Available online at http://www.cna-journal.com
Commun. Nonlinear Anal. 6(1) (2019), 18–22
Research Article

Communications in Nonlinear Analysis
Journal Homepage: www.cna-journal.com

Some notes on the paper [Further discussion on modified multivalued α_*-ψ-contractive type mappings]

B. Mohammadi[a,*], V. Parvaneh[b]

[a]*Department of Mathematics, Marand Branch, Islamic Azad University, Marand, Iran.*
[b]*Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran*

Abstract

In this paper, we show that the claim of the paper [Ali *et al.*, Further discussion on modified multivalued α_*-ψ-contractive type mappings, Filomat 29 (2015)] which says that the notion of α_*-η-ψ-contractive multi-valued mappings can not be rduced into α_*-ψ-contractive multi-valued mappings, is not true. Also, we provide a common fixed point result for an α_*-admissible countable family of multi-valued mappings. Finally, we show that the common fixed point result of Ali *et al.* for a countable family of multi-valued mappings using α_*-admissible mappings with respect to η can be reduced to α_*-admissible mappings without using the auxiliary function η.

Keywords: α_*-ψ-contractive, α_*-η-ψ-contractive, common fixed point, multi-valued mapping.
2010 MSC: 47H10, 47H04

1. Introduction

Recently, Samet *et al.* [19] introduced the notion of α-ψ-contractive self-mappings via α-admissible self mappings. In fact, in this paper, the authors proved existence and uniqueness of a fixed point for mappings satisfying only a locally contraction. In [17] Salimi *et al.* introduced the notion of modified α-ψ-contractive mappings using another auxiliary function η. In this paper, the authors established some fixed point theorems for such (single-valued) mappings in the setting of complete metric spaces. Later, Mohammadi and Rezapour [16] and independently, Berzig and Karapinar[10], noticed that modified (single-valued) α-ψ-contractive type mappings can be considered as a particular case of α-ψ-contractive mappings. In 2013, Mohammadi and Rezapour [15], introduced the notion of α-ψ-contractive multi-valued mappings via α-admissible multi-valued mappings. In the same year, Asl *et al.* [9] provided the notion of α_*-ψ-contractive multi-valued mappings via α_*-admissible multi-valued mappings. In [6], Ali *et al.* introduced the notion

*Corresponding author
Email addresses: bmohammadi@marandiau.ac.ir (B. Mohammadi), zam.dalahoo@gmail.com (V. Parvaneh)

Received 2019-04-18

of modified α_*-ψ-contractive (α_*-η-ψ-contractive) multi-valued mappings and α_*-admissible multi-valued mappings with respect to η and claimed that this notion is a proper generalization of the notion α_*-ψ-contractive multi-valued mappings. In other words, they claimed that the notion of modified multivalued α_*-ψ-contractive mapping can not be reduced into multivalued α_*-ψ-contractive mapping. In addition, they investigated the existence of a common fixed point for a sequence of multivalued α_*-η-ψ-contractive mappings. In this paper, we show that this notion can be considered as a particular case of the old result α_*-ψ-contractive multi-valued mappings. So, the auxiliary function η is not needed.

2. Preliminaries

Let (X, d) be a metric space. Througout this paper denote by $CL(X)$ the set of all nonempty closed subsets of X. Denote by Ψ the family of all nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for all $t > 0$. It is well known that $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$. For any $A, B \in CL(X)$, let the mapping $H : CL(X) \times CL(X) \to [0, \infty]$ defined by

$$H(A,B) = \begin{cases} \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}, & \textit{if the maximum exists} \\ \infty, & \textit{otherwise}, \end{cases}$$

be the generalized Pompeiu-Hausdorff metric induced by d, where $d(a, B) = \inf\{d(a, b) : b \in B\}$ is the distance from a to $B \subseteq X$.

