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Optimal Coincidence Best Approximation Solution in B-Fuzzy Metric Spaces

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Abstract

In this paper, we prove the existence of optimal coincidence point and best proximity point in b-fuzzy metric space for two mappings satisfying certain contractive conditions and prove some proximal theorems which provide the existence of an optimal approximate solution to some operator equations which are not solvable. We also provide an application to the fixed point theory of our obtained results.

Keywords: Fuzzy metric space, b-Fuzzy metric space, Optimal approximate solution, Fuzzy expansive, Fuzzy isometry, s-increasing sequence, t-norm.
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1. Introduction and Preliminaries

Introduction Zadeh [15] introduced the concept of a fuzzy set A in a universal set X by a mapping $f : A \to [0, 1]$. Kramosil and Michalek [9] defined fuzzy metric space which was further modified by George and Veeramani [5] with the help of a continuous $t-$norm.

On the other hand, Bakhtin [2] introduced the notion of b-metric spaces which can be viewed as a generalization of metric space (see also,[3]). Sedghi and Shobe [14] combined the concepts of fuzzy set and b-metric space to introduce a b-fuzzy metric space. They obtained a fixed point result of a continuous mapping. Hussain, Salimi and Parvaneh [8] derived some new fixed point results of mappings defined on triangular partially ordered fuzzy b-metric spaces. They also studied the relationship between parametric b-metric and fuzzy b-metric. In 2016, S. Nadban [10] studied the concepts of fuzzy quasi b-metric and fuzzy quasi pseudo b-metric spaces. In 2017, Dosenovic,Javaheri and Shobe [4] proved coupled coincidence fixed point theorems in complete b-fuzzy metric spaces.

Consistent with [1] [11], [12], [13] and [14], we recall the following definitions:

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Definition 1.1. [13] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be continuous t-norm if for any $a, b, c, d \in [0,1]$, the following conditions hold:

1) $*$ is associative and commutative;

2) $*$ is continuous;

3) $a * 1 = a$;

4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

Three examples of a continuous t-norm are $a * b = a \cdot b$ (usual product t-norm); $a * b = \min(a,b)$ (Minimum t-norm) and $a * b = \max\{a + b - 1, 0\}$ (Lukasiewicz t-norm).

Definition 1.2. [14] Let X be a nonempty set, $*$ a continuous $t - norm$ and $b \geq 1$ a given real number. A fuzzy set M on $X \times X \times (0,\infty)$ is said to be a $b-$fuzzy metric if for any $x, y, z \in X$ and $t, s > 0$, the following conditions hold:

(bM1) $M(x, y, t) > 0$;

(bM2) $M(x, y, t) = 1$ if and only if $x = y$;

(bM3) $M(x, y, t) = M(y, x, t)$;

(bM4) $M(x, z, t + s) \geq M\left(x, y, \frac{t}{b}\right) * M\left(y, z, \frac{s}{b}\right)$;

(bM5) $M(x, y, \cdot) : (0,\infty) \to [0,1]$ is continuous.

The quadruple $(X, M, *, b)$ is called a $b-$fuzzy metric space. Note that a $b-$ fuzzy metric is a fuzzy metric if we take $b = 1$ but the converse does not hold in general ([14]).

Definition 1.3. [14] Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is called b-nondecreasing if any $x, y \in \mathbb{R}$, $x > by$ implies that $f(x) \geq f(y)$.

Sedghi and Shobe [14] proved the following Lemma:

Lemma 1.4. *[14] Let $(X, M, *, b)$ be a b-fuzzy metric space. Then $M(x, y, \cdot) : (0,\infty) \to [0,1]$ is b-nondecreasing. Also,*

$$M(x, y, b^n t) \geq M(x, y, t) \text{ for all } n \in \mathbb{N}.$$

Remark 1.5. In general, a b-fuzzy metric is not continuous but M is a continuous function on $X^2 \times (0,\infty)$ [6].

Let us recall some topological properties of b-fuzzy metric space studied by Sedghi and Shobe [14] .

Let $(X, M, *, b)$ be a b-fuzzy metric space. An open ball $B(x, r, t)$ with center $x_0 \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}, \text{ where } t > 0.$$

A set A in X is called an open set if for any $x \in A$, there exits $t > 0$ and $r \in (0,1)$ such that $B(x, r, t) \subset A$. A collection of all such sets is a topology τ induced by the b-fuzzy metric M.

A sequence $\{x_n\}$ in $(X, M, *, b)$ converges to a point x in X if and only if for each $t > 0$, $M(x_n, x, t) \to 1$ as $n \to \infty$. Furthermore, a sequence $\{x_n\}$ in $(X, M, *, b)$ is called the Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$M(x_n, x_m, t) > 1 - \epsilon$$

for all $n, m \geq n_0$.

