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Common coupled fixed point theorem for two pairs of hybrid maps in complex valued metric spaces

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Abstract

In this paper, we prove a common coupled fixed point theorem for two hybrid pairs of maps with greatest lower bound property in complex valued metric spaces. We also give an example to illustrate our main theorem.

Keywords: Complex valued metric, w-compatible maps, hybrid pair, g.l.b property.
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1. Introduction and Preliminaries

Azam et al. [6] introduced the notion of a complex valued metric space which is a generalization of the classical metric space and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a rational contractive conditions. Later on several authors proved fixed and common fixed point theorems in complex valued metric spaces, for example, we refer the readers to [2, 4, 9, 12–14, 16, 18, 25–29]. Recently, Azam et al.[7] and Ahmad et al.[5] obtained some new fixed point results for multi-valued mappings in the setting of complex valued metric spaces.
The purpose of this paper is to study the common coupled fixed points for a pair of hybrid mappings satisfying a rational inequality in the frame work of a complex valued metric space. We also give an example to illustrate our main theorem. The proved result generalizes and extends the Theorems 9 and 15 of [5], Theorem 10 of [7], Theorem 9 of [15] and Theorem 2.1 of [23].
Throughout this paper $\mathcal{R}$, $\mathcal{R}^+$, $\mathcal{N}$ and $\mathbb{C}$ denote the set of all real numbers, non-negative real numbers, positive integers and complex numbers respectively. First, we refer the following preliminaries.
Let $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ follows:

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$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2).$$

Thus $z_1 \precsim z_2$ if one of the following holds:

(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$.

(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$.

(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

Clearly $0 \precsim z_1 \precsim z_2 \Rightarrow |z_1| \leq |z_2|$. We will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (2), (3) and (4) is satisfied. Also we will write $z_1 \prec z_2$ if only (4) is satisfied.

Remark 1.1. One can easily check that the following statements:

(i) if $0 \precsim z_1 \precnsim z_2$ then $|z_1| < |z_2|$;

(ii) if $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 1.2 ([6])**.** Let X be a non empty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X, if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$.

(iii) $d(x, y) \precsim d(x, z) + d(z, y)$.

The pair (X, d) is called a complex valued metric space.

Remark 1.3. Let (X, d) be a complex valued metric space. Then

(i) $|d(x,y)| \, or \, |d(u,v)| < |1 + d(x,y) + d(u,v)|, \forall x, y, u, v \in X$.

(ii) $|d(x, y)| > 0$, if $x \neq y$.

Definition 1.4. ([6]) Let (X, d) be a complex valued metric space.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called a limit point of a set $A \subseteq X$ whenever for each $0 \prec r \in \mathbb{C}$ such that $B(x, r) \cap (X - A) \neq \phi$.

(iii) A subset $B \subseteq X$ is called open whenever each point of B is an interior point of B.

(iv) A subset $B \subseteq X$ is called closed whenever each limit point of B is in B.

(v) The family $F = \{B(x, r) : x \in X \; and \;\; 0 \prec r\}$ is a sub basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdorff.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \preceq c$ there is $n_0 \in \mathcal{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent to x and x is the limit point of $\{x_n\}$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathcal{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathcal{N}$, then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d) then (X, d) is called a complete complex valued metric space. We require the follwing lemmas.

Lemma 1.5 ([6]). *Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X.Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Lemma 1.6 ([6]). *Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X.Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n,m \to \infty$.*

One can easily prove the following lemma.

Lemma 1.7. *Let (X,d) be a complex valued metric space and let $\{x_n\}$ and $\{y_n\}$ be sequences in X converging to x and y respectively. Then $|d(x_n,y_n)| \to |d(x,y)|$ as $n \to \infty$.*

Now, we follow the notations and definitions given in [5].
Let (X,d) be a complex valued metric space. We denote the family of nonempty, closed and bounded subsets of a complex valued metric space X by $CB(X)$. From now onwards, we denote for $z_1 \in \mathbb{C}$,

$$s(z_1) = \{z \in \mathbb{C} : z_1 \precsim z\}$$

and for $a \in X$ and $B \in CB(X)$,

$$s(a,B) = \bigcup_{b \in B} s(d(a,b)) = \bigcup_{b \in B} \{z \in \mathbb{C} : d(a,b) \precsim z\}.$$

For $A, B \in CB(X)$, we denote

$$s(A,B) = \left(\bigcap_{a \in A} s(a,B)\right) \cap \left(\bigcap_{b \in B} s(b,A)\right).$$

Remark 1.8 ([5]). Let (X,d) be a complex valued metric space and $T : X \to CB(X)$ be a multivalued map, for $x \in X$ and $A \in CB(X)$, define $W_x(A) = \{d(x,a) : a \in A\}$. Thus, for $x, y \in X$, we have $W_x(Ty) = \{d(x,u) : u \in Ty\}$.

Remark 1.9 ([5]). Let (X,d) be a complex valued metric space. If $\mathbb{C} = \mathcal{R}$ then (X,d) is a metric space. Moreover, for $A, B \in CB(X)$, $H(A,B) = inf\{s(A,B)\}$ is the Hausdorff distance induced by d.

Definition 1.10 ([5]). Let (X,d) be a complex valued metric space. A nonempty subset A of X is called bounded from below, if there exists some $z \in \mathbb{C}$ such that $z \precsim a$ for all $a \in A$.
A multivalued mapping $F : X \to 2^{\mathbb{C}}$ is called bounded from below if for each $x \in X$ there exists $z_x \in \mathbb{C}$ such that $z_x \preceq u$ for all $u \in Fx$.

Definition 1.11 ([5]). The multivalued mapping $T : X \to CB(X)$ is said to have the lower bound property (l.b.Property) on (X,d) if for any $x \in X$, the multi-valued mapping $F_x : X \to 2^{\mathbb{C}}$ defined by $F_x(y) = W_x(Ty)$ is bounded from below.That is, for $x, y \in X$, there exists an element $l_x(Ty) \in \mathbb{C}$ such that $l_x(Ty) \precsim u$, for all $u \in W_x(Ty)$, where $l_x(Ty)$ is called a lower bound of T associated with (x,y).

Definition 1.12 ([5]). Let (X,d) be a complex valued metric space.The multivalued mapping $T : X \to CB(X)$ is said to have the greatest lower bound property (g.l.b.Property) on (X,d) if the greatest lower bound of $W_x(Ty)$ exists in $\mathbb{C}$ for all $x, y \in X$. We denote by $d(x,Ty)$ the g.l.b.of $W_x(Ty)$. That is $d(x,Ty) = inf\{d(x,u) : u \in Ty\}$.

Bhaskar and Lakshmikantham [8] introduced the concept of coupled fixed points and Lakshmikantham and Ciric [17] defined the common coupled fixed points. Later on several authors obtained coupled fixed point theorems in various spaces, for example, [1, 19–22, 24] and the references therein.

Definition 1.13 ([10]). Let the mappings $F : X \times X \to CB(X)$ and $f : X \to X$ be given. An element $(x,y) \in X \times X$ is called

(i) A coupled coincidence point of a pair (F, f), if $fx \in F(x, y)$ and $fy \in F(y, x)$,

(ii) A common coupled fixed point of a pair (F, f), if $x = fx \in F(x, y)$ and $y = fy \in F(y, x)$.

Definition 1.14 ([3]). Let $F : X \times X \to 2^X$ be a multivalued map and f be a self map on X. The hybrid pair (F, f) is called w-compatible if $f(F(x, y)) \subseteq F(fx, fy)$ whenever (x, y) is a coupled coincidence point of F and f.

2. Main Result

Now we prove our main result.