Definition 2.1. [9] Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $G : X \to CL(X)$ is called α_*-admissible if

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha_*(Gx, Gy) \geq 1,$$

where

$$\alpha_*(Gx, Gy) = \inf\{\alpha(a, b)|a \in Gx, b \in Gy\}.$$

Definition 2.2. [9] Let (X, d) be a metric space. A mapping $G : X \to CL(X)$ is called α_*-ψ-contractive if there exist $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha_*(Gx, Gy)H(Gx, Gy) \leq \psi(d(x, y)),$$

for all $x, y \in X$.

Hussain *et al.* [11] extended the notions of α_*-admissiblity with respect to η and modified α_*-ψ-contractivity to multi-valued mappings as follows.

Definition 2.3. [11] Let (X, d) be a metric space and $\alpha, \eta : X \times X \to [0, \infty)$ be two functions. A mapping $G : X \to CL(X)$ is called α_*-admissible with respect to η if

$$\alpha(x, y) \geq \eta(x, y) \Longrightarrow \alpha_*(Gx, Gy) \geq \eta_*(Gx, Gy), \tag{2.1}$$

where

$$\alpha_*(Gx, Gy) = \inf\{\alpha(a, b)|a \in Gx, b \in Gy\}$$

and

$$\eta_*(Gx, Gy) = \sup\{\eta(a, b)|a \in Gx, b \in Gy\}.$$

Definition 2.4. [11] Let (X, d) be a metric space. A mapping $G : X \to CL(X)$ is called modified α_*-ψ-contractive (α_*-η-ψ-contractive) if there exist $\alpha, \eta : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha_*(Gx, Gy) \geq \eta_*(Gx, Gy) \Longrightarrow H(Gx, Gy) \leq \psi(d(x, y)),$$

for all $x, y \in X$.

3. Main results

In [6] Ali *et al.* considering a multi-valued mapping $G : X \to CL(X)$ observed that the value of $H(Gx, Gy)$ may be infinite for some choice of $x, y \in X$. Consequently, they claimed that an α_*-η-ψ-contractive multi-valued mapping may not imply an α_*-ψ-contractive multi-valued mapping, in general. Indeed, they claimed that if one define

$$\beta(x,y) = \begin{cases} 1, & if\ \alpha(x,y) \geq \eta(x,y) \\ 0, & otherwise, \end{cases}$$

then (2.1) reduces into

$$\beta_*(Gx, Gy)H(Gx, Gy) \leq \psi(d(x,y)),$$

for all $x, y \in X$. For $\beta_*(Gx, Gy) = 1$, we have $H(Gx, Gy) \leq \psi(d(x,y))$. For $\beta_*(Gx, Gy) = 0$, we have $0.H(Gx, Gy) \leq \psi(d(x,y))$. Now, here is the point. Ali *et al.* claimed that if $H(Gx, Gy) = \infty$, then $0.\infty$ is an indeterminate form. But it should be noted that the form $0.\infty$ is an indeterminate form if $0 = 0^+$ be a limitative amount. In the case where 0 be exactly 0 and not a limitative amount, we have $0.\infty = 0$. This is the point that Ali *et al.* did not note. Thus, we think that the case $0.H(Gx, Gy) = 0 \leq \psi(d(x,y))$ is true even if $H(Gx, Gy) = \infty$. Therefore, the notion of α_*-η-ψ-contractive multi-valued mappings can be considered as a particular case of α_*-ψ-contractive multi-valued mappings.

In [6] Ali *et al.* gave the following definition.

Definition 3.1. Let $\{G_i : X \to CL(X)\}_{i=1}^{\infty}$ be a sequence of multi-valued mappings on a metric space (X, d). Let $\alpha, \eta : X \times X \to [0, \infty)$ be two functions. Then, $\{G_i\}_{i=1}^{\infty}$ is called α_*-admissible with respect to η if

$$\alpha(x,y) \geq \eta(x,y) \Longrightarrow \alpha(u,v) \geq \eta(u,v), \forall u \in G_i x, v \in G_j y, \tag{3.1}$$

for each $i, j \in \mathbb{N}$.

