-1)a_1\right] + |2\alpha a_2 + (n-1)a_1|}{(n-1)|n\alpha a_n + a_{n-1}|}$$

are simple.

Proof: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, then

$$\begin{aligned} D_\alpha P(z) &= (n\alpha a_n + a_{n-1})z^{n-1} + ((n-1)\alpha a_{n-1} + 2a_{n-2})z^{n-2} \\ &\quad + ... + (3\alpha a_3 + (n-2)a_2)z^2 + (2\alpha a_2 + (n-1)a_1)z + (\alpha a_1 + na_0). \end{aligned}$$

So

$$\begin{aligned} D'_\alpha P(z) &= (n-1)(n\alpha a_n + a_{n-1})z^{n-2} + (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})z^{n-3} + ... \\ &\quad + 2(3\alpha a_3 + (n-2)a_2)z + (2\alpha a_2 + (n-1)a_1). \end{aligned}$$

Now consider the polynomial

$Q(z) = (1-z)D'_\alpha P(z)$

That is

$$\begin{aligned} Q(z) &= (1-z)\{(n-1)(n\alpha a_n + a_{n-1})z^{n-2} + (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})z^{n-3} + ... \\ &\quad + 2(3\alpha a_3 + (n-2)a_2)z + (2\alpha a_2 + (n-1)a_1)\} \end{aligned}$$

which implies

$$\begin{aligned} Q(z) &= (n-1)(n\alpha a_n + a_{n-1})z^{n-2} + (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})z^{n-3} + ... \\ &\quad + 2(3\alpha a_3 + (n-2)a_2)z + (2\alpha a_2 + (n-1)a_1) - (n-1)(n\alpha a_n + a_{n-1})z^{n-1} \\ &\quad - (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})z^{n-2} - ... - 2(3\alpha a_3 + (n-2)a_2)z^2 \\ &\quad - (2\alpha a_2 + (n-1)a_1)z \end{aligned}$$

or

$$\begin{aligned} Q(z) &= -(n-1)(n\alpha a_n + a_{n-1})z^{n-1} + \{(n-1)(n\alpha a_n + a_{n-1}) - (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})\}z^{n-2} \\ &\quad + \{(n-2)((n-1)\alpha a_{n-1} + 2a_{n-2}) - (n-3)((n-2)\alpha a_{n-2} + 3a_{n-3})\}z^{n-3} + ... \\ &\quad + \{3(4\alpha a_4 + (n-3)a_3) - 2(3\alpha a_3 + (n-2)a_2)\}z^2 + \{2(3\alpha a_3 + (n-2)a_2) \\ &\quad - (2\alpha a_2 + (n-1)a_1)\}z + \{2\alpha a_2 + (n-1)a_1\}. \end{aligned}$$

Therefore

$$\begin{aligned} |Q(z)| &\geq |(n-1)(n\alpha a_n + a_{n-1})||z|^{n-1} - \{|(n-1)(n\alpha a_n + a_{n-1}) - (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})||z|^{n-2} \\ &\quad + |(n-2)((n-1)\alpha a_{n-1} + 2a_{n-2}) - (n-3)((n-2)\alpha a_{n-2} + 3a_{n-3})||z|^{n-3} + ... \\ &\quad + |3(4\alpha a_4 + (n-3)a_3) - 2(3\alpha a_3 + (n-2)a_2)||z^2| + |2(3\alpha a_3 + (n-2)a_2) - (2\alpha a_2 + (n-1)a_1)||z| \\ &\quad + |2\alpha a_2 + (n-1)a_1|\} \end{aligned}$$