Theorem 2.1. . *Let (X, d) be a complex valued metric space and $F, G : X \times X \to CB(X)$ be multi valued maps with g.l.b property and $f : X \to X$ be a self mapping satisfying the following*

(2.1.1) $F(X \times X) \bigcup G(X \times X) \subseteq f(X)$,

(2.1.2) $f(X)$ *is a complete subspace of* X,

(2.1.3) $a_1 d(fx, fu) + a_2 d(fy, fv) + a_3 d(fx, F(x, y)) + a_4 d(fu, G(u, v))$

$$\begin{aligned}
&+ a_5 d(fx, G(u, v)) + a_6 d(fu, F(x, y)) + a_7 \frac{d(fx, F(x, y)) d(fu, G(u, v))}{1 + d(fx, fu) + d(fy, fv)} \\
&+ a_8 \frac{d(fx, G(u, v)) d(fu, F(x, y))}{1 + d(fx, fu) + d(fy, fv)} + a_9 \frac{d(fx, F(x, y)) d(fx, G(u, v))}{1 + d(fx, fu) + d(fy, fv)} \\
&+ a_{10} \frac{d(fu, F(x, y)) d(fu, G(u, v))}{1 + d(fx, fu) + d(fy, fv)} \in s(F(x, y), G(u, v)),
\end{aligned}$$

for all $x, y, u, v \in X$, where $a_i (i = 1, 2, \cdots, 10)$ are non-negative real numbers such that $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + 2a_9 < 1$.
Then F, G and f have a common coupled coincidence point in $X \times X$.

(2.1.4) *Furthermore, assume that the pairs (F, f) and (G, f) are w-compatible and there exist $p, q \in X$ such that $\lim_{n\to\infty} f^n x = p$ and $\lim_{n\to\infty} f^n y = q$, whenever (x, y) is a common coupled coincidence point of F, G and f. Also assume that f is continuous at p and q. Then F, G and f have a common coupled fixed point in $X \times X$.*

Proof. Let $(x_0, y_0) \in X \times X$. From (2.1.1), we can choose $x_1, y_1 \in X$ such that $fx_1 \in F(x_0, y_0)$ and $fy_1 \in F(y_0, x_0)$ for some $x_1, y_1 \in X$. From (2.1.3), we have

$$\begin{aligned}
&a_1 d(fx_0, fx_1) + a_2 d(fy_0, fy_1) + a_3 d(fx_0, F(x_0, y_0)) + a_4 d(fx_1, G(x_1, y_1)) \\
&+ a_5 d(fx_0, G(x_1, y_1)) + a_6 d(fx_1, F(x_0, y_0)) + a_7 \frac{d(fx_0, F(x_0, y_0)) d(fx_1, G(x_1, y_1))}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} \\
&+ a_8 \frac{d(fx_0, G(x_1, y_1)) d(fx_1, F(x_0, y_0))}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} + a_9 \frac{d(fx_0, F(x_0, y_0)) d(fx_0, G(x_1, y_1))}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} \\
&+ a_{10} \frac{d(fx_1, F(x_0, y_0)) d(fx_1, G(x_1, y_1))}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} \in s(F(x_0, y_0), G(x_1, y_1)) \\
&\subseteq \bigcap_{x \in F(x_0, y_0)} s(x, G(x_1, y_1)).
\end{aligned}$$

Since $fx_1 \in F(x_0, y_0)$ we have

$$\begin{aligned}
&a_1 d(fx_0, fx_1) + a_2 d(fy_0, fy_1) + a_3 d(fx_0, F(x_0, y_0)) + a_4 d(fx_1, G(x_1, y_1)) \\
&+ a_5 d(fx_0, G(x_1, y_1)) + a_6 d(fx_1, F(x_0, y_0)) + a_7 \frac{d(fx_0, F(x_0, y_0)) d(fx_1, G(x_1, y_1))}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} \\
&+ a_8 \frac{d(fx_0, G(x_1, y_1)) d(fx_1, F(x_0, y_0))}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} + a_9 \frac{d(fx_0, F(x_0, y_0)) d(fx_0, G(x_1, y_1))}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} \\
&+ a_{10} \frac{d(fx_1, F(x_0, y_0)) d(fx_1, G(x_1, y_1))}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} \in s(fx_1, G(x_1, y_1)) \\
&= \bigcup_{y \in G(x_1, y_1)} s(d(fx_1, y)).
\end{aligned}$$

From (2.1.1), there exists $x_2 \in X$ with $fx_2 \in G(x_1, y_1)$ such that

$$\begin{aligned}
&a_1 d(fx_0, fx_1) + a_2 d(fy_0, fy_1) + a_3 d(fx_0, F(x_0, y_0)) + a_4 d(fx_1, G(x_1, y_1)) \\
&+ a_5 d(fx_0, G(x_1, y_1)) + a_6 d(fx_1, F(x_0, y_0)) + a_7 \frac{d(fx_0, F(x_0, y_0)) d(fx_1, G(x_1, y_1))}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} \\
&+ a_8 \frac{d(fx_0, G(x_1, y_1)) d(fx_1, F(x_0, y_0))}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} + a_9 \frac{d(fx_0, F(x_0, y_0)) d(fx_0, G(x_1, y_1))}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} \\
&+ a_{10} \frac{d(fx_1, F(x_0, y_0)) d(fx_1, G(x_1, y_1))}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} \in s(d(fx_1, fx_2)).
\end{aligned}$$

Thus

$$\begin{aligned}
d(fx_1, fx_2) \precsim\ & a_1 d(fx_0, fx_1) + a_2 d(fy_0, fy_1) + a_3 d(fx_0, F(x_0, y_0)) \\
&+ a_4 d(fx_1, G(x_1, y_1)) + a_5 d(fx_0, G(x_1, y_1)) + a_6 d(fx_1, F(x_0, y_0)) \\
&+ a_7 \tfrac{d(fx_0, F(x_0, y_0)) d(fx_1, G(x_1, y_1))}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} + a_8 \tfrac{d(fx_0, G(x_1, y_1)) d(fx_1, F(x_0, y_0))}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} \\
&+ a_9 \tfrac{d(fx_0, F(x_0, y_0)) d(fx_0, G(x_1, y_1))}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} + a_{10} \tfrac{d(fx_1, F(x_0, y_0)) d(fx_1, G(x_1, y_1))}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)}.
\end{aligned}$$

By using the g.l.b property of F and G, we get

$$\begin{aligned}
d(fx_1, fx_2) \precsim\ & a_1 d(fx_0, fx_1) + a_2 d(fy_0, fy_1) + a_3 d(fx_0, fx_1) \\
&+ a_4 d(fx_1, fx_2) + a_5 d(fx_0, fx_2) + a_6 d(fx_1, fx_1) \\
&+ a_7 \frac{d(fx_0, fx_1) d(fx_1, fx_2)}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} + a_8 \frac{d(fx_0, fx_2) d(fx_1, fx_1)}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} \\
&+ a_9 \frac{d(fx_0, fx_1) d(fx_0, fx_2)}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)} + a_{10} \frac{d(fx_1, fx_1) d(fx_1, fx_2)}{1 + d(fx_0, fx_1) + d(fy_0, fy_1)}.
\end{aligned}$$

Now, we have

$$\begin{aligned}
|d(fx_1, fx_2)| \leq\ & a_1 |d(fx_0, fx_1)| + a_2 |d(fy_0, fy_1)| + a_3 |d(fx_0, fx_1)| + a_4 |d(fx_1, fx_2)| \\
&+ a_5 [|d(fx_0, fx_1)| + |d(fx_1, fx_2)|] + a_6(0) + a_7 \frac{|d(fx_0, fx_1)|\ |d(fx_1, fx_2)|}{|1 + d(fx_0, fx_1) + d(fy_0, fy_1)|} \\
&+ a_8(0) + a_9 \frac{|d(fx_0, fx_1)|\ [|d(fx_0, fx_1)| + |d(fx_1, fx_2)|]}{|1 + d(fx_0, fx_1) + d(fy_0, fy_1)|} + a_{10}(0) \\
&< a_1 |d(fx_0, fx_1)| + a_2 |d(fy_0, fy_1)| + a_3 |d(fx_0, fx_1)| + a_4 |d(fx_1, fx_2)| \\
&+ a_5 |d(fx_0, fx_1)| + a_5 |d(fx_1, fx_2)| + a_7 |d(fx_1, fx_2)| \\
&+ a_9 |d(fx_0, fx_1)| + a_9 |d(fx_1, fx_2)|.
\end{aligned}$$

Thus

$$|d(fx_1, fx_2)| < \frac{a_1 + a_2 + a_3 + a_5 + a_9}{1 - a_4 - a_5 - a_7 - a_9} \max\{|d(fx_0, fx_1)|, |d(fy_0, fy_1)|\}. \tag{2.1}$$