A b-fuzzy metric space X is said to be complete if every Cauchy sequence in X is convergent.

A subset A in X is said to be F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Lemma 1.6. *[14] Let $(X, M, *, b)$ be a b-fuzzy metric space.*

(i) *If a sequence $\{x_n\}$ in X converges to a point x in X, then x is unique.*

(ii) *If a sequence $\{x_n\}$ in X is convergent, then $\{x_n\}$ is a Cauchy sequence.*

A set A in X is said to be closed if for each b-convergent sequence $\{x_n\}$ in A with $x_n \to x$, we have $x \in A$.

Definition 1.7. Let A and B be two nonempty subsets of a b-fuzzy metric space $(X, M, *, b)$. Define:

$$\begin{aligned} A_0(t) &= \{x \in A : M(x, y, t) = M(A, B, t) \text{ for some } y \in B\}, \\ B_0(t) &= \{y \in B : M(x, y, t) = M(A, B, t) \text{ for some } x \in A\}, \end{aligned}$$

where

$$M(A, B, t) = \sup\{M(a, b, t) : a \in A, \ b \in B\}.$$

A distance of a point $x \in X$ from a nonempty set A is defined by

$$M(x, A, t) = \sup_{a \in A} M(x, a, t), \text{ where } t > 0.$$

Definition 1.8. [12] A point x in A is said to be optimal coincidence point of a pair (g, T) of mappings $T : A \longrightarrow B$ and $g : A \longrightarrow A$ if

$$M(gx, Tx, t) = M(A, B, t),$$

holds.

Lemma 1.9. *[11]Let $T : A \to B$. If for any $t > 0$, $A_0(t) \neq \phi$ and $T(A_0(t)) \subseteq B_0(t)$. Then, there exist a sequence $\{x_n\}$ in $A_0(t)$ such that*

$$M(x_{n+1}, Tx_n, t) = M(A, B, t).$$

Definition 1.10. [11]A sequence $\{x_n\}$ in $A_0(t)$ satisfying $M(x_{n+1}, Tx_n, t) = M(A, B, t)$ is called proximal fuzzy Picard sequence starting with $x_0 \in A_0(t)$.

Definition 1.11. [11] A set $A_0(t)$ is fuzzy proximal T-orbitally complete if and only if every Cauchy proximal fuzzy Picard sequence in $A_0(t)$ starting with some $x_0 \in A_0(t)$ converges to an element of $A_0(t)$.

Definition 1.12. [12]Let A, B be nonempty subsets of a fuzzy metric space $(X, M, *)$. A set B is said to be fuzzy approximately compact with respect to A if for every sequence $\{y_n\}$ in B, if $M(x, y_n, t) \to M(x, B, t)$ for some $x \in A$, then $x \in A_0(t)$.

Definition 1.13. [12]Let A be a nonempty subset of a fuzzy metric space $(X, M, *)$. A mapping $g : A \to A$ is said to be fuzzy isometry if for any $x, y \in A$ and $t > 0$, we have

$$M(gx, gy, t) = M(x, y, t).$$

Definition 1.14. [12]Let A be a nonempty subset of a fuzzy metric space $(X, M, *)$. A mapping $g : A \to A$ is said to be fuzzy expansive if for any $x, y \in A$ and $t > 0$, we have

$$M(gx, gy, t) \leq M(x, y, t).$$

Definition 1.15. [7] A sequence $\{t_n\}$ of positive real numbers is an s-increasing if there exists $n_0 \in \mathbb{N}$ such that

$$t_{n+1} \geq t_n + 1, \text{ for all } n \geq n_0.$$

Definition 1.16. [12] A fuzzy metric space $(X, M, *)$ is said to satisfy property-T if for an s-increasing sequence $\{t_n\}$, we have

$$\prod_{n\geq n_0}^{\infty} M(x, y, t_n) \geq 1-\epsilon \text{ for all } n \geq n_0.$$

Suppose that $\eta : (0,1] \to [0,\infty)$ such that

C1 η is continuous and decreasing.
C2 $\eta(t) = 0$ if and only if $t = 1$.
C3 for any $r, s \in (0,1]$ with $r * s > 0$, we have $\eta(r * s) \leq \eta(r) + \eta(s)$, where $*$ is any continuous t-norm.

$\Omega = \{\eta : (0,1] \to [0,\infty)$ such that η satisfies $(C1), (C2)$ and $(C3)\}$.
Define $\eta : (0,1] \to [0,\infty)$ by $\eta(t) = \frac{1}{t} - 1$ and take, $* = \wedge$. Then $\eta \in \Omega \neq \phi$.