which implies

$$\begin{aligned} |Q(z)| &\geq (n-1)|n\alpha a_n + a_{n-1}||z|^{n-2}\Big[|z| - \big((n-1)|n\alpha a_n + a_{n-1}|\big)^{-1}\{|(n-1)(n\alpha a_n + a_{n-1}) \\ &\quad - (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})| + |(n-2)((n-1)\alpha a_{n-1} + 2a_{n-2}) \\ &\quad - (n-3)((n-2)\alpha a_{n-2} + 3a_{n-3})||z|^{-1} + ... + |3(4\alpha a_4 + (n-3)a_3) - 2(3\alpha a_3 + (n-2)a_2)||z|^{-(n-4)} \\ &\quad + |2(3\alpha a_3 + (n-2)a_2) - (2\alpha a_2 + (n-1)a_1)||z|^{-(n-3)} + |2\alpha a_2 + (n-1)a_1||z|^{-(n-2)}\}\Big]. \end{aligned}$$

Now if $|z| > 1$, then $\frac{1}{|z|} < 1$,

$$\begin{aligned}|Q(z)| \geq (n-1)|n\alpha a_n + a_{n-1}|\Big[|z| - \big((n-1)|n\alpha a_n + a_{n-1}|\big)^{-1}\{|(n-1)(n\alpha a_n + a_{n-1})\\ - (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})| + |(n-2)((n-1)\alpha a_{n-1} + 2a_{n-2}) - (n-3)((n-2)\alpha a_{n-2}\\ + 3a_{n-3})| + ... + |3(4\alpha a_4 + (n-3)a_3) - 2(3\alpha a_3 + (n-2)a_2)| + |2(3\alpha a_3\\ + (n-2)a_2) - (2\alpha a_2 + (n-1)a_1)| + |2\alpha a_2 + (n-1)a_1|\}\Big]\end{aligned}$$

that is

$$\begin{aligned}|Q(z)| \geq (n-1)|n\alpha a_n + a_{n-1}||z|^{n-2}\Big[|z| - \big((n-1)|n\alpha a_n + a_{n-1}|\big)^{-1}\{(n-1)(n\alpha a_n + a_{n-1})\\ - (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2}) + (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2}) - (n-3)((n-2)\alpha a_{n-2} + 3a_{n-3})\\ + ... + 3(4\alpha a_4 + (n-3)a_3) - 2(3\alpha a_3 + (n-2)a_2) + 2(3\alpha a_3 + (n-2)a_2) - (2\alpha a_2 + (n-1)a_1)\\ + |2\alpha a_2 + (n-1)a_1|\}\Big]\end{aligned}$$

or

$$\begin{aligned}|Q(z)| \geq (n-1)|n\alpha a_n + a_{n-1}||z|^{n-2}\Big[|z| - \big((n-1)|n\alpha a_n + a_{n-1}|\big)^{-1}\{(n-1)(n\alpha a_n + a_{n-1})\\ - (2\alpha a_2 + (n-1)a_1) + |2\alpha a_2 + (n-1)a_1|\}\Big].\end{aligned}$$

Hence, $|Q(z)| > 0$ if

$$|z| > \frac{(n-1)(n\alpha a_n + a_{n-1}) - (2\alpha a_2 + (n-1)a_1) + |2\alpha a_2 + (n-1)a_1|}{(n-1)|n\alpha a_n + a_{n-1}|}$$

This shows that all the zeros of $Q(z)$ whose modulus is greater than 1 lie in

$$|z| \leq \frac{(n-1)(n\alpha a_n + a_{n-1}) - (2\alpha a_2 + (n-1)a_1) + |2\alpha a_2 + (n-1)a_1|}{(n-1)|n\alpha a_n + a_{n-1}|}.$$

Since the zeros of $Q(z)$ whose modulus is less than or equal to 1 already lie in

$$|z| \leq \frac{(n-1)(n\alpha a_n + a_{n-1}) - (2\alpha a_2 + (n-1)a_1) + |2\alpha a_2 + (n-1)a_1|}{(n-1)|n\alpha a_n + a_{n-1}|},$$

it follows that all the zeros of $Q(z)$ lie in

$$|z| \leq \frac{(n-1)(n\alpha a_n + a_{n-1}) - (2\alpha a_2 + (n-1)a_1) + |2\alpha a_2 + (n-1)a_1|}{(n-1)|n\alpha a_n + a_{n-1}|}.$$