Now from (2.1.3), we have

$$\begin{aligned}
&a_1 d(fy_0, fy_1) + a_2 d(fx_0, fx_1) + a_3 d(fy_0, F(y_0, x_0)) + a_4 d(fy_1, G(y_1, x_1)) \\
&+ a_5 d(fy_0, G(y_1, x_1)) + a_6 d(fy_1, F(y_0, x_0)) + a_7 \frac{d(fy_0, F(y_0, x_0)) d(fy_1, G(y_1, x_1))}{1 + d(fy_0, fy_1) + d(fx_0, fx_1)} \\
&+ a_8 \frac{d(fy_0, G(y_1, x_1)) d(fy_1, F(y_0, x_0))}{1 + d(fy_0, fy_1) + d(fx_0, fx_1)} + a_9 \frac{d(fy_0, F(y_0, x_0)) d(fy_0, G(y_1, x_1))}{1 + d(fy_0, fy_1) + d(fx_0, fx_1)} \\
&+ a_{10} \frac{d(fy_1, F(y_0, x_0)) d(fy_1, G(y_1, x_1))}{1 + d(fy_0, fy_1) + d(fx_0, fx_1)} \in s(F(y_0, x_0), G(y_1, x_1)) \\
&\subseteq \bigcap_{y \in F(y_0, x_0)} s(x, G(y_1, x_1)).
\end{aligned}$$

Since $fy_1 \in F(y_0, x_0)$, we have

$$\begin{aligned}
&a_1 d(fy_0, fy_1) + a_2 d(fx_0, fx_1) + a_3 d(fy_0, F(y_0, x_0)) + a_4 d(fy_1, G(y_1, x_1)) \\
&+ a_5 d(fy_0, G(y_1, x_1)) + a_6 d(fy_1, F(y_0, x_0)) + a_7 \frac{d(fy_0, F(y_0, x_0)) d(fy_1, G(y_1, x_1))}{1 + d(fy_0, fy_1) + d(fx_0, fx_1)} \\
&+ a_8 \frac{d(fy_0, G(y_1, x_1)) d(fy_1, F(y_0, x_0))}{1 + d(fy_0, fy_1) + d(fx_0, fx_1)} + a_9 \frac{d(fy_0, F(y_0, x_0)) d(fy_0, G(y_1, x_1))}{1 + d(fy_0, fy_1) + d(fx_0, fx_1)} \\
&+ a_{10} \frac{d(fy_1, F(y_0, x_0)) d(fy_1, G(y_1, x_1))}{1 + d(fy_0, fy_1) + d(fx_0, fx_1)} \in s(fy_1, G(y_1, x_1)) \\
&= \bigcup_{y \in G(y_1, x_1)} s(d(fy_1, y)).
\end{aligned}$$

From (2.1.1), there exists $y_2 \in X$ with $fy_2 \in G(y_1, x_1)$ such that

$$\begin{aligned}
&a_1 d(fy_0, fy_1) + a_2 d(fx_0, fx_1) + a_3 d(fy_0, F(y_0, x_0)) + a_4 d(fy_1, G(y_1, x_1)) \\
&+ a_5 d(fy_0, G(y_1, x_1)) + a_6 d(fy_1, F(y_0, x_0)) + a_7 \frac{d(fy_0, F(y_0, x_0)) d(fy_1, G(y_1, x_1))}{1 + d(fy_0, fy_1) + d(fx_0, fx_1)} \\
&+ a_8 \frac{d(fy_0, G(y_1, x_1)) d(fy_1, F(y_0, x_0))}{1 + d(fy_0, fy_1) + d(fx_0, fx_1)} + a_9 \frac{d(fy_0, F(y_0, x_0)) d(fy_0, G(y_1, x_1))}{1 + d(fy_0, fy_1) + d(fx_0, fx_1)} \\
&+ a_{10} \frac{d(fy_1, F(y_0, x_0)) d(fy_1, G(y_1, x_1))}{1 + d(fy_0, fy_1) + d(fx_0, fx_1)} \in s(d(fy_1, fy_2)).
\end{aligned}$$

Thus

$$\begin{aligned}
d(fy_1, fy_2) \precsim\ & a_1 d(fy_0, fy_1) + a_2 d(fx_0, fx_1) + a_3 d(fy_0, F(y_0, x_0)) \\
&+ a_4 d(fy_1, G(y_1, x_1)) + a_5 d(fy_0, G(y_1, x_1)) + a_6 d(fy_1, F(y_0, x_0)) \\
&+ a_7 \frac{d(fy_0, F(y_0, x_0)) d(fy_1, G(y_1, x_1))}{1 + d(fy_0, fy_1) + d(fx_0, fx_1)} + a_8 \frac{d(fy_0, G(y_1, x_1)) d(fy_1, F(y_0, x_0))}{1 + d(fy_0, fy_1) + d(fx_0, fx_1)} \\
&+ a_9 \frac{d(fy_0, F(y_0, x_0)) d(fy_0, G(y_1, x_1))}{1 + d(fy_0, fy_1) + d(fx_0, fx_1)} + a_{10} \frac{d(fy_1, F(y_0, x_0)) d(fy_1, G(y_1, x_1))}{1 + d(fy_0, fy_1) + d(fx_0, fx_1)}
\end{aligned}$$

By using the g.l.b property of F and G, we get

$$\begin{aligned} d(fy_1,fy_2) &\precsim a_1 d(fy_0,fy_1)+a_2 d(fx_0,fx_1)+a_3 d(fy_0,fy_1)\\ &+a_4 d(fy_1,fy_2)+a_5 d(fy_0,fy_2)+a_6 d(fy_1,fy_1)\\ &+a_7\frac{d(fy_0,fy_1)d(fy_1,fy_2)}{1+d(fy_0,fy_1)+d(fx_0,fx_1)}+a_8\frac{d(fy_0,fy_2)d(fy_1,fy_1)}{1+d(fy_0,fy_1)+d(fx_0,fx_1)}\\ &+a_9\frac{d(fy_0,fy_1)d(fy_0,fy_2)}{1+d(fy_0,fy_1)+d(fx_0,fx_1)}+a_{10}\frac{d(fy_1,fy_1)d(fy_1,fy_2)}{1+d(fy_0,fy_1)+d(fx_0,fx_1)}. \end{aligned}$$

Now, we have

$$\begin{aligned} |d(fy_1,fy_2)| &\le a_1|d(fy_0,fy_1)|+a_2|d(fx_0,fx_1)|+a_3|d(fy_0,fy_1)|+a_4|d(fy_1,fy_2)|\\ &+a_5[|d(fy_0,fy_1)|+|d(fy_1,fy_2)|]+a_6(0)+a_7\frac{|d(fy_0,fy_1)|\ |d(fy_1,fy_2)|}{|1+d(fy_0,fy_1)+d(fx_0,fx_1)|}\\ &+a_8(0)+a_9\frac{|d(fy_0,fy_1)|\ [|d(fy_0,fy_1)|+|d(fy_1,fy_2)|]}{|1+d(fy_0,fy_1)+d(fx_0,fx_1)|}+a_{10}(0)\\ &< a_1|d(fy_0,fy_1)|+a_2|d(fx_0,fx_1)|+a_3|d(fy_0,fy_1)|\\ &+a_4|d(fy_1,fy_2)|+a_5|d(fy_0,fy_1)|+a_5|d(fy_1,fy_2)|\\ &+a_7|d(fy_1,fy_2)|+a_9|d(fy_0,fy_1)|+a_9|d(fy_1,fy_2)|. \end{aligned}$$

Thus

$$|d(fy_1,fy_2)|<\frac{a_1+a_2+a_3+a_5+a_9}{1-a_4-a_5-a_7-a_9}\max\{|d(fx_0,fx_1)|,|d(fy_0,fy_1)|\}. \tag{2.2}$$

From (2.1) and (2.2), we have

$$\max\left\{\begin{array}{c}|d(fx_1,fx_2)|,\\ |d(fy_1,fy_2)|\end{array}\right\}\le\lambda\max\left\{\begin{array}{c}|d(fx_0,fx_1)|,\\ |d(fy_0,fy_1)|\end{array}\right\},$$

where $\lambda=\frac{a_1+a_2+a_3+a_5+a_9}{1-a_4-a_5-a_7-a_9}$.