Definition 1.17. A sequence $\{t_n\}$ of positive real numbers is said to be eventual increasing if there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $t_{n+1} \geq t_n$.

Example 1.18. Let $t_n = e^n$, for all $n \in \mathbb{N}$. It satisfies both the definitions of s-increasing sequence and eventual increasing. That is,

$$t_{n+1} = e^{n+1} \geq e^n + 1 \text{ and } t_{n+1} \geq t_n.$$

Remark 1.19. Every s-increasing sequence is increasing sequence but the converse is not true.

Following is the counter example to prove this.

Example 1.20. Let $t_n = -e^{-n}$, which is an increasing sequence but it is not s-increasing.

$$\begin{aligned} t_0 &= -e^{-0} = -1 \\ t_1 &= -e^{-1} = -0.36 \\ t_2 &= -e^{-2} = -0.1353 \\ &\vdots \end{aligned}$$

Hence,

$$t_{n+1} \ngeq t_n + 1.$$

Definition 1.21. A mapping $T : A \to B$ is said to be b-fuzzy proximal contraction of type-I with respect to $\eta \in \Omega$, if there exist a $k \in (0,1)$ such that for any $x, y, u, v \in A$ and $t > 0$, we have

$$\left.\begin{array}{l} M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\} \Longrightarrow \eta\left[M(u, v, t)\right] \leq k\eta\left[M(x, y, t)\right].$$

Definition 1.22. Let $T : A \to B$ and $g : A \to A$. A pair (T, g) is said to be b-fuzzy $\Omega-$generalized proximal contraction pair with respect to $\eta \in \Omega$ if there exist a $k \in (0,1)$ such that for any $x, y, u, v \in A$ and $t > 0$, we have

$$\left.\begin{array}{l} M(gu, Tx, t) = M(A, B, t) \\ M(gv, Ty, t) = M(A, B, t) \end{array}\right\} \Longrightarrow \eta\left[M(gu, gv, t)\right] \leq k\eta\left[M(x, y, t)\right].$$

Definition 1.23. A mapping $T : A \to B$ is said to be b-fuzzy proximal contraction of type-II if there exists $k \in (0,1)$ such that for any $x, y, u, v \in A$, and $t > 0$, we have

$$\left.\begin{array}{l} M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\} \Longrightarrow M(u, v, t) \geq M\left(x, y, \frac{t}{k}\right).$$

Definition 1.24. Let $T : A \to B$ and $g : A \to A$. A pair (T, g) is said to be b-fuzzy general proximal contraction if there exists $k \in (0,1)$ such that for any $x, y, u, v \in A$ and $t > 0$, we have

$$\left.\begin{array}{l} M(gu, Tx, t) = M(A, B, t) \\ M(gv, Ty, t) = M(A, B, t) \end{array}\right\} \Longrightarrow M(gu, gv, t) \geq M\left(x, y, \frac{t}{k}\right).$$

2. Main Result

Theorem 2.1. *Let $(X, M, *, b)$ be a complete b-fuzzy metric space, B a fuzzy approximately compact with respect to nonempty closed subset A in X. Suppose that a pair of mapping (g, T) is b-fuzzy $\Omega-$generalized proximal contraction where $T : A \to B$ and $g : A \to A$ is a fuzzy expansive mapping. If $T(A_0(t)) \subseteq B_0(t)$ and $\phi \neq A_0(t) \subseteq g(A_0(t))$ for each $t > 0$. Then, there exists an element $x^* \in A$ such that $M(gx^*, Tx^*, t) = M(A, B, t)$.*

Proof. Let x_0 be an arbitrary element in $A_0(t)$. As $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, we may choose an element $x_1 \in A_0(t)$ such that

$$M(gx_1, Tx_0, t) = M(A, B, t).$$

Also since $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$, and $A_0(t) \subseteq g(A_0(t))$, it follows that there exists an element $x_2 \in A_0(t)$ such that the following equation holds:

$$M(gx_2, Tx_1, t) = M(A, B, t).$$

Continuing this way, we can obtain a sequence $\{x_n\}$ in $A_0(t)$ such that it satisfies:

$$M(gx_n, Tx_{n-1}, t) = M(A, B, t) \quad \text{and } M(gx_{n+1}, Tx_n, t) = M(A, B, t), \tag{2.1}$$

for each positive integer n and for $k \in (0, 1)$. As $\{g, T\}$ is b-fuzzy $\Omega-$generalized proximal contraction, we have

$$\eta[M(gx_n, gx_{n+1}, t)] \leq k\eta[M(x_{n-1}, x_n, t)],$$

for all $n \geq 0$. As g is fuzzy expansive and η is a decreasing mapping on $[0, \infty)$, we obtain that

$$\eta[M(x_n, x_{n+1}, t)] \leq k\eta[M(x_{n-1}, x_n, t)].$$

Thus,

$$\begin{aligned}
\eta[M(x_n, x_{n+1}, t)] &\leq k\eta[M(x_{n-1}, x_n, t)] \\
&\leq k^2\eta[M(x_{n-2}, x_{n-1}, t)] \\
&\leq \cdots \\
&\leq k^n\eta[M(x_0, x_1, t)], \text{ for each } t > 0.
\end{aligned}$$

