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Existence of positive solutions to a coupled system with three-point boundary conditions via degree theory

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Abstract

In this paper we study the existence of solutions of nonlinear fractional hybrid differential equations. By using the topological degree theory, some results on the existence of solutions are obtained. The results are demonstrated by a proper example.

Keywords: Coupled system, fractional derivative, green function, growth condition, topological degree theory, fixed-point theorem.
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1. Introduction

The interest in the study of differential equations of fractional order lies in the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. Fractional differential equation can describe many phenomena in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, electromagnetic, physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, etc. There are large number of papers dealing with the solvability of nonlinear fractional differential equations. The papers [4, 5, 31, 32] considered boundary value problems for fractional differential equations. For more recent development on this hot topic, one can see the monographs of Baleanu et al. [6] , Diethelm [10], Kilbas *et al.* [14], Lakshmikantham *et al.* [15], Miller and Ross [17], Michalski [18], Podlubny [19] and Tarasov [24]. Fractional differential equations involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have been paid more attention. Existence of solutions to boundary value problems for coupled systems of fractional order differential equations involving Riemann-Liouville or Caputo derivative have attracted more attentions, we refer to [2, 8, 22, 23, 29, 33]. In these papers, classical fixed

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point theorems such as Banach contraction principle and Schauder fixed point theorem have been used to develop conditions for existence of solutions. Many results can be found in literature dealing with existence and uniqueness of solutions via different techniques of functional analysis; we refer the reader to [1, 3, 7, 11, 13, 16, 20, 26, 27, 30].
Degree theory for the solution of fractional order differential equations was first applied by Isaia [12]. In the said paper, the priori estimate method is used together with the degree for condensing maps to establish conditions for the existence of solution to the integral equations as given by

$$u(t) = \phi(t, u(t)) + \int_a^b \psi(t, s, u(s))ds, t \in [a, b].$$

Wang *et al* [25], studied the existence and uniqueness of solutions via topological degree method to a class of nonlocal Cauchy problems of the form

$$\begin{cases} \mathcal{D}^q u(t) = f(t, u(t)), \ t \in [0, T], \\ u(0) + g(u) = u_0, \end{cases}$$

where $\mathcal{D}^\alpha$ is the Caputo fractional derivative of order $q \in (0, 1]$, $u_0 \in \mathbb{R}$, and $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous.
Wang *et al* [28], obtained sufficient conditions for the existence and uniqueness of positive solutions to the following coupled system of nonlinear three-point boundary values problem

$$\begin{cases} \mathcal{D}^p u(t) = f(t, u(t)), \ \mathcal{D}^q u(t) = g(t, v(t)), \ 0 < t < 1, \\ u(0) = 0, \ v(0) = 0, \ u(1) = au(\eta), \ v(1) = bv(\eta). \end{cases}$$

Topological theory was also applied by Shah *et al.* [21], to obtain sufficient conditions for the existence and uniqueness of solutions to more general coupled systems of nonlinear multi-point boundary value problems.

$$\begin{cases} \mathcal{D}^\alpha x(t) = \phi(t, x(t), y(t)), \\ \mathcal{D}^\beta y(t) = \psi(t, x(t), y(t)), t \in [0, 1], \\ x(0) = 0, \ y(0) = 0, \\ x(1) = \delta x(\eta), \ y(1) = \gamma y(\eta), \end{cases}$$

where $1 < \alpha \leq 2$ is a real number, $\mathcal{D}^\alpha$ is the Caputo fractional differential operator of order α.
Motivated by the above results, we use topological degree theory approach and fixed point theorem to study sufficient conditions for existence and uniqueness of solutions to some non linear boundary value problem with boundary conditions of the form

$$\begin{cases} \mathcal{D}^\alpha (x(t) - f(t, x(t))) = g(t, y(t), I^\alpha y(t)), \\ \mathcal{D}^\alpha (y(t) - f(t, y(t))) = g(t, x(t), I^\alpha x(t)), \\ x(0) = 0, \ x(1) = h(x(\eta)), \\ x(0) = 0, \ y(1) = h(y(\eta)), \end{cases} \tag{1.1}$$

where $\mathcal{D}^\alpha$ denotes the Riemann-Liouville fractional differential operator of order , α, $1 < \alpha \leq 2$, where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ and $g : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

The rest of this paper is organized as follows. In Section 2, we give some notations, recall some concepts and important results needed in this paper. In Section 3, we introduce a concept of existence of at least solutions for the problem (4.1) by using a new fixed point theorem which is linking degree theory for condensing maps.

2. Preliminaries

Here, in this section we give some fundamental definitions and results from fractional calculus and topological degree theory. For detail see [9, 12, 14].

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $u : (0, \infty) \to \mathbb{R}$ is given by

$$\mathcal{I}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. The fractional derivative of order $\alpha > 0$, of a continuous function $u : (0, \infty) \to \mathbb{R}$ is given by

$$\mathcal{D}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, $n = [\alpha]$ denotes the integer part of α, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Let X be a Banach spaces throughout this paper and $\mathbf{B} \in P(X)$ be the family of all its bounded sets. Then, we recall the following results.