Continuing in this way, we get the sequences $\{x_n\}$ and $\{y_n\}$ in X such that $fx_{2n+1}\in F(x_{2n},y_{2n})$, $fx_{2n+2}\in G(x_{2n+1},y_{2n+1})$, $fy_{2n+1}\in F(y_{2n},x_{2n})$ and $fy_{2n+2}\in G(y_{2n+1},x_{2n+1})$ for $n=0,1,2,\cdots$ and

$$\begin{aligned} \max\left\{\begin{array}{c}|d(fx_n,fx_{n+1})|,\\ |d(fy_n,fy_{n+1})|\end{array}\right\} &\le\lambda\max\left\{\begin{array}{c}|d(fx_{n-1},fx_n)|,\\ |d(fy_{n-1},fy_n)|\end{array}\right\}\\ &\ \vdots\\ &\le\lambda^n\max\left\{\begin{array}{c}|d(fx_0,fx_1)|,\\ |d(fy_0,fy_1)|\end{array}\right\} \end{aligned} \tag{2.3}$$

For $m>n$ and using (2.3) we get

$$\begin{aligned} |d(fx_n,fx_m)| &\le |d(fx_n,fx_{n+1})|+|d(fx_{n+1},fx_{n+2})|+\ldots..+|d(fx_{m-1},fx_m)|\\ &\le(\lambda^n+\lambda^{n+1}+\ldots..+\lambda^{m-1})\max\left\{\begin{array}{c}|d(fx_0,fx_1)|,\\ |d(fy_0,fy_1)|\end{array}\right\}\\ &\le\frac{\lambda^n}{1-\lambda}\max\left\{\begin{array}{c}|d(fx_0,fx_1)|,\\ |d(fy_0,fy_1)|\end{array}\right\}\\ &\to 0\ as\ n,m\to\infty. \end{aligned}$$

Thus $\{fx_n\}$ is a Cauchy sequence in $f(X)$. Similarly, we can show that $\{fy_n\}$ is also a Cauchy sequence in $f(X)$. Since $f(X)$ is a complete, there exist $u,v\in X$ such that $fx_n\to fu$ and $fy_n\to fv$ as $n\to\infty$.

From (2.1.3), we have

$$\begin{aligned}
&a_1 d(fu, fx_{2n+1}) + a_2 d(fv, fy_{2n+1}) + a_3 d(fu, F(u,v)) + a_4 d(fx_{2n+1}, G(x_{2n+1}, y_{2n+1})) \\
&+ a_5 d(fu, G(x_{2n+1}, y_{2n+1})) + a_6 d(fx_{2n+1}, F(u,v)) + a_7 \frac{d(fu, F(u,v)) d(fx_{2n+1}, G(x_{2n+1}, y_{2n+1}))}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} \\
&+ a_8 \frac{d(fu, G(x_{2n+1}, y_{2n+1})) d(fx_{2n+1}, F(u,v))}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} + a_9 \frac{d(fu, F(u,v)) d(fu, G(x_{2n+1}, y_{2n+1}))}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} \\
&+ a_{10} \frac{d(fx_{2n+1}, F(u,v)) d(fx_{2n+1}, G(x_{2n+1}, y_{2n+1}))}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} \in s(F(u,v), G(x_{2n+1}, y_{2n+1})) \\
&\subseteq \bigcap_{y \in G(x_{2n+1}, y_{2n+1})} s(F(u,v), y).
\end{aligned}$$

Since $fx_{2n+2} \in G(x_{2n+1}, y_{2n+1})$ we have

$$\begin{aligned}
&a_1 d(fu, fx_{2n+1}) + a_2 d(fv, fy_{2n+1}) + a_3 d(fu, F(u,v)) + a_4 d(fx_{2n+1}, G(x_{2n+1}, y_{2n+1})) \\
&+ a_5 d(fu, G(x_{2n+1}, y_{2n+1})) + a_6 d(fx_{2n+1}, F(u,v)) + a_7 \frac{d(fu, F(u,v)) d(fx_{2n+1}, G(x_{2n+1}, y_{2n+1}))}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} \\
&+ a_8 \frac{d(fu, G(x_{2n+1}, y_{2n+1})) d(fx_{2n+1}, F(u,v))}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} + a_9 \frac{d(fu, F(u,v)) d(fu, G(x_{2n+1}, y_{2n+1}))}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} \\
&+ a_{10} \frac{d(fx_{2n+1}, F(u,v)) d(fx_{2n+1}, G(x_{2n+1}, y_{2n+1}))}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} \in s(F(u,v), fx_{2n+2}) \\
&= \bigcup_{u' \in F(u,v)} s(d(u', fx_{2n+2})).
\end{aligned}$$

From (2.1.1), there exists $s_n \in X$ with $fs_n \in F(u,v)$ such that

$$\begin{aligned}
&a_1 d(fu, fx_{2n+1}) + a_2 d(fv, fy_{2n+1}) + a_3 d(fu, F(u,v)) + a_4 d(fx_{2n+1}, G(x_{2n+1}, y_{2n+1})) \\
&+ a_5 d(fu, G(x_{2n+1}, y_{2n+1})) + a_6 d(fx_{2n+1}, F(u,v)) + a_7 \frac{d(fu, F(u,v)) d(fx_{2n+1}, G(x_{2n+1}, y_{2n+1}))}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} \\
&+ a_8 \frac{d(fu, G(x_{2n+1}, y_{2n+1})) d(fx_{2n+1}, F(u,v))}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} + a_9 \frac{d(fu, F(u,v)) d(fu, G(x_{2n+1}, y_{2n+1}))}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} \\
&+ a_{10} \frac{d(fx_{2n+1}, F(u,v)) d(fx_{2n+1}, G(x_{2n+1}, y_{2n+1}))}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} \in s(d(fs_n, fx_{2n+2}))
\end{aligned}$$

Thus

$$\begin{aligned}
d(fs_n, fx_{2n+2}) \precsim\ & a_1 d(fu, fx_{2n+1}) + a_2 d(fv, fy_{2n+1}) + a_3 d(fu, F(u,v)) \\
&+ a_4 d(fx_{2n+1}, G(x_{2n+1}, y_{2n+1})) + a_5 d(fu, G(x_{2n+1}, y_{2n+1})) \\
&+ a_6 d(fx_{2n+1}, F(u,v)) + a_7 \frac{d(fu, F(u,v)) d(fx_{2n+1}, G(x_{2n+1}, y_{2n+1}))}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} \\
&+ a_8 \frac{d(fu, G(x_{2n+1}, y_{2n+1})) d(fx_{2n+1}, F(u,v))}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} + a_9 \frac{d(fu, F(u,v)) d(fu, G(x_{2n+1}, y_{2n+1}))}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} \\
&+ a_{10} \frac{d(fx_{2n+1}, F(u,v)) d(fx_{2n+1}, G(x_{2n+1}, y_{2n+1}))}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})}.
\end{aligned}$$

By using the g.l.b property of F and G, we have

$$\begin{aligned} d(fs_n, fx_{2n+2}) \precsim\ & a_1 d(fu, fx_{2n+1}) + a_2 d(fv, fy_{2n+1}) + a_3 d(fu, fs_n) \\ & + a_4 d(fx_{2n+1}, fx_{2n+2}) + a_5 d(fu, fx_{2n+2}) + a_6 d(fx_{2n+1}, fs_n) \\ & a_7 \frac{d(fu, fs_n) d(fx_{2n+1}, fx_{2n+2})}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} + a_8 \frac{d(fu, fx_{2n+2}) d(fx_{2n+1}, fs_n)}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} \\ & + a_9 \frac{d(fu, fs_n) d(fu, fx_{2n+2})}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})} + a_{10} \frac{d(fx_{2n+1}, fs_n) d(fx_{2n+1}, fx_{2n+2})}{1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})}. \end{aligned}$$