Theorem 3.2. *[6] Let (X, d) be a complete metric space and let the sequence $\{G_i : X \to CL(X)\}_{i=1}^{\infty}$ be α_*-admissible with respect to η such that*

$$x, y \in X, \alpha(x,y) \geq \eta(x,y) \Longrightarrow H(G_i x, G_j y) \leq \psi(d(x,y)), \tag{3.2}$$

for each $i, j \in \mathbb{N}$ and ψ be strictly increasing function in Ψ. Assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $y_i \in G_i x_0$ for each $i \in \mathbb{N}$ such that $\alpha(x_0, y_i) \geq \eta(x_0, y_i)$;

(ii) if $\{x_i\}$ is a sequence in X with $x_i \to x$ as $i \to \infty$ and $\alpha(x_{i-1}, x_i) \geq \eta(x_{i-1}, x_i)$ for each $i \in \mathbb{N}$, then we have $\alpha(x_{i-1}, x) \geq \eta(x_{i-1}, x)$ for each $i \in \mathbb{N}$.

Then, the mappings $\{G_i\}$ for $i \in \mathbb{N}$, have a common fixed point.

Now, we give the following definition and theorem.

Definition 3.3. Let $\{G_i : X \to CL(X)\}_{i=1}^{\infty}$ be a sequence of multi-valued mappings on a metric space (X, d). Let $\alpha : X \times X \to [0, \infty)$ be a function. Then, $\{G_i\}_{i=1}^{\infty}$ is called α_*-admissible if

$$\alpha(x,y) \geq 1 \Longrightarrow \alpha(u,v) \geq 1, \forall u \in G_i x, v \in G_j y, \tag{3.3}$$

for each $i, j \in \mathbb{N}$.

Theorem 3.4. *Let (X, d) be a complete metric space and let the sequence $\{G_i : X \to CL(X)\}_{i=1}^{\infty}$ be α_*-admissible such that*

$$x, y \in X, \alpha(x,y) \geq 1 \Longrightarrow H(G_i x, G_j y) \leq \psi(d(x,y)), \tag{3.4}$$

for each $i, j \in \mathbb{N}$ and ψ be strictly increasing function in Ψ. Assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $y_i \in G_i x_0$ for each $i \in \mathbb{N}$ such that $\alpha(x_0, y_i) \geq 1$;

(ii) if $\{x_i\}$ is a sequence in X with $x_i \to x$ as $i \to \infty$ and $\alpha(x_{i-1}, x_i) \geq 1$ for each $i \in \mathbb{N}$, then we have $\alpha(x_{i-1}, x) \geq 1$ for each $i \in \mathbb{N}$.

Then, the mappings $\{G_i\}$ for $i \in \mathbb{N}$, have a common fixed point.

Proof. By hypothesis, there exist $x_0 \in X$ and $x_1 \in G_1 x_0$ such that $\alpha(x_0, x_1) \geq 1$. If $x_1 \in G_i x_1$ for each $i \in \mathbb{N}$, then x_1 is a common fixed point of G_i. Let $x_1 \notin G_2 x_1$. Then, from (3.4), we have

$$0 < d(x_1, G_2 x_1) < q d(x_1, G_2 x_1) \leq q H(G_1 x_0, G_2 x_1) \leq q \psi(d(x_0, x_1)).$$

Thus, there exists $x_2 \in G_2 x_1$ such that

$$0 < d(x_1, x_2) < q\psi(d(x_0, x_1)).$$

Since, ψ is strictly increasing, we get $\psi(d(x_1, x_2)) < \psi(q\psi(d(x_0, x_1)))$.

Put $q_1 = \frac{\psi(q\psi(d(x_0,x_1)))}{\psi(d(x_1,x_2))}$. Then $q_1 > 1$. Since the sequence $\{G_i\}$ is α_*-admissible, then $\alpha(x_1, x_2) \geq 1$. If $x_2 \in G_i x_2$ for each $i \in \mathbb{N}$, then x_2 is a common fixed point of $\{G_i\}$. Let $x_2 \notin G_3 x_2$. Then, we have

$$0 < d(x_2, G_3 x_2) < q_1 d(x_2, G_3 x_2) \leq q_1 H(G_2 x_1, G_3 x_2) \leq q_1 \psi(d(x_1, x_2)) = \psi(q\psi(d(x_0, x_1))).$$