On taking limit as $n \to \infty$ on both sides of the above inequality, we have

$$\lim_{n\to\infty} \eta(M(x_n, x_{n+1}, t)) = 0.$$

Now we show that $\{x_n\}$ is a Cauchy sequence. Suppose that there exists some $n_0 \in \mathbb{N}$ with $m > n > n_0$ such that,

$$\begin{aligned}
\eta(M(x_n, x_m, t)) &\leq \eta(M\left(x_n, x_n, \frac{t}{b} - \sum_{i=n}^{m-1} \frac{a_i t}{b}\right) * M\left(x_n, x_m, \sum_{i=n}^{m-1} \frac{a_i t}{b}\right)) \\
&= \eta\left(1 * M\left(x_n, x_m, \sum_{i=n}^{m-1} \frac{a_i t}{b}\right)\right) \\
&\leq \eta(1) + \eta(M\left(x_n, x_m, \sum_{i=n}^{m-1} \frac{a_i t}{b}\right)) \\
&= \eta(M\left(x_n, x_m, \sum_{i=n}^{m-1} \frac{a_i t}{b}\right)),
\end{aligned}$$

where $\{a_i\}$ is a decreasing sequence of positive numbers satisfying $\sum\limits_{i=n}^{m-1} a_i = 1$. Moreover, we obtain that

$$\begin{aligned}
\eta(M(x_n, x_m, t) &\leq \eta(M\left(x_n, x_m, \sum_{i=n}^{m-1} \frac{a_i t}{b}\right)) \\
&\leq \eta(M\left(x_n, x_{n+1}, \frac{a_n t}{b^2}\right) * M\left(x_{n+1}, x_{n+2}, \frac{a_{n+1} t}{b^2}\right) * \\
&\quad \cdots * M\left(x_{m-1}, x_m, \frac{a_{m-1} t}{b^2}\right)) \\
&\leq \eta(M\left(x_n, x_{n+1}, \frac{a_n t}{b^2}\right)) + \eta(M\left(x_{n+1}, x_{n+2}, \frac{a_{n+1} t}{b^2}\right)) + \\
&\quad \cdots + \eta(M\left(x_{m-1}, x_m, \frac{a_{m-1} t}{b^2}\right)) \\
&\leq k^n \eta(M\left(x_0, x_1, \frac{a_n t}{b^2}\right)) + k^{n+1} \eta(M\left(x_0, x_1, \frac{a_{n+1} t}{b^2}\right)) + \\
&\quad \cdots + k^{m-1} \eta(M\left(x_0, x_1, \frac{a_{m-1} t}{b^2}\right)).
\end{aligned}$$

Hence

$$\begin{aligned}
\eta(M(x_n, x_m, t) &\leq k^n \eta(M\left(x_0, x_1, \frac{a_n t}{b^2}\right)) + k^{n+1} \eta(M\left(x_0, x_1, \frac{a_{n+1} t}{b^2}\right)) + \\
&\quad \cdots + k^{m-1} \eta(M\left(x_0, x_1, \frac{a_{m-1} t}{b^2}\right)).
\end{aligned} \tag{2.2}$$

Assume that

$$\max\{\eta(M\left(x_0, x_1, \frac{a_n t}{b^2}\right)), \eta(M\left(x_0, x_1, \frac{a_{n+1} t}{b^2}\right)), \cdots, \eta(M\left(x_0, x_1, \frac{a_{m-1} t}{b^2}\right))\} = \eta\left(M\left(x_0, x_1, qt\right)\right),$$

for some $q \in \{\frac{a_i}{b^2} : n \leq i \leq m-1 \text{ and } b \geq 1\}$, then the inequality (2.2) becomes

$$\begin{aligned}
\eta(M(x_n, x_m, t) &\leq k^n \eta(M(x_0, x_1, qt)) + k^{n+1} \eta(M(x_0, x_1, qt)) + \cdots + k^{m-1} \eta(M(x_0, x_1, qt)) \\
&\leq (k^n + k^{n+1} + \cdots + k^{m-1}) \eta(M(x_0, x_1, qt)) \\
&= k^n (1 + k + \cdots + k^{m-n-1}) \eta(M(x_0, x_1, qt)) \\
&\leq \frac{k^n}{1-k} \eta(M(x_0, x_1, qt)),
\end{aligned}$$