Thus all the zeros of $D'_\alpha P(z)$ lie in

$$|z| \leq \frac{(n-1)(n\alpha a_n + a_{n-1}) - (2\alpha a_2 + (n-1)a_1) + |2\alpha a_2 + (n-1)a_1|}{(n-1)|n\alpha a_n + a_{n-1}|}.$$

In another words, all the zeros of $D_\alpha P(z)$ whose modulus is greater than or equal to

$$\frac{(n-1)(n\alpha a_n + a_{n-1}) - (2\alpha a_2 + (n-1)a_1) + |2\alpha a_2 + (n-1)a_1|}{(n-1)|n\alpha a_n + a_{n-1}|}$$

are simple.

Corollary 2.2. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that*

$$(n-1)a_{n-1} \geq 2(n-2)a_{n-2} \geq ... \geq 3(n-3)a_3 \geq 2(n-2)a_2 \geq (n-1)a_1.$$

Then all the zeros of $D_0P(z) = nP(z) - zP'(z)$ *whose modulus is greater than or equal to*

$$\frac{|a_1| - a_1 + a_{n-1}}{|a_{n-1}|}$$

are simple.

Corollary 2.3. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that*

$$(n-1)a_{n-1} \geq 2(n-2)a_{n-2} \geq ... \geq 3(n-3)a_3 \geq 2(n-2)a_2 \geq (n-1)a_1 > 0.$$

Then all the zeros of $D_0P(z) = nP(z) - zP'(z)$ *whose modulus is greater than or equal to 1 are simple.*

Theorem 2.4. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n*. Let* $\alpha \in \mathbb{R}$ *such that*

$$\begin{aligned}(n-1)\left[n\alpha a_n + a_{n-1}\right] &\leq (n-2)\left[(n-1)\alpha a_{n-1} + 2a_{n-2}\right] \leq ... \leq 3\left[4\alpha a_4 + (n-3)a_3\right] \\ &\leq 2\left[3\alpha a_3 + (n-2)a_2\right] \leq \left[2\alpha a_2 + (n-1)a_1\right].\end{aligned}$$

Then all the zeros of $D_\alpha P(z)$ *whose modulus is greater than or equal to*

$$\frac{\left[2\alpha a_2 + (n-1)a_1\right] + |2\alpha a_2 + (n-1)a_1| - (n-1)\left[n\alpha a_n + a_{n-1}\right]}{(n-1)|n\alpha a_n + a_{n-1}|}$$

are simple.

Proof:Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, then

$$\begin{aligned}D_\alpha P(z) = &(n\alpha a_n + a_{n-1})z^{n-1} + ((n-1)\alpha a_{n-1} + 2a_{n-2})z^{n-2} + ... + (3\alpha a_3 + (n-2)a_2)z^2 \\ &+ (2\alpha a_2 + (n-1)a_1)z + (\alpha a_1 + na_0).\end{aligned}$$

So

$$\begin{aligned}D'_\alpha P(z) = &(n-1)(n\alpha a_n + a_{n-1})z^{n-2} + (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})z^{n-3} + ... \\ &+ 2(3\alpha a_3 + (n-2)a_2)z + (2\alpha a_2 + (n-1)a_1).\end{aligned}$$

Now consider the polynomial

$Q(z) = (1-z)D'_\alpha P(z)$

That is

$$\begin{aligned}Q(z) = (1-z)\{&(n-1)(n\alpha a_n + a_{n-1})z^{n-2} + (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})z^{n-3} + ... \\ &+ 2(3\alpha a_3 + (n-2)a_2)z + (2\alpha a_2 + (n-1)a_1)\}\end{aligned}$$