Now consider

$$\begin{aligned} |d(fu, fs_n)| \leq\ & |d(fu, fx_{2n+2})| + |d(fx_{2n+2}, fs_n)| \\ \leq\ & |d(fu, fx_{2n+2})| + a_1 |d(fu, fx_{2n+1})| + a_2 |d(fv, fy_{2n+1})| \\ & + a_3 |d(fu, fs_n)| + a_4 |d(fx_{2n+1}, fx_{2n+2})| \\ & + a_5 |d(fu, fx_{2n+2})| + a_6 [|d(fx_{2n+1}, fu)| + |d(fu, fs_n)|\ ] \\ & + a_7 \frac{|d(fu, fs_n)|\ |d(fx_{2n+1}, fx_{2n+2})|}{|1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})|} + a_8 \frac{|d(fu, fx_{2n+2})|\ |d(fx_{2n+1}, fs_n)|}{|1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})|} \\ & + a_9 \frac{|d(fu, fs_n)|\ |d(fu, fx_{2n+2})|}{|1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})|} + a_{10} \frac{|d(fx_{2n+1}, fs_n)|\ |d(fx_{2n+1}, fx_{2n+2})|}{|1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})|} \end{aligned}$$

and

$$\begin{aligned} (1 - a_3 - a_6)|d(fu, fs_n)| \leq\ & |d(fu, fx_{2n+2})| + a_1 |d(fu, fx_{2n+1})| + a_2 |d(fv, fy_{2n+1})| \\ & + a_4 |d(fx_{2n+1}, fx_{2n+2})| + a_5 |d(fu, fx_{2n+2})| + a_6 |d(fx_{2n+1}, fu)| \\ & + a_7 \frac{|d(fu, fs_n)|\ |d(fx_{2n+1}, fx_{2n+2})|}{|1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})|} + a_8 \frac{|d(fu, fx_{2n+2})|\ |d(fx_{2n+1}, fs_n)|}{|1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})|} \\ & + a_9 \frac{|d(fu, fs_n)|\ |d(fu, fx_{2n+2})|}{|1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})|} + a_{10} \frac{|d(fx_{2n+1}, fs_n)|\ |d(fx_{2n+1}, fx_{2n+2})|}{|1 + d(fu, fx_{2n+1}) + d(fv, fy_{2n+1})|} \to 0, \end{aligned}$$

as $n \to \infty$. Thus $fs_n \to fu$ as $n \to \infty$.
Since $F(u, v)$ is closed and $fs_n \in F(u, v)$, we have

$$fu \in F(u, v) \tag{2.4}$$

Similarly, we can show that

$$fv \in F(v, u). \tag{2.5}$$

From (2.1.3), we have

$$\begin{aligned} & a_1 d(fx_{2n}, fu) + a_2 d(fy_{2n}, fv) + a_3 d(fx_{2n}, F(x_{2n}, y_{2n})) + a_4 d(fu, G(u, v)) \\ & + a_5 d(fx_{2n}, G(u, v)) + a_6 d(fu, F(x_{2n}, y_{2n})) + a_7 \frac{d(fx_{2n}, F(x_{2n}, y_{2n})) d(fu, G(u, v))}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} \\ & + a_8 \frac{d(fx_{2n}, G(u, v)) d(fu, F(x_{2n}, y_{2n}))}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} + a_9 \frac{d(fx_{2n}, F(x_{2n}, y_{2n})) d(fx_{2n}, G(u, v))}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} \\ & + a_{10} \frac{d(fu, F(x_{2n}, y_{2n})) d(fu, G(u, v))}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} \in s(F(x_{2n}, y_{2n}), G(u, v)) \\ & \subseteq \bigcap_{x \in F(x_{2n}, y_{2n})} s(x, G(u, v)). \end{aligned}$$

Since $fx_{2n+1} \in F(x_{2n}, y_{2n})$, we have

$$\begin{aligned}
&a_1 d(fx_{2n}, fu) + a_2 d(fy_{2n}, fv) + a_3 d(fx_{2n}, F(x_{2n}, y_{2n})) + a_4 d(fu, G(u,v)) \\
&+ a_5 d(fx_{2n}, G(u,v)) + a_6 d(fu, F(x_{2n}, y_{2n})) + a_7 \frac{d(fx_{2n}, F(x_{2n}, y_{2n}))d(fu, G(u,v))}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} \\
&+ a_8 \frac{d(fx_{2n}, G(u,v))d(fu, F(x_{2n}, y_{2n}))}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} + a_9 \frac{d(fx_{2n}, F(x_{2n}, y_{2n}))d(fx_{2n}, G(u,v))}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} \\
&+ a_{10} \frac{d(fu, F(x_{2n}, y_{2n}))d(fu, G(u,v))}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} \in s(fx_{2n+1}, G(u,v)) \\
&= \bigcup_{u' \in G(u,v)} s(d(fx_{2n+1}, u')).
\end{aligned}$$

From (2.1.1), there exists $t_n \in X$ with $ft_n \in G(u,v)$ such that

$$\begin{aligned}
&a_1 d(fx_{2n}, fu) + a_2 d(fy_{2n}, fv) + a_3 d(fx_{2n}, F(x_{2n}, y_{2n})) + a_4 d(fu, G(u,v)) \\
&+ a_5 d(fx_{2n}, G(u,v)) + a_6 d(fu, F(x_{2n}, y_{2n})) + a_7 \frac{d(fx_{2n}, F(x_{2n}, y_{2n}))d(fu, G(u,v))}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} \\
&+ a_8 \frac{d(fx_{2n}, G(u,v))d(fu, F(x_{2n}, y_{2n}))}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} + a_9 \frac{d(fx_{2n}, F(x_{2n}, y_{2n}))d(fx_{2n}, G(u,v))}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} \\
&+ a_{10} \frac{d(fu, F(x_{2n}, y_{2n}))d(fu, G(u,v))}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} \in s(d(fx_{2n+1}, ft_n)).
\end{aligned}$$

Thus

$$\begin{aligned}
d(fx_{2n+1}, ft_n) \precsim\ & a_1 d(fx_{2n}, fu) + a_2 d(fy_{2n}, fv) + a_3 d(fx_{2n}, F(x_{2n}, y_{2n})) \\
&+ a_4 d(fu, G(u,v)) + a_5 d(fx_{2n}, G(u,v)) + a_6 d(fu, F(x_{2n}, y_{2n})) \\
&+ a_7 \frac{d(fx_{2n}, F(x_{2n}, y_{2n}))d(fu, G(u,v))}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} + a_8 \frac{d(fx_{2n}, G(u,v))d(fu, F(x_{2n}, y_{2n}))}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} \\
&+ a_9 \frac{d(fx_{2n}, F(x_{2n}, y_{2n}))d(fx_{2n}, G(u,v))}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} + a_{10} \frac{d(fu, F(x_{2n}, y_{2n}))d(fu, G(u,v))}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)}
\end{aligned}$$

By using the g.l.b property of F and G, we obtain

$$\begin{aligned}
d(fx_{2n+1}, ft_n) \precsim\ & a_1 d(fx_{2n}, fu) + a_2 d(fy_{2n}, fv) + a_3 d(fx_{2n}, fx_{2n+1}) \\
&+ a_4 d(fu, ft_n) + a_5 d(fx_{2n}, ft_n) + a_6 d(fu, fx_{2n+1}) \\
&+ a_7 \frac{d(fx_{2n}, fx_{2n+1})d(fu, ft_n)}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} + a_8 \frac{d(fx_{2n}, ft_n)d(fu, fx_{2n+1})}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} \\
&+ a_9 \frac{d(fx_{2n}, fx_{2n+1})d(fx_{2n}, ft_n)}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)} + a_{10} \frac{d(fu, fx_{2n+1})d(fu, ft_n)}{1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)}.
\end{aligned}$$