that is, for all $n \in \mathbb{N}$,

$$\eta(M(x_n, x_m, t) \leq \frac{k^n}{1-k} \eta(M(x_0, x_1, qt)).$$

On taking limit as $n \to \infty$ on both sides of the above inequality, we have

$$0 \leq \lim_{n,m \to \infty} \eta(M(x_n, x_m, t)) \leq 0.$$

By the continuity of η, we have

$$\lim_{n,m \to \infty} M(x_n, x_m, t) = 1.$$

Thus, $\{x_n\}$ is a Cauchy sequence in a closed subset A of a complete b-fuzzy metric space $(X, M, *, b)$. Hence there exists some $x^* \in A$ such that

$$\lim_{n \to +\infty} M(x_n, x^*, t) = 1, \text{ for all } t > 0.$$

Note that,

$$M(gx^*, B, t) \geq M(gx^*, Tx^*, t) = M(A, B, t) \geq M(gx^*, B, t).$$

As g is continuous and the sequence $\{x_n\}$ converges to x^*, we obtain that

$$M(gx^*, Tx_n, t) \to M(gx^*, B, t).$$

If, we consider $Tx^* = y_n$ (say) in B. Since $\{Tx_n\} \subseteq B$, and B is a fuzzy approximately compact with respect to A, $\{Tx_n\}$ has a subsequence which converges to some y in B, therefore $M(gx^*, y, t) = M(A, B, t)$, and hence $gx^* \in A_0(t)$. As $A_0 \subseteq g(A_0)$ implies that $gu = gx^*$ for some $u \in A_0(t)$, we have

$$M(gu, Tx^*, t) = M(A, B, t) = M(gx_{n+1}, Tx_n, t), \ \forall n \in \mathbb{N}.$$

As $\{g, T\}$ is b-fuzzy $\Omega-$generalized proximal contraction and g is fuzzy expansive mapping, we have

$$\eta(M(u, x_{n+1}, t)) \leq \eta(M(gu, gx_{n+1}, t)) \leq k\eta(M(x^*, x_n, t)).$$

Taking limit as $n \to \infty$ on both sides of the above inequality, we obtain that

$$\eta(M(u, x^*, t)) \leq k\eta(1)$$

and hence $M(u, x^*, t) = 1$ which implies $u = x^*$. Thus

$$M(gx^*, Tx^*, t) = M(gu, Tx^*, t) = M(A, B, t),$$

gives that x^* is the optimal coincidence point of the pair $\{g, T\}$. If there is another optimal coincidence point $y^* \neq x^*$ of the pair $\{g, T\}$ in $A_0(t)$, then we have

$$M(gx^*, Tx^*, t) = M(A, B, t) \text{ and } M(gy^*, Ty^*, t) = M(A, B, t).$$

Since $\{g, T\}$ is b-fuzzy $\Omega-$generalized proximal contraction and g is fuzzy expansive, so

$$\begin{aligned} \eta(M(x^*, y^*, t)) &\leq \eta(M(gx^*, gy^*, t)) \\ &\leq k\eta(M(x^*, y^*, t)) \\ &< \eta(M(x^*, y^*, t)), \end{aligned}$$

gives a contradiction. Hence the result. □

Example 2.2. Let $X = \mathbb{R} \times [-1, 1]$ and $d : X \times X \to [0, \infty)$ be a $b-$ metric defined by

$$d((x_1, y_1), (x_2, y_2)) = (|x_1 - x_2| + |y_1 - y_2|)^2.$$

Let $A = \{(x, 1) : x \in \mathbb{R}\}$ and $B = \{(x, -1) : x \in \mathbb{R}\}$. It is known that $(X, M_d, \wedge)$ is a complete b-fuzzy metric space (1.2), where M_d is a standard b-fuzzy metric induced by d. Note that

$$M_d(A, B, t) = \frac{t}{t+4}$$

Note that $A_0(t) = A$ and $B_0(t) = B$. Define a mapping $T : A \to B$ as:

$$T(x, 1) = \left(\frac{x}{3}, -1\right)$$

and $g : A \to A$ as:

$$g(x, 1) = (3x, 1).$$

Clearly g is fuzzy expansive mapping. Also the pair (g, T) satisfies $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) = g(A_0(t))$. If we take $u = (x_1, 1)$, $v = (x_2, 1) \in A$, then $x = (x_3, 1)$ and $y = (x_4, 1) \in A$ satisfy