which implies

$$\begin{aligned}Q(z) = &(n-1)(n\alpha a_n + a_{n-1})z^{n-2} + (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})z^{n-3} + ... + 2(3\alpha a_3 + (n-2)a_2)z \\ &+ (2\alpha a_2 + (n-1)a_1) - (n-1)(n\alpha a_n + a_{n-1})z^{n-1} - (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})z^{n-2} - ... \\ &- 2(3\alpha a_3 + (n-2)a_2)z^2 - (2\alpha a_2 + (n-1)a_1)z\end{aligned}$$

or

$$\begin{aligned}Q(z) = &-(n-1)(n\alpha a_n + a_{n-1})z^{n-1} + \{(n-1)(n\alpha a_n + a_{n-1}) - (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})\}z^{n-2}\\ &+ \{(n-2)((n-1)\alpha a_{n-1} + 2a_{n-2}) - (n-3)((n-2)\alpha a_{n-2} + 3a_{n-3})\}z^{n-3} + ...\\ &+ \{3(4\alpha a_4 + (n-3)a_3) - 2(3\alpha a_3 + (n-2)a_2)\}z^2 + \{2(3\alpha a_3 + (n-2)a_2)\\ &- (2\alpha a_2 + (n-1)a_1)\}z + \{2\alpha a_2 + (n-1)a_1\}.\end{aligned}$$

Therefore

$$\begin{aligned}|Q(z)| \geq\ &|(n-1)(n\alpha a_n + a_{n-1})||z|^{n-1} - \{|(n-1)(n\alpha a_n + a_{n-1}) - (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})||z|^{n-2}\\ &+ |(n-2)((n-1)\alpha a_{n-1} + 2a_{n-2}) - (n-3)((n-2)\alpha a_{n-2} + 3a_{n-3})||z|^{n-3} + ...\\ &+ |3(4\alpha a_4 + (n-3)a_3) - 2(3\alpha a_3 + (n-2)a_2)||z^2| + |2(3\alpha a_3 + (n-2)a_2) - (2\alpha a_2 + (n-1)a_1)||z|\\ &+ |2\alpha a_2 + (n-1)a_1|\}\end{aligned}$$

which implies

$$\begin{aligned}|Q(z)| \geq (n-1)|n\alpha a_n + a_{n-1}||z|^{n-2}\Big[&|z| - \big((n-1)|n\alpha a_n + a_{n-1}|\big)^{-1}\{|(n-1)(n\alpha a_n + a_{n-1})\\ &- (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})| + |(n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})\\ &- (n-3)((n-2)\alpha a_{n-2} + 3a_{n-3})||z|^{-1} + ... + |3(4\alpha a_4 + (n-3)a_3) - 2(3\alpha a_3\\ &+ (n-2)a_2)||z|^{-(n-4)} + |2(3\alpha a_3 + (n-2)a_2) - (2\alpha a_2 + (n-1)a_1)||z|^{-(n-3)}\\ &+ |2\alpha a_2 + (n-1)a_1||z|^{-(n-2)}\}\Big].\end{aligned}$$

Now if $|z| > 1$, then $\frac{1}{|z|} < 1$, so

$$\begin{aligned}|Q(z)| \geq (n-1)|n\alpha a_n + a_{n-1}|\Big[&|z| - \big((n-1)|n\alpha a_n + a_{n-1}|\big)^{-1}\{|(n-1)(n\alpha a_n + a_{n-1})\\ &- (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})| + |(n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})\\ &- (n-3)((n-2)\alpha a_{n-2} + 3a_{n-3})| + ... + |3(4\alpha a_4 + (n-3)a_3) - 2(3\alpha a_3 + (n-2)a_2)|\\ &+ |2(3\alpha a_3 + (n-2)a_2) - (2\alpha a_2 + (n-1)a_1)| + |2\alpha a_2 + (n-1)a_1|\}\Big]\end{aligned}$$

