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Note on some Iyengar integral inequalities

Khaled Boukerrioua[a,*], Badreddine Meftah[b], Tarik Chiheb[b]

[a]*Lanos Laboratatory , University of Badji-Mokhtar, Annaba, Algeria.*
[b]*Laboratoire des Tlcommunications, Facult des Sciences et de la Technologie, Universit 8 Mai 1945 de Guelma,. P.O. Box 401, 24000 Guelma, Algeria.*

Abstract

In this short note, some Iyengar integral inequalities are established via new extension of Montgomery identity.

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1. introduction

In 1938, Iyengar [1] proved the following interesting integral inequality.

Theorem 1.1. *Let f be differentiable function on $[a,b]$ and $|f'(x)| \leq M$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}\right| \leq \frac{M(b-a)}{4} - \frac{(f(b)-f(a))^2}{4M(b-a)}. \tag{1.1}$$

Inequality (1.1) has attracted many researchers, various generalizations, extensions and variants have appeared in the literature. Recently, [1, 2] proved the following inequalities involving bounded of the second-order derivatives.

Theorem 1.2. *Let $f \in C^2[a,b]$ and $|f''(x)| \leq M$. Then*

$$|I| \leq \frac{M}{24}(b-a)^3 - \frac{\left|f'(a) - 2f'(\frac{a+b}{2}) + f'(b)\right|^3}{24M^2},$$

*Corresponding author
Email addresses: `khaledv2004@yahoo.fr` (Khaled Boukerrioua), `badrimeftah@yahoo.fr` (Badreddine Meftah), `tchiheb@yahoo.fr` (Tarik Chiheb)

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where

$$I=\int_a^b f(t)dt-\frac{1}{2}(b-a)(f(a)+f(b))+\frac{1}{8}(b-a)^2\left(f'(b)-f'(a)\right).$$

The following lemmas are very useful in our main results.

Lemma 1.3 (Mean Value Theorem)**.** *Suppose that f is continuous function on a closed interval $I:=[a,b]$, and that f has a derivative in the open interval (a,b), then there exists at least one point c in (a,b) such that*

$$f(b)-f(a)=f'(c)(b-a). \tag{1.2}$$

Lemma 1.4 ([3])**.** *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on $[a,b]$, and $f':[a,b]\to\mathbb{R}$ integrable on $[a,b]$, then the Montgomery identity holds*

$$f(x)=\frac{1}{b-a}\int_a^b f(t)dt+\int_a^b p(x,t)f'(t)dt, \tag{1.3}$$

where $p(x,t)$ is the Peano kernel, defined as follows

$$p(x,t)=\begin{cases}\frac{t-a}{b-a} & a\le t\le x,\\ \frac{t-b}{b-a} & x<t\le b.\end{cases} \tag{1.4}$$

The main purpose of this work is to obtain a new Iyengar type inequalities by using new extension of the Montgomery identity.

2. Main result

Lemma 2.1. *Let $f\in C^2[a,b]$, then we have*

$$f(x)-\frac{1}{b-a}\int_a^b f(t)dt+\frac{(b-x)^2f'(b)-(x-a)^2f'(a)}{2(b-a)}=\frac{(x-a)^3f''(c_1)+(b-x)^3f''(c_2)}{3(b-a)}, \tag{2.1}$$

where $a<c_1<t$ and $t<c_2<b$.

Proof. From Lemma 1.4, we have

$$\begin{aligned}f(x)&=\frac{1}{b-a}\int_a^b f(t)dt+\int_a^b p(x,t)f'(t)dt\\&=\frac{1}{b-a}\int_a^b f(t)dt+\frac{1}{b-a}\left[\int_a^x (t-a)f'(t)dt+\int_x^b (t-b)f'(t)dt\right].\end{aligned} \tag{2.2}$$

Applying Lemma 1.3 to f'on $[a,x]$, it yields

$$f'(t)=(t-a)f''(c_1)+f'(a),\qquad \text{where}\ \ a<c_1<t, \tag{2.3}$$

now, using Lemma 1.3 on $[x,b]$, we get

$$f'(t)=(t-b)f''(c_2)+f'(b),\qquad \text{where}\ \ t<c_2<b. \tag{2.4}$$

Substituting (2.4) and (2.3) in (2.2), we obtain

$$\begin{aligned} f(x) &= \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{b-a}\left[\int_a^x \left[(t-a)^2 f''(c_1) + (t-a)f'(a)\right]dt \right. \\ &\left. + \int_x^b \left[(t-b)^2 f''(c_2) + (t-b)f'(b)\right]dt\right], \qquad (2.5) \end{aligned}$$

where $a < c_1 < t < x$ and $x < t < c_2 < b$. Thus, (2.5) gives

$$\begin{aligned} f(x) &= \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{b-a}\left[\frac{(x-a)^3}{3}f''(c_1) + \frac{(x-a)^2}{2}f'(a)\right. \\ &\left. -\frac{(x-b)^3}{3}f''(c_2) - \frac{(x-b)^2}{2}f'(b)\right]. \qquad (2.6) \end{aligned}$$