$$\begin{aligned} M(gu, Tx, t) &= M(A, B, t), \text{ and} \\ M(gv, Ty, t) &= M(A, B, t), \end{aligned}$$

provided that $x_1 = \frac{x_3}{9}$ and $x_2 = \frac{x_4}{9}$ for x_1, x_2, x_3 and $x_4 \in \mathbb{R}$.
If we choose

$$\eta(t) = \frac{1}{t} - 1 \text{ and } k \in (0,1),$$

then

$$\eta[M(gu, gv, t)] \leq k\eta[M(x, y, t)],$$

holds true. So all the conditions of the Theorem (2.1) are satisfied. Moreover, $(0,1)$ is the unique optimal coincidence point in the set $A_0(t)$ of the pair (g, T).

Corollary 2.3. *Let $(X, M, *, b)$ be a complete b-fuzzy metric space, and $T : A \to B$ a b-fuzzy proximal contraction of type-I with $T(A_0(t)) \subseteq B_0(t)$ for each $t > 0$. If B a is fuzzy approximately compact with respect to nonempty closed subset A in X. Then, there exists an element $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$.*

Proof. The result follows from the Theorem (2.1) if we take $g = I_A$ (an identity mapping on A). □

Corollary 2.4. *Let $(X, M, *, b)$ be a complete b-fuzzy metric space, B a fuzzy approximately compact with respect to nonempty closed subset A in X. Suppose that a pair of mapping (g, T) is b-fuzzy $\Omega-$generalized proximal contraction where $T : A \to B$ and $g : A \to A$ is a fuzzy isometry mapping. If $T(A_0(t)) \subseteq B_0(t)$ and $\phi \neq A_0(t) \subseteq g(A_0(t))$ for each $t > 0$. Then, there exists an element $x^* \in A$ such that $M(gx^*, Tx^*, t) = M(A, B, t)$.*

Proof. Every fuzzy isometry $g : A \to A$ is fuzzy expansive mapping such that

$$M(x, y, t) = M(gx, gy, t)$$

rest of the proof is on the same lines as 2.3. □

Theorem 2.5. *Let $(X, M, *, b)$ be a complete b-fuzzy metric space, B a fuzzy approximately compact with respect to nonempty closed subset A in X. Suppose that a pair of mapping (g, T) is b-fuzzy $\Omega-$general proximal contraction where $T : A \to B$ and $g : A \to A$ is a fuzzy expansive mapping. If $T(A_0(t)) \subseteq B_0(t)$ and $\phi \neq A_0(t) \subseteq g(A_0(t))$ for each $t > 0$. Then, there exists an element $x^* \in A$ such that $M(gx^*, Tx^*, t) = M(A, B, t)$.*

Proof. Let x_0 be an arbitrary element in $A_0(t)$. As $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, we may choose an element $x_1 \in A_0(t)$ such that

$$M(gx_1, Tx_0, t) = M(A, B, t).$$

Also since $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$, and $A_0(t) \subseteq g(A_0(t))$, it follows that there exists an element $x_2 \in A_0(t)$ such that the following holds:

$$M(gx_2, Tx_1, t) = M(A, B, t).$$

Continuing this way, we can obtain a sequence $\{x_n\}$ in $A_0(t)$ such that it satisfies:

$$M(gx_n, Tx_{n-1}, t) = M(A, B, t) \quad \text{and } M(gx_{n+1}, Tx_n, t) = M(A, B, t), \tag{2.3}$$

for each positive integer n and $k \in (0, 1)$. As $\{g, T\}$ is b-fuzzy $\Omega-$general proximal contraction, we have

$$M(gx_n, gx_{n+1}, t) \geq M\left(x_{n-1}, x_n, \frac{t}{k}\right).$$

As g is fuzzy expansive, we obtain that

$$M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, \frac{t}{k}).$$

Thus,

$$\begin{aligned} M(x_n, x_{n+1}, t) \geq M\left(x_{n-1}, x_n, \frac{t}{k}\right) &\geq M\left(x_{n-2}, x_{n-1}, \frac{t}{k^2}\right) \\ &\geq M\left(x_{n-3}, x_{n-2}, \frac{t}{k^3}\right) \\ &\vdots \\ &\geq M\left(x_0, x_1, \frac{t}{k^n}\right), \text{ for each } t > 0. \end{aligned}$$