that is

$$\begin{aligned}|Q(z)| \geq (n-1)|n\alpha a_n + a_{n-1}|\Big[&|z| - \big((n-1)|n\alpha a_n + a_{n-1}|\big)^{-1}\{(n-2)((n-1)\alpha a_{n-1} + 2a_{n-2})\\ &- (n-1)(n\alpha a_n + a_{n-1}) + (n-3)((n-2)\alpha a_{n-2} + 3a_{n-3}) - (n-2)((n-1)\alpha a_{n-1} + 2a_{n-2}) + ...\\ &+ 2(3\alpha a_3 + (n-2)a_2) - 3(4\alpha a_4 + (n-3)a_3) + (2\alpha a_2 + (n-1)a_1) - 2(3\alpha a_3 + (n-2)a_2)\\ &+ |2\alpha a_2 + (n-1)a_1|\}\Big]\end{aligned}$$

or

$$\begin{aligned}|Q(z)| \geq (n-1)|n\alpha a_n + a_{n-1}|\Big[&|z| - \big((n-1)|n\alpha a_n + a_{n-1}|\big)^{-1}\{(2\alpha a_2 + (n-1)a_1) + |2\alpha a_2 + (n-1)a_1|\\ &- (n-1)(n\alpha a_n + a_{n-1})\}\Big].\end{aligned}$$

Hence $|Q(z)| > 0$ if

$$|z| \geq \frac{(2\alpha a_2 + (n-1)a_1) + |2\alpha a_2 + (n-1)a_1| - (n-1)(n\alpha a_n + a_{n-1})}{(n-1)|n\alpha a_n + a_{n-1}|}.$$

This shows that all the zeros of $Q(z)$ whose modulus is greater than 1 lie in

$$|z| \leq \frac{(2\alpha a_2 + (n-1)a_1) + |2\alpha a_2 + (n-1)a_1| - (n-1)(n\alpha a_n + a_{n-1})}{(n-1)|n\alpha a_n + a_{n-1}|}.$$

Since the zeros of $Q(z)$ whose modulus is less than or equal to 1 already lie in

$$|z| \leq \frac{(2\alpha a_2 + (n-1)a_1) + |2\alpha a_2 + (n-1)a_1| - (n-1)(n\alpha a_n + a_{n-1})}{(n-1)|n\alpha a_n + a_{n-1}|},$$

it follows that all the zeros of Q(z) lie in

$$|z| \leq \frac{(2\alpha a_2 + (n-1)a_1) + |2\alpha a_2 + (n-1)a_1| - (n-1)(n\alpha a_n + a_{n-1})}{(n-1)|n\alpha a_n + a_{n-1}|}.$$

Thus all the zeros of $D'_\alpha P(z)$ lie in

$$|z| \leq \frac{(2\alpha a_2 + (n-1)a_1) + |2\alpha a_2 + (n-1)a_1| - (n-1)(n\alpha a_n + a_{n-1})}{(n-1)|n\alpha a_n + a_{n-1}|}.$$

In another words, all the zeros of $D_\alpha P(z)$ whose modulus is greater than or equal to

$$\frac{(2\alpha a_2 + (n-1)a_1) + |2\alpha a_2 + (n-1)a_1| - (n-1)(n\alpha a_n + a_{n-1})}{(n-1)|n\alpha a_n + a_{n-1}|}$$

are simple.

Corollary 2.5. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that*

$$(n-1)a_{n-1} \leq 2(n-2)a_{n-2} \leq ... \leq 3(n-3)a_3 \leq 2(n-2)a_2 \leq (n-1)a_1.$$

Then all the zeros of $D_0P(z) = nP(z) - zP'(z)$ whose modulus is greater than or equal to

$$\frac{|a_1| + a_1 - a_{n-1}}{|a_{n-1}|}$$

are simple.

Corollary 2.6. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that*

$$(n-1)a_{n-1} \leq 2(n-2)a_{n-2} \leq ... \leq 3(n-3)a_3 \leq 2(n-2)a_2 \leq (n-1)a_1 > 0.$$

Then all the zeros of $D_0P(z) = nP(z) - zP'(z)$ whose modulus is greater than or equal to 1 are simple

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