Now, there exists $x_3 \in G_3 x_2$ such that

$$0 < d(x_2, x_3) < \psi(q\psi(d(x_0, x_1))).$$

Since, ψ is strictly increasing, we get $\psi(d(x_2, x_3)) < \psi^2(q\psi(d(x_0, x_1)))$. Put $q_2 = \frac{\psi^2(q\psi(d(x_0,x_1)))}{\psi(d(x_2,x_3))}$. Then $q_2 > 1$. Also, since the sequence $\{G_i\}$ is α_*-admissible, then $\alpha(x_2, x_3) \geq 1$. If $x_3 \in G_i x_3$ for each $i \in \mathbb{N}$, then x_3 is a common fixed point of $\{G_i\}$. Let $x_3 \notin G_4 x_3$. For $q_2 > 1$, we have

$$0 < d(x_3, G_4 x_3) < q_2 d(x_3, G_4 x_3) \leq q_2 H(G_3 x_2, G_4 x_3) \leq q_2 \psi(d(x_2, x_3)) = \psi^2(q\psi(d(x_0, x_1))).$$

Now, there exists $x_4 \in G_4 x_3$ such that

$$0 < d(x_3, x_4) < \psi^2(q\psi(d(x_0, x_1))).$$

Continuing in the same way, we get a sequence $\{x_i\}$ in X such that $x_i \in G_i x_{i-1}, \alpha(x_{i-1}, x_i) \geq 1$ and $d(x_i, x_{i+1}) < \psi^{i-1}(q\psi(d(x_0, x_1)))$ for each $i \in \mathbb{N}$. Let $j > i$. we have

$$d(x_i, x_j) \leq \Sigma_{n=i}^{j-1} d(x_n, x_{n+1}) < \Sigma_{n=i}^{j-1} \psi^{n-1}(q\psi(d(x_0, x_1))).$$

Since $\psi \in \Psi$, then we have $\lim_{i,j\to\infty} d(x_i, x_j) = 0$. Hence $\{x_i\}$ is a Cauchy sequence in (X, d). By completeness of (X, d), there exists $x \in X$ such that $x_i \to x$ as $i \to \infty$. By hypothesis (ii), we have $\alpha(x_{i-1}, x) \geq 1$ for each $i \in \mathbb{N}$. Now, for each $n = 1, 2, 3, \ldots$ we have

$$d(x_i, G_n x^*) \leq H(G_i x_{i-1}, G_n x^*) \leq \psi(d(x_{i-1}, x^*)).$$

Letting $i \to \infty$ in above inequality, we have $d(x^*, G_n x^*) = 0$ for each $n \in \mathbb{N}$. Since $G_n x^*$ is closed, we conclude that $x^* \in G_n x^*$ for each $n \in \mathbb{N}$, that is, x^* is a common fixed point of $\{G_i\}$. □

Now, we show that Theorem 3.2 is a particular case of 3.4. To see this, define

$$\beta(x, y) = \begin{cases} 1, & if\ \alpha(x, y) \geq \eta(x, y) \\ 0, & otherwise. \end{cases}$$

Let $\{G_i\}_{i=1}^{\infty}$ is α_*-admissible with respect to η. Now, if $\beta(x, y) \geq 1$, then $\alpha(x, y) \geq \eta(x, y)$ and so $\alpha(u, v) \geq \eta(u, v), \forall u \in G_i x, v \in G_j y$. Thus, $\beta(u, v) \geq 1, \forall u \in G_i x, v \in G_j y$, that is, $\{G_i\}_{i=1}^{\infty}$ is β_*-admissible. Now, let (3.2) holds. Then, if $\beta(x, y) \geq 1$, we get $\alpha(x, y) \geq \eta(x, y)$. By (3.2), we obtain $H(G_i x, G_j y) \leq \psi(d(x, y))$. Thus (3.4) holds. Obviously, the conditions (i) and (ii) in Theorem 3.2 reduces into the conditions (i) and (ii) of Theorem 3.4 with function β instead of α. This, arguments show that the conditions of Theorem 3.2 leads to the conditions of Theorem 3.4, that is, Theorem 3.2 is a particular case of 3.4.