Consider

$$\begin{aligned}
|d(fu, ft_n)| \leq\ & |d(fu, fx_{2n+1})| + |d(fx_{2n+1}, ft_n)| \\
\leq\ & |d(fu, fx_{2n+1})| + a_1 |d(fx_{2n}, fu)| + a_2 |d(fy_{2n}, fv)| + a_3 |d(fx_{2n}, fx_{2n+1})| \\
&+ a_4 |d(fu, ft_n)| + a_5 [|d(fx_{2n}, fu)| + |d(fu, ft_n)|\ ] + a_6 |d(fu, fx_{2n+1})| \\
&+ a_7 \frac{|d(fx_{2n}, fx_{2n+1})|\ |d(fu, ft_n)|}{|1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)|} + a_8 \frac{|d(fx_{2n}, ft_n)|\ |d(fu, fx_{2n+1})|}{|1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)|} \\
&+ a_9 \frac{|d(fx_{2n}, fx_{2n+1})|\ |d(fx_{2n}, ft_n)|}{|1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)|} + a_{10} \frac{|d(fu, fx_{2n+1})|\ |d(fu, ft_n)|}{|1 + d(fx_{2n}, fu) + d(fy_{2n}, fv)|}
\end{aligned}$$

and

$$\begin{aligned}(1-a_4-a_5)|d(fu,ft_n)| &\leq |d(fu,fx_{2n+1})|+a_1|d(fx_{2n},fu)|+a_2|d(fy_{2n},fv)|\\ &+a_3|d(fx_{2n},fx_{2n+1})|+a_5|d(fx_{2n},fu)|+a_6|d(fu,fx_{2n+1})|\\ &+a_7\frac{|d(fx_{2n},fx_{2n+1})|\,|d(fu,ft_n)|}{|1+d(fx_{2n},fu)+d(fy_{2n},fv)|}+a_8\frac{|d(fx_{2n},ft_n)|\,|d(fu,fx_{2n+1})|}{|1+d(fx_{2n},fu)+d(fy_{2n},fv)|}\\ &+a_9\frac{|d(fx_{2n},fx_{2n+1})|\,|d(fx_{2n},ft_n)|}{|1+d(fx_{2n},fu)+d(fy_{2n},fv)|}+a_{10}\frac{|d(fu,fx_{2n+1})|\,|d(fu,ft_n)|}{|1+d(fx_{2n},fu)+d(fy_{2n},fv)|}\to 0,\end{aligned}$$

as $n\to\infty$. Thus $ft_n\to fu$ as $n\to\infty$. Since $G(u,v)$ is closed and $ft_n\in G(u,v)$, we have

$$fu\in G(u,v). \tag{2.6}$$

Similarly we can show that

$$fv\in G(v,u). \tag{2.7}$$

From $(2.4),(2.5),(2.6)$ and (2.7), it follows that (u,v) is a common coupled coincidence point of F,G and f. Now assume (2.1.4). Since (u,v) is a common coupled coincidence point of F,G and f, from (2.1.4), there exist $p,q\in X$ such that

$$\lim_{n\to\infty}f^nu=p \textit{ and } \lim_{n\to\infty}f^nv=q. \tag{2.8}$$

Since f is continuous at p and q, we have

$$fp=p \textit{ and } fq=q. \tag{2.9}$$

Since the pair (F,f) is w-compatible and from (2.4) and (2.5), we have $f^2u\in f(F(u,v))\subseteq F(fu,fv)$ and $f^2v\in f(F(v,u))\subseteq F(fv,fu)$. Since the pair (G,f) is w-compatible and from (6) and (7), we have $f^2u\in f(G(u,v))\subseteq G(fu,fv)$ and $f^2v\in f(G(v,u))\subseteq G(fv,fu)$. Thus (fu,fv) is a common coupled coincidence point of F,G and f. Continuing in this way, we can show that (f^nu,f^nv) is a common coupled coincidence point of F,G and f. It is also clear that $f^nu\in F(f^{n-1}u,f^{n-1}v)$, $f^nv\in F(f^{n-1}v,f^{n-1}u)$, $f^nu\in G(f^{n-1}u,f^{n-1}v)$ and $f^nv\in G(f^{n-1}v,f^{n-1}u)$. From (2.1.3), we have

$$\begin{aligned}&a_1d(fp,f^nu)+a_2d(fq,f^nv)+a_3d(fp,F(p,q))+a_4d(f^nu,G(f^{n-1}u,f^{n-1}v))\\ &+a_5d(fp,G(f^{n-1}u,f^{n-1}v))+a_6d(f^nu,F(p,q))+a_7\frac{d(fp,F(p,q))d(f^nu,G(f^{n-1}u,f^{n-1}v))}{1+d(fp,f^nu)+d(fq,f^nv)}\\ &+a_8\frac{d(fp,G(f^{n-1}u,f^{n-1}v))d(f^nu,F(p,q))}{1+d(fp,f^nu)+d(fq,f^nv)}+a_9\frac{d(fp,F(p,q))d(fp,G(f^{n-1}u,f^{n-1}v))}{1+d(fp,f^nu)+d(fq,f^nv)}\\ &+a_{10}\frac{d(f^nu,F(p,q))d(f^nu,G(f^{n-1}u,f^{n-1}v))}{1+d(fp,f^nu)+d(fq,f^nv)}\in s(F(p,q),G(f^{n-1}u,f^{n-1}v))\\ &\subseteq\bigcap_{y\in G(f^{n-1}u,f^{n-1}v)}s(F(p,q),y).\end{aligned}$$

Since $f^nu\in G(f^{n-1}u,f^{n-1}v)$ we have

$$\begin{aligned}&a_1d(fp,f^nu)+a_2d(fq,f^nv)+a_3d(fp,F(p,q))+a_4d(f^nu,G(f^{n-1}u,f^{n-1}v))\\ &+a_5d(fp,G(f^{n-1}u,f^{n-1}v))+a_6d(f^nu,F(p,q))+a_7\frac{d(fp,F(p,q))d(f^nu,G(f^{n-1}u,f^{n-1}v))}{1+d(fp,f^nu)+d(fq,f^nv)}\\ &+a_8\frac{d(fp,G(f^{n-1}u,f^{n-1}v))d(f^nu,F(p,q))}{1+d(fp,f^nu)+d(fq,f^nv)}+a_9\frac{d(fp,F(p,q))d(fp,G(f^{n-1}u,f^{n-1}v))}{1+d(fp,f^nu)+d(fq,f^nv)}\\ &+a_{10}\frac{d(f^nu,F(p,q))d(f^nu,G(f^{n-1}u,f^{n-1}v))}{1+d(fp,f^nu)+d(fq,f^nv)}\in s(F(p,q),f^nu)\\ &=\bigcup_{z\in F(p,q)}s(d(z,f^nu).\end{aligned}$$

From (2.1.1), there exists $z_n \in X$ with $fz_n \in F(p,q)$ such that

$$\begin{aligned}
&a_1 d(fp, f^n u) + a_2 d(fq, f^n v) + a_3 d(fp, F(p,q)) + a_4 d(f^n u, G(f^{n-1}u, f^{n-1}v)) \\
&+ a_5 d(fp, G(f^{n-1}u, f^{n-1}v)) + a_6 d(f^n u, F(p,q)) + a_7 \frac{d(fp, F(p,q)) d(f^n u, G(f^{n-1}u, f^{n-1}v))}{1 + d(fp, f^n u) + d(fq, f^n v)} \\
&+ a_8 \frac{d(fp, G(f^{n-1}u, f^{n-1}v)) d(f^n u, F(p,q))}{1 + d(fp, f^n u) + d(fq, f^n v)} + a_9 \frac{d(fp, F(p,q)) d(fp, G(f^{n-1}u, f^{n-1}v))}{1 + d(fp, f^n u) + d(fq, f^n v)} \\
&+ a_{10} \frac{d(f^n u, F(p,q)) d(f^n u, G(f^{n-1}u, f^{n-1}v))}{1 + d(fp, f^n u) + d(fq, f^n v)} \in s(d(fz_n, f^n u)).
\end{aligned}$$

Thus

$$\begin{aligned}
d(fz_n, f^n u) \precsim\ & a_1 d(fp, f^n u) + a_2 d(fq, f^n v) + a_3 d(fp, F(p,q)) \\
&+ a_4 d(f^n u, G(f^{n-1}u, f^{n-1}v)) + a_5 d(fp, G(f^{n-1}u, f^{n-1}v)) + a_6 d(f^n u, F(p,q)) \\
&+ a_7 \frac{d(fp, F(p,q)) d(f^n u, G(f^{n-1}u, f^{n-1}v))}{1 + d(fp, f^n u) + d(fq, f^n v)} + a_8 \frac{d(fp, G(f^{n-1}u, f^{n-1}v)) d(f^n u, F(p,q))}{1 + d(fp, f^n u) + d(fq, f^n v)} \\
&+ a_9 \frac{d(fp, F(p,q)) d(fp, G(f^{n-1}u, f^{n-1}v))}{1 + d(fp, f^n u) + d(fq, f^n v)} + a_{10} \frac{d(f^n u, F(p,q)) d(f^n u, G(f^{n-1}u, f^{n-1}v))}{1 + d(fp, f^n u) + d(fq, f^n v)}.
\end{aligned}$$