Suppose that $n < m$, for some $m, n \in \mathbb{N}$. Let $a_i = \dfrac{1}{i(i+1)}$, where $i \in \mathbb{N}$. Note that $\{a_i\}$ is a decreasing sequence of positive numbers satisfying $\sum\limits_{i=1}^{\infty} a_i = 1$. Now we show that $\{x_n\}$ is a Cauchy sequence. Suppose that there exists some $n_0 \in \mathbb{N}$ with $m > n > n_0$ such that

$$\begin{aligned} M(x_n, x_m, t) &\geq M\left(x_n, x_n, \frac{t}{b} - \sum_{i=n}^{m-1} \frac{a_i t}{b}\right) * M\left(x_n, x_m, \sum_{i=n}^{m-1} \frac{a_i t}{b}\right) \\ &= M\left(x_n, x_m, \sum_{i=n}^{m-1} \frac{a_i t}{b}\right) \\ &\geq M\left(x_n, x_{n+1}, \frac{a_n t}{b^2}\right) * M\left(x_{n+1}, x_{n+2}, \frac{a_{n+1} t}{b^2}\right) * \cdots * M\left(x_{m-1}, x_m, \frac{a_{m-1} t}{b^2}\right) \\ &\geq M\left(x_0, x_1, \frac{a_n t}{b^2 k^n}\right) * M\left(x_0, x_1, \frac{a_{n+1} t}{b^2 k^{n+1}}\right) * \cdots * M\left(x_0, x_1, \frac{a_{m-1} t}{b^2 k^{m-1}}\right) \\ &\geq M\left(x_0, x_1, \frac{t}{[n(n+1)] b^2 k^n}\right) * \cdots * M\left(x_0, x_1, \frac{t}{[m(m-1)] b^2 k^{m-1}}\right). \\ &\geq \prod_{n=1}^{\infty} M\left(x_0, x_1, \frac{t}{[n(n+1)] b^2 k^n}\right). \end{aligned}$$

Now, if we write $t_n = \frac{t}{[n(n+1)]b^2k^n}$, then $\lim_{n\to\infty}(t_{n+1} - t_n) \to \infty$ as $n \to \infty$. Therefore $\{t_n\}$ is an $s-$increasing sequence and satisfying the property-T and hence there exist some n_0 in $\mathbb{N}$ and $\epsilon > 0$ such that $\prod\limits_{n=n_0}^{\infty} M(x_0, x_1, t_i) > 1 - \epsilon$ for all $n, m \geq n_0$. Thus, $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$. So, $\{x_n\}$ is a Cauchy sequence in a closed subset A of a complete b-fuzzy metric space $(X, M, *, b)$. Hence there exists some $x^* \in A$ such that $\lim_{n\to\infty} M(x_n, x^*, t) = 1$ for all $t > 0$. Note that

$$M(gx^*, B, t) \geq M(gx^*, Tx^*, t) = M(A, B, t) \geq M(gx^*, B, t).$$

If, we consider $Tx^* = y_n$ (say) in B.As g is continuous and the sequence $\{x_n\}$ converges to x^*, we obtain that

$$M(gx^*, Tx_n, t) \to M(gx^*, B, t) \text{ when } n \to \infty.$$

Since $\{Tx_n\} \subseteq B$, and B is a fuzzy approximately compact with respect to A, $\{Tx_n\}$ has a subsequence which converges to some y in B, therefore $M(gx^*, y, t) = M(A, B, t)$, and hence $gx^* \in A_0(t)$. As $A_0 \subseteq g(A_0)$ implies that $gu = gx^*$ for some $u \in A_0(t)$, we have

$$M(gu, Tx^*, t) = M(A, B, t) = M(gx_{n+1}, Tx_n, t), \ \forall n \in \mathbb{N}.$$

As (g, T) is b-fuzzy $\Omega-$general proximal contraction and g is fuzzy expansive mapping, we have

$$M(u, x_{n+1}, t) \geq M(gu, gx_{n+1}, t) \geq M\left(x^*, x_n, \frac{t}{k}\right).$$

Taking limit as $n \to \infty$ on both sides of the above inequality, we obtain that

$$M(u, x^*, t) \geq 1,$$

and hence $M(u, x^*, t) = 1$ which implies that $u = x^*$. Thus

$$M(gx^*, Tx^*, t) = M(gu, Tx^*, t) = M(A, B, t),$$

gives that x^* is the optimal coincidence point of the pair (g, T). If there is another optimal coincidence point $y^* \neq x^*$ of the pair (g, T) in $A_0(t)$, then we have

$$\begin{aligned} M(gx^*, Tx^*, t) &= M(A, B, t) \text{ and} \\ M(gy^*, Ty^*, t) &= M(A, B, t). \end{aligned}$$