4. Consequences

In this paper, we showed that the auxiliary function η in the notion α_*-η-ψ-contractive multi-valued mappings is not need in general, as well as, olready we have shown it in [16] for self-mappings.

Acknowledgement

This study was supported by Marand Branch, Islamic Azad University, Marand, Iran.

References

[1] M. U. Ali, T. Kamran, *Multivalued F-contractions and related fixed point theorems with an application*, Filomat, **4** (2016), 3779-3793.

[2] M. U. Ali, T. Kamran, E. Karapinar, *$(\alpha,\ \psi,\ \xi)$-contractive multi-valued mappings*, Fixed Point Theory Applications, 2014 **2014** :7.

[3] M. U. Ali, T. Kamran, E. Karapinar,*A new approach to $(\alpha,\ \psi)$-contractive nonself multivalued mappings* , Journal Inequalities and Applications, 2014 **2014** :71.

[4] M. U. Ali, T. Kamran,*On $(\alpha^*,\ \psi)$-contractive multi-valued mappings* , Fixed Point Theory and Applications, 2013 **2013** :137.

[5] M. U. Ali, T. Kamran,W. Sintunavarat, P. Katchang, *Mizoguchi-Takahashis fixed point theorem with α, η functions*, Abstract and Applied Analysis, **2013** (2013), Article ID 418798.

[6] M. U. Ali, T. Kamran, E. Karapinar, *Further discussion on modified multivalued α_*-ψ-contractive type mappings*, Filomat, **29** (2015), 1893-1900. 1, 3, 3.2

[7] H. Alikhani, Sh. Rezapour , N. Shahzad, *Fixed points of a new type of contractive mappings and multifunctions*, Filomat, **27** (2013), 1315-1319.

[8] P. Amiri, S. Rezapour, N. Shahzad,*Fixed points of generalized α-ψ-contractions*, Revista de la Real Academia de Ciencias Exactas Fisicas y Naturales Serie A Mate doi: 10.1007/s13398-013-0123-9.

[9] J. H. Asl, S. Rezapour, N. Shahzad,*On fixed points of α-ψ-contractive multifunctions* , Fixed Point Theory and Applications, 2012 **2012** :212. 1, 2.1, 2.2

[10] M. Berzig, E. Karapinar, *Note on "Modified α-ψ-contractive mappings with application"*, Thai Journal of Mathematics (2014) In press. 1

[11] N. Hussain, P. Salimi, A. Latif,*Fixed point results for single and set-valued α-η-ψ-contractive mappings*, Fixed Point Theory and Applications, 2013 **2013** :212. 2, 2.3, 2.4

[12] E. Karapinar, *α-ψ-Geraghty contraction type mappings and some related fixed point results*, Filomat, **28** (2014) 3748.

[13] E. Karapinar, B. Samet, *Generalized α-ψ-contractive type mappings and related fixed point theorems with applications*, Abstract Applied Analysis, **2012** (2012) Article id: 793486.

[14] G. Minak, I. Altun,*Some new generalizations of Mizoguchi-Takahashi type fixed point theorem* , Journal Inequalities and Applications, 2013 **2013** :493.

[15] B. Mohammadi, S. Rezapour, N Shahzad, *Some results on fixed points of α-ψ-Ciric generalized multifunctions*, Fixed Point Theory and Applications, 2013 **2013** :24. 1

[16] B. Mohammadi, S. Rezapour,*On modified α-ϕ-contractions, Journal of Advanced Mathematical Studies*, **6** (2013), 162-166. 1, 4

[17] P. Salimi, A. Latif, N. Hussain,*Modified α-ψ-contractive mappings with applications* , Fixed Point Theory and Applications, 2013 **2013**:151. 1

[18] P. Salimi, C. Vetro, P. Vetro,*Fixed point theorems for twisted (α, β)-ψ-contractive type mappings and applications*, Filomat, **27** (2013), 605-615.

[19] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Analysis, **75** (2012), 2154-2165. 1