By using the g.l.b property of F and G, we have

$$\begin{aligned}
d(fz_n, f^n u) \precsim\ & a_1 d(fp, f^n u) + a_2 d(fq, f^n v) + a_3 d(fp, fz_n) + a_4 d(f^n u, f^n u) \\
&+ a_5 d(fp, f^n u) + a_6 d(f^n u, fz_n) + a_7 \frac{d(fp, fz_n) d(f^n u, f^n u)}{1 + d(fp, f^n u) + d(fq, f^n v)} \\
&+ a_8 \frac{d(fp, f^n u) d(f^n u, fz_n)}{1 + d(fp, f^n u) + d(fq, f^n v)} + a_9 \frac{d(fp, fz_n) d(fp, f^n u)}{1 + d(fp, f^n u) + d(fq, f^n v)} + a_{10} \frac{d(f^n u, fz_n) d(f^n u, f^n u)}{1 + d(fp, f^n u) + d(fq, f^n v)}.
\end{aligned}$$

Now

$$\begin{aligned}
|d(fp, fz_n)| &\le |d(fp, f^n u)| + |d(f^n u, fz_n)| \\
&\le |d(fp, f^n u)| + a_1 |d(fp, f^n u)| + a_2 |d(fq, f^n v)| + a_3 |d(fp, fz_n)| + a_4(0) \\
&+ a_5 |d(fp, f^n u)| + a_6 [|d(f^n u, fp)| + |d(fp, fz_n)|\ ] + a_7(0) \\
&+ a_8 \frac{|d(fp, f^n u)|\ |d(f^n u, fz_n)|}{|1 + d(fp, f^n u) + d(fq, f^n v)|} + a_9 \frac{|d(fp, fz_n)|\ |d(fp, f^n u)|}{|1 + d(fp, f^n u) + d(fq, f^n v)|} + a_{10}(0).
\end{aligned}$$

$$\begin{aligned}
(1 - a_3 - a_6)|d(fp, fz_n)| \le\ & |d(fp, f^n u)| + a_1 |d(fp, f^n u)| + a_2 |d(fq, f^n v)| \\
&+ a_5 |d(fp, f^n u)| + a_6 |d(f^n u, fp)| + a_7(0) \\
&+ a_8 \frac{|d(fp, f^n u)|\ |d(f^n u, fz_n)|}{|1 + d(fp, f^n u) + d(fq, f^n v)|} + a_9 \frac{|d(fp, fz_n)|\ |d(fp, f^n u)|}{|1 + d(fp, f^n u) + d(fq, f^n v)|} + a_{10}(0) \to 0.
\end{aligned}$$

as $n \to \infty$ from (2.8) and (2.9). Thus $fz_n \to fp$ as $n \to \infty$. Since $F(p,q)$ is closed and $fz_n \in F(p,q)$, we have

$$fp \in F(p,q). \tag{2.10}$$

Similarly, we will show that

$$fq \in F(q,p). \tag{2.11}$$

From (2.1.3), we have

$$\begin{aligned}
&a_1 d(f^n u, fp) + a_2 d(f^n v, fq) + a_3 d(fp, G(p,q)) + a_4 d(f^n u, F(f^{n-1}u, f^{n-1}v)) \\
&+ a_5 d(fp, F(f^{n-1}u, f^{n-1}v)) + a_6 d(f^n u, G(p,q)) + a_7 \frac{d(fp, G(p,q)) d(f^n u, F(f^{n-1}u, f^{n-1}v))}{1 + d(f^n u, fp) + d(f^n v, fq)} \\
&+ a_8 \frac{d(fp, F(f^{n-1}u, f^{n-1}v)) d(f^n u, G(p,q))}{1 + d(f^n u, fp) + d(f^n v, fq)} + a_9 \frac{d(f^n u, F(f^{n-1}u, f^{n-1}v)) d(f^n u, G(p,q))}{1 + d(f^n u, fp) + d(f^n v, fq)} \\
&+ a_{10} \frac{d(fp, F(f^{n-1}u, f^{n-1}v)) d(fp, G(p,q))}{1 + d(f^n u, fp) + d(f^n v, fq)} \in s(F(f^{n-1}u, f^{n-1}v), G(p,q)) \\
&\subseteq \bigcap_{x \in F(f^{n-1}u, f^{n-1}v)} s(x, G(p,q)).
\end{aligned}$$

Since $f^n u \in F(f^{n-1}u, f^{n-1}v)$, we have

$$\begin{aligned}
&a_1 d(f^n u, fp) + a_2 d(f^n v, fq) + a_3 d(fp, G(p,q)) + a_4 d(f^n u, F(f^{n-1}u, f^{n-1}v)) \\
&+ a_5 d(fp, F(f^{n-1}u, f^{n-1}v)) + a_6 d(f^n u, G(p,q)) + a_7 \frac{d(fp, G(p,q)) d(f^n u, F(f^{n-1}u, f^{n-1}v))}{1 + d(f^n u, fp) + d(f^n v, fq)} \\
&+ a_8 \frac{d(fp, F(f^{n-1}u, f^{n-1}v)) d(f^n u, G(p,q))}{1 + d(f^n u, fp) + d(f^n v, fq)} + a_9 \frac{d(f^n u, F(f^{n-1}u, f^{n-1}v)) d(f^n u, G(p,q))}{1 + d(f^n u, fp) + d(f^n v, fq)} \\
&+ a_{10} \frac{d(fp, F(f^{n-1}u, f^{n-1}v)) d(fp, G(p,q))}{1 + d(f^n u, fp) + d(f^n v, fq)} \in s(f^n u, G(p,q)) \\
&= \bigcup_{z \in G(p,q)} s(d(f^n u, z)).
\end{aligned}$$

From (2.1.1), there exists $w_n \in X$ with $fw_n \in G(p,q)$ such that

$$\begin{aligned}
&a_1 d(f^n u, fp) + a_2 d(f^n v, fq) + a_3 d(fp, G(p,q)) + a_4 d(f^n u, F(f^{n-1}u, f^{n-1}v)) \\
&+ a_5 d(fp, F(f^{n-1}u, f^{n-1}v)) + a_6 d(f^n u, G(p,q)) + a_7 \frac{d(fp, G(p,q)) d(f^n u, F(f^{n-1}u, f^{n-1}v))}{1 + d(f^n u, fp) + d(f^n v, fq)} \\
&+ a_8 \frac{d(fp, F(f^{n-1}u, f^{n-1}v)) d(f^n u, G(p,q))}{1 + d(f^n u, fp) + d(f^n v, fq)} + a_9 \frac{d(f^n u, F(f^{n-1}u, f^{n-1}v)) d(f^n u, G(p,q))}{1 + d(f^n u, fp) + d(f^n v, fq)} \\
&+ a_{10} \frac{d(fp, F(f^{n-1}u, f^{n-1}v)) d(fp, G(p,q))}{1 + d(f^n u, fp) + d(f^n v, fq)} \in s(d(f^n u, fw_n)).
\end{aligned}$$

Thus

$$\begin{aligned}
d(f^n u, fw_n) \precsim\ & a_1 d(f^n u, fp) + a_2 d(f^n v, fq) + a_3 d(fp, G(p,q)) + a_4 d(f^n u, F(f^{n-1}u, f^{n-1}v)) \\
&+ a_5 d(fp, F(f^{n-1}u, f^{n-1}v)) + a_6 d(f^n u, G(p,q)) + a_7 \frac{d(fp, G(p,q)) d(f^n u, F(f^{n-1}u, f^{n-1}v))}{1 + d(f^n u, fp) + d(f^n v, fq)} \\
&+ a_8 \frac{d(fp, F(f^{n-1}u, f^{n-1}v)) d(f^n u, G(p,q))}{1 + d(f^n u, fp) + d(f^n v, fq)} + a_9 \frac{d(f^n u, F(f^{n-1}u, f^{n-1}v)) d(f^n u, G(p,q))}{1 + d(f^n u, fp) + d(f^n v, fq)} \\
&+ a_{10} \frac{d(fp, F(f^{n-1}u, f^{n-1}v)) d(fp, G(p,q))}{1 + d(f^n u, fp) + d(f^n v, fq)}.
\end{aligned}$$