The desired identity follows from (2.6). □

Theorem 2.2. *Let $f \in C^2[a,b]$ such that $f''(x)$ is bounded for all $x \in [a,b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}(f(b)+f(a)) - (b-a)\frac{f'(b) - f'(a)}{4}\right| \le \frac{(b-a)^2}{3}\left\|f''\right\|_\infty. \qquad (2.7)$$

Proof. From Lemma 2.1 and property of modulus, we have

$$\begin{aligned} \left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt + \frac{(b-x)^2 f'(b) - (x-a)^2 f'(a)}{2(b-a)}\right| &= \left|\frac{(x-a)^3 f''(c_1) + (b-x)^3 f''(c_2)}{3(b-a)}\right| \\ &\le \left|\frac{(x-a)^3 + (b-x)^3}{3(b-a)}\right| \left\|f''\right\|_\infty \\ &\le \frac{(x-a)^3 + (b-x)^3}{3(b-a)}\left\|f''\right\|_\infty, \qquad (2.8) \end{aligned}$$

the substitution of x by a in (2.8), gives

$$\left|f(a) - \frac{1}{b-a}\int_a^b f(t)dt + \frac{(b-a)f'(b)}{2}\right| \le \frac{(b-a)^2}{3}\left\|f''\right\|_\infty, \qquad (2.9)$$

now, substituting x by b in (2.8), we obtain

$$\left|f(b) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{(b-a)f'(a)}{2}\right| \le \frac{(b-a)^2}{3}\left\|f''\right\|_\infty. \qquad (2.10)$$

The addition of (2.9) and (2.10), gives

$$\left|\frac{2}{b-a}\int_a^b f(t)dt - (f(b)+f(a)) - \frac{(b-a)}{2}\left(f'(b) - f'(a)\right)\right| \le \frac{2(b-a)^2}{3}\left\|f''\right\|_\infty, \qquad (2.11)$$

dividing both sides of (2.11) by 2, we obtain the desired result in (2.7), which completes the proof. □

Corollary 2.3. *Let $f \in C^2[a,b]$ such that the second derivative is bounded on $[a,b]$, then the following inequality holds*

$$\left| \frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}(f(b)+f(a)) \right| \leq \frac{7(b-a)^2}{12}\|f''\|_\infty . \tag{2.12}$$

Proof. From Theorem 2.2, we have

$$\begin{aligned} -\frac{(b-a)^2}{3}\|f''\|_\infty &\leq \frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}(f(b)+f(a)) - (b-a)\frac{f'(b)-f'(a)}{4} \\ &\leq \frac{(b-a)^2}{3}\|f''\|_\infty , \end{aligned} \tag{2.13}$$

the above double inequality gives

$$\begin{aligned} (b-a)\frac{f'(b)-f'(a)}{4} - \frac{(b-a)^2}{3}\|f''\|_\infty &\leq \frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}(f(b)+f(a)) \\ &\leq \frac{(b-a)^2}{3}\|f''\|_\infty + (b-a)\frac{f'(b)-f'(a)}{4}. \end{aligned} \tag{2.14}$$

From Lemma 1.3, we have

$$f'(b) - f'(a) = f''(\eta)(b-a),$$

where $\eta \in (a,b)$. Using the properties of modulus, the above equality becomes

$$|f'(b) - f'(a)| = |b-a|\left|f''(\eta)\right| \leq (b-a)\|f''\|_\infty . \tag{2.15}$$

From (2.15), we get

$$\begin{aligned} -\frac{7(b-a)^2}{12}\|f''\|_\infty &= -\left[\frac{(b-a)^2}{4} + \frac{(b-a)^2}{3}\right]\|f''\|_\infty \\ &\leq (b-a)\frac{f'(b)-f'(a)}{4} - \frac{(b-a)^2}{3}\|f''\|_\infty \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \frac{(b-a)^2}{3}\|f''\|_\infty + (b-a)\frac{f'(b)-f'(a)}{4} &\leq \left[\frac{(b-a)^2}{3} + \frac{(b-a)^2}{4}\right]\|f''\|_\infty \\ &= \frac{7(b-a)^2}{12}\|f''\|_\infty . \end{aligned} \tag{2.17}$$

Combining (2.14), (2.16) and (2.17), we obtain the desired inequality in (2.12). The proof is achieved. □

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