Since (g, T) is b-fuzzy $\Omega-$general proximal contraction and g is fuzzy expansive, we have

$$M(x^*, y^*, t) \geq M(gx^*, gy^*, t) \geq M\left(x^*, y^*, \frac{t}{k}\right),$$

which gives

$$M(x^*, y^*, t) \geq M\left(x^*, y^*, \frac{t}{k}\right),$$

a contradiction. Hence the optimal coincidence point of the pair (g, T) is unique. □

Example 2.6. Let $X = \mathbb{R} \times [-1, 1]$. Suppose that $d : X \times X \to [0, \infty)$ is defined by

$$d((x_1, y_1), (x_2, y_2)) = (|x_1 - x_2| + |y_1 - y_2|)^2$$

which is a b-metric on X. Let $A = \{(x, 1) : x \in \mathbb{R}\}$ and $B = \{(x, -1) : x \in \mathbb{R}\}$. Note that $(X, M_d, \wedge)$ is a complete b-fuzzy metric space (1.2), where M_d is a standard b-fuzzy metric induced by d. Note that

$$M_d(A, B, t) = \frac{t}{t+4}, \; A_0(t) = A \text{ and } B_0(t) = B.$$

Define a mapping $T : A \to B$ by $T(x, 1) = \left(\frac{x}{2}, -1\right)$ and $g : A \to A$ by $g(x, 1) = (2x, 1)$. Clearly g is fuzzy expansive mapping. $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) = g(A_0(t))$. Elements $u = (x_1, 1)$, $v = (x_2, 1)$, $x = (x_3, 1)$ and $y = (x_4, 1)$ in A satisfy

$$\begin{aligned} M(gu, Tx, t) &= M(A, B, t), \\ M(gv, Ty, t) &= M(A, B, t), \end{aligned}$$

provided that $x_1 = \frac{x_3}{4}$ and $x_2 = \frac{x_4}{4}$. Also,

$$M(gu, gv, t) \geq M\left(x, y, \frac{t}{k}\right),$$

holds true for all $k \geq \frac{2}{5}$. Thus all the conditions of Theorem (2.5) are satisfied. Moreover $(0, 1)$ is the unique optimal coincidence point of the pair (g, T) in $A_0(t)$.

Corollary 2.7. *Let $(X, M, *, b)$ be a complete b-fuzzy metric space, B a fuzzy approximately compact with respect to nonempty closed subset A in X. Suppose that a mapping T is b-fuzzy proximal contraction of type-II, where $T : A \to B$. If $T(A_0(t)) \subseteq B_0(t)$ for each $t > 0$. Then, there exists an element $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A, B, t)$.*

Proof. The result follow from the Theorem (2.5) if $g = I_A$ (an identity mapping on A). □

Corollary 2.8. *Let $(X, M, *, b)$ be a complete b-fuzzy metric space, B a fuzzy approximately compact with respect to nonempty closed subset A in X. Suppose that a pair of mapping (g, T) is b-fuzzy $\Omega-$general proximal contraction where $T : A \to B$ and $g : A \to A$ is a fuzzy isometry mapping. If $T(A_0(t)) \subseteq B_0(t)$ and $\phi \neq A_0(t) \subseteq g(A_0(t))$ for each $t > 0$. Then, there exists an element $x^* \in A$ such that $M(gx^*, Tx^*, t) = M(A, B, t)$.*

3. Application

As an application of our results, we prove some new fixed point theorems as follows. We start with the following fixed point theorem:

Theorem 3.1. *Let $(X, M, *, b)$ be a complete b-fuzzy metric space, let $T : X \to X$ be a mapping satisfying*

i) $\eta[M(Tx, Ty, t)] \leq k\eta[M(x, y, t)]$.

ii) T is continuous.

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

Proof. Let $A = B = X$, first we will prove that T is b-fuzzy proximal contraction of type-I. Let $x, y, u, v \in X$, satisfy the following conditions:

$$\begin{cases} M(u, Tx, t) = M(A, B, t), \\ M(v, Ty, t) = M(A, B, t). \end{cases}$$

Since $M(A, B, t) = 1$, so we have $u = Tx$ and $v = Ty$. Since T satisfies condition (i) hence

$$\eta(M(u, v, t)) = \eta(M(Tx, Ty, t)) \leq k\eta(M(x, y, t))$$

which implies that T is b-fuzzy proximal contraction of type-I with respect to $\eta \in \Omega$. If we choose $y = Tx$, then

$$M(A, B, t) = M(y, Tx, t) = M(Tx, Tx, t).$$

Set B is approximative compact with respect to A, the conditions of Corollary 2.3 are satisfied, so there exists $x^* \in X$ such that $M(x^*, Tx^*, t) = 1 = M(A, B, t)$, which implies that $Tx^* = x^*$. □

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