By using the g.l.b property of F and G, we get

$$\begin{aligned}
d(f^n u, fw_n) \precsim\ & a_1 d(f^n u, fp) + a_2 d(f^n v, fq) + a_3 d(fp, fw_n) + a_4 d(f^n u, f^n u) + a_5 d(fp, f^n u) \\
&+ a_6 d(f^n u, fw_n) + a_7 \frac{d(fp, fw_n) d(f^n u, f^n u)}{1 + d(f^n u, fp) + d(f^n v, fq)} + a_8 \frac{d(fp, f^n u) d(f^n u, fw_n)}{1 + d(f^n u, fp) + d(f^n v, fq)} \\
&+ a_9 \frac{d(f^n u, f^n u) d(f^n u, fw_n)}{1 + d(f^n u, fp) + d(f^n v, fq)} + a_{10} \frac{d(fp, f^n u) d(fp, fw_n)}{1 + d(f^n u, fp) + d(f^n v, fq)}
\end{aligned}$$

Now

$$\begin{aligned}|d(fp,fw_n)| &\le |d(fp,f^nu)| + |d(f^nu,fw_n)|\\ &\le |d(fp,f^nu)| + a_1|d(f^nu,fp)| + a_2|d(f^nv,fq)| + a_3|d(fp,fw_n)|\\ &+ a_4(0) + a_5|d(fp,f^nu)| + a_6[|d(f^nu,fp)| + |d(fp,fw_n)|\ ] + a_7(0)\\ &+ a_8\frac{|d(fp,f^nu)|\ |d(f^nu,fw_n)|}{|1+d(f^nu,fp)+d(f^nv,fq)|} + a_9(0) + a_{10}\frac{|d(fp,f^nu)|\ |d(fp,fw_n)|}{|1+d(f^nu,fp)+d(f^nv,fq)|}\end{aligned}$$

and

$$\begin{aligned}(1-a_3-a_6)|d(fp,fw_n)| &\le |d(fp,f^nu)| + a_1|d(f^nu,fp)| + a_2|d(f^nv,fq)|\\ &+ a_5|d(fp,f^nu)| + a_6|d(f^nu,fp)|\\ &+ a_7(0) + a_8\frac{|d(fp,f^nu)|\ |d(f^nu,fw_n)|}{|1+d(f^nu,fp)+d(f^nv,fq)|}\\ &+ a_9(0) + a_{10}\frac{|d(fp,f^nu)|\ |d(fp,fw_n)|}{|1+d(f^nu,fp)+d(f^nv,fq)|} \to 0,\end{aligned}$$

as $n \to \infty$ from (2.8) and (2.9). Thus $fw_n \to fp$ as $n \to \infty$. Since $G(p,q)$ is closed and $fw_n \in G(p,q)$, we have

$$fp \in G(p,q). \tag{2.12}$$

Similarly, we will show that

$$fq \in G(q,p). \tag{2.13}$$

From $(2.10), (2.11), (2.12)$ and (2.13), it follows that (p,q) is a common coupled fixed point of F, G and f. □

Now we give an example to support Theorem 2.1.

Example 2.2. Let $X = [0,1]$ be equipped with the metric $d : X \times X \to \mathbb{C}$ defined as $d(x,y) = max\{x,y\}e^{i\frac{\pi}{6}}$ and $d(x,x) = 0$, for all $x, y \in X$. Let $F, G : X \times X \to CB(X)$ and $f : X \to X$ be defined as $F(x,y) = [0, \frac{x^2+y^2}{4}], G(x,y) = [0, \frac{x+y}{8}]$ and $fx = x$, for all $x, y \in X$.
Case(i): Suppose $\frac{x^2+y^2}{4} \le \frac{u+v}{8}$, then

$$\begin{aligned}\frac{1}{4}[d(fx,fu)+d(fy,fv)] &\succeq \frac{1}{8}[d(fx,fu)+d(fy,fv)]\\ &= \frac{1}{8}e^{i\frac{\pi}{6}}[\max\{x,u\}+\max\{y,v\}]\\ &\succeq e^{i\frac{\pi}{6}}\frac{u+v}{8}\\ &= \max\{\frac{x^2+y^2}{4}, \frac{u+v}{8}\}e^{i\frac{\pi}{6}}\\ &= s([0, \frac{x^2+y^2}{4}], [0, \frac{x+y}{8}])\\ &= s(F(x,y), G(u,v)).\end{aligned}$$

Case(ii): Suppose $\frac{x^2+y^2}{4} > \frac{u+v}{8}$, then

$$\begin{aligned}\frac{1}{4}[d(fx,fu)+d(fy,fv)] &\succeq \frac{1}{4}e^{i\frac{\pi}{6}}[\max\{x,u\}+\max\{y,v\}]\\ &\succeq e^{i\frac{\pi}{6}}\frac{x+y}{4}\\ &\succeq e^{i\frac{\pi}{6}}\frac{x^2+y^2}{4}\\ &= \max\{\frac{x^2+y^2}{4},\frac{u+v}{8}\}e^{i\frac{\pi}{6}}\\ &= s([0,\frac{x^2+y^2}{4}],[0,\frac{x+y}{8}])\\ &= s(F(x,y),G(u,v)).\end{aligned}$$

Thus (2.1.3) is satisfied. One can easily prove the remaining conditions. Clearly $(0,0)$ is a common coupled fixed point of F,G and f.

Remark 2.3. We observed that in Example 2.2 of Rao et al. [23], the metric $d(x,y)=|x-y|e^{i\theta}$, for all $x,y\in X$, when $\theta=tan^{-1}(\frac{y}{x})$ is not a complex valued metric since $d(x,y)\neq d(y,x)$ for $x\neq y$ in general. This Example 2.2 of Rao et al. [23] is valid if we replace $\theta=tan^{-1}(\frac{y}{x})$ by $\theta=\frac{\pi}{3}$.

Remark 2.4. Set $F(x,y)=Sx, G(x,y)=Tx$ and $fx=x$ for all $x,y\in X$ in Theorem 2.1

(*i*) If $a_3=0,a_4=0,a_5=0,a_6=0,a_9=0$ and $a_{10}=0$, we get a result which generalizes Theorem 9 of [5].

(*ii*) If $a_1=0,a_2=0,a_5=0,a_6=0,a_8=0,a_9=0$ and $a_{10}=0$, we get a result which generalizes Theorem 15 of [5].

(*iii*) If $a_1=0,a_2=0,a_3=0,a_4=0,a_7=0,a_9=0$ and $a_{10}=0$, we get a result which generalizes Theorem 10 of [7].

(*iv*) If $a_3=0,a_4=0,a_5=0$ and $a_6=0$, we get a result which generalizes Theorem 9 of [15].

Remark 2.5. If we set $G(x,y)=F(x,y)$ for all $x,y\in X$ and $a_3=0,a_4=0,a_5=0,a_6=0,a_9=0$ and $a_{10}=0$ in Theorem 2.1, we get a result which generalizes Theorem 2.1 of [23].

Corollary 2.6. *Let* (X,d) *be a complete metric space and* $F,G: X\times X\to CB(X)$ *be multi valued mappings and* $f:X\to X$ *be a self mapping satisfying* (2.1.1), (2.1.2) *and* (2.5.1)

$$\begin{aligned}H(F(x,y),G(u,v)) \le{}& a_1d(fx,fu)+a_2d(fy,fv)+a_3d(fx,F(x,y))+a_4d(fu,G(u,v))\\ &+a_5d(fx,G(u,v))+a_6d(fu,F(x,y))+a_7\frac{d(fx,F(x,y))d(fu,G(u,v))}{1+d(fx,fu)+d(fy,fv)}\\ &+a_8\frac{d(fx,G(u,v))d(fu,F(x,y))}{1+d(fx,fu)+d(fy,fv)}+a_9\frac{d(fx,F(x,y))d(fx,G(u,v))}{1+d(fx,fu)+d(fy,fv)}\\ &+a_{10}\frac{d(fu,F(x,y))d(fu,G(u,v))}{1+d(fx,fu)+d(fy,fv)},\end{aligned}$$

for all $x,y,u,v\in X$, *where* $a_i(i=1,2,\cdots,10)$ *are non-negative real numbers such that* $a_1+a_2+a_3+a_4+a_5+a_6+a_7+2a_9<1$. *Then* F,G *and* f *have a common coupled coincidence point in* $X\times X$. *Furthermore, assume that* (2.1.4) *Then* F,G *and* f *have a common coupled fixed point in* $X\times X$.

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