Definition 2.3. The Kuratowski measure of noncompactness $\alpha : \mathbf{B} \to \mathbb{R}^+$ is defined as

$$\alpha(B) = \inf\{d > 0, \text{ where } B \in \mathbf{B} \text{ admits a finite cover by sets of diameter at most } d\}.$$

Proposition 2.4. The Kuratowski measure α satisfies the following properties:

(i) $\alpha(B) = 0$ if and only if B is relatively compact;

(ii) α is a seminorm, that is, $\alpha(\lambda B) = |\lambda|\ \alpha(B), \lambda \in \mathbb{R}$ and $\alpha(B_1 + B_2) \le \alpha(B_1) + \alpha(B_2)$;

(iii) $B_1 \subset B_2$ implies $\alpha(B_1) \le \alpha(B_2)$; $\alpha(B_1 \cup B_2) = \max\{\alpha(B_1), \alpha(B_2)\}$.

(iv) $\alpha(convB) = \alpha(B)$;

(v) $\alpha(\overline{B}) = \alpha(B)$.

Definition 2.5. Consider $\Omega \subset X$ and $\mathcal{F} : \Omega \to X$ a continuous bounded map. We say that $\mathcal{F}$ is α-Lipschitz if there exists $k \ge 0$ such that $\alpha(\mathcal{F}(B)) \le \alpha(B)$ for all $B \subset \Omega$ bounded. If, in addition, $k < 1$, then we say that $\mathcal{F}$ is a strict α-contraction.
We say that $\mathcal{F}$ is α-condensing if $\alpha(\mathcal{F}(B)) < \alpha(B)$ for all $B \subset \Omega$ bounded with $\alpha(B) > 0$. In other words, $\alpha(\mathcal{F}(B)) = \alpha(B)$ implies $\alpha(B) = 0$. The class of all strict α-contractions $\mathcal{F} : \Omega \to X$ is denoted by $\Im C_{\alpha}(\Omega)$ and the class of all α-condensing maps $\mathcal{F} : \Omega \to X$ is denoted by $C_a(\Omega)$.
We remark that $\Im C_{\alpha}(\Omega) \subset C_{\alpha}(\Omega)$ and every $\mathcal{F} \in C_{\alpha}(\Omega)$ is α-Lipschitz with constant $k = 1$. We also recall that $\mathcal{F} : \Omega \to X$ is Lipschitz if there exists $k > 0$ such that $\|\mathcal{F}x - \mathcal{F}y\| \le k\|x - y\|$ for all $x, y \in \Omega$ and that $\mathcal{F}$ is a strict contraction if $k < 1$. Next, we collect some properties of the applications defined above.

Proposition 2.6. If $\mathcal{F}, G : \Omega \to X$ are α-Lipschitz maps with constants k,respectively k' then $\mathcal{F} + G : \Omega \to X$ are α-Lipschitz with constants $k + k'$.

Proposition 2.7. If $\mathcal{F} : \Omega \to X$ is compact, then $\mathcal{F}$ is α-Lipschitz with constant $k = 0$.

Proposition 2.8. If $\mathcal{F} : \Omega \to X$ is Lipschitz with constant k, then $\mathcal{F}$ is α-Lipschitz with the same constant k.

The theorem below asserts the existence and the basic properties of the topological degree for α-condensing perturbations of the identity.

Theorem 2.9. *Let*

$$\Theta = \begin{cases} (I-\mathcal{F},\Omega,y): \ \Omega \subset X \text{ open and bounded}, \\ \mathcal{F} \in C_\alpha(\overline{\Omega}), \ y \in X \ (I-\mathcal{F})(\partial\Omega), \end{cases}$$

be the family of the admissible triplets. There exists one degree function $D : \Theta \to Z$ which satisfies the properties:

(D_1) $D(I,\Omega,y) = 1$ *for every $y \in \Omega$ (Normalization);*

(D_2) *For every disjoint, open sets $\Omega_1, \Omega_2 \subset \Omega$ and every $y \notin (I-\mathcal{F})(\overline{\Omega} \setminus (\Omega_1 \bigcup \Omega_2))$*

$$D(I-\mathcal{F},\Omega,y) = D(I-\mathcal{F},\Omega_1,y) + D(I-\mathcal{F},\Omega_2,y);$$

(D_3) *D(I-H(t,),Ω,y(t)) is independent of t∈[0,1] for every continuous, bounded map $H : [0,1] \times \overline{\Omega} \to X$ which satisfies*

$$\alpha(H([0,1]\times B)) < \alpha(B) \ \forall \ B \subset \overline{\Omega} \text{ with } \alpha(B) > 0$$

and every continuous function $y : [0,1] \to X$ which satisfies

$$y(t) \neq x - H(t,x) \ \forall \ t \in [0,1], \ for \ all \ x \in \partial\Omega;$$

(D_4) $D(I-\mathcal{F},\Omega,y) \neq 0$ *implies $y \in (I-\mathcal{F})(\Omega)$.*

Now we state a fixed point theorem which will be used in the proofs of the main results.

Theorem 2.10. *Let $\mathcal{F} : X \to X$ be α-condensing and $S = \{x \in X \ : \text{there exists } \lambda \in [0,1] \text{ such that } x = \lambda\mathcal{F}x\}$. If S is a bounded set in X, so there exists $r > 0$ such that $S \subset B_r(0)$, then $\mathcal{F}$ has at least one fixed point and the set of the fixed points of $\mathcal{F}$ lies in $B_r(0)$.*
Let $X = C([0,1],\mathbb{R})$ the Banach space of all continuous functions from $C[0,1] \to \mathbb{R}$ with the topological norm $\|x\| = \max\{|x| : t \in [0,1]\}$. Then the product space $X \times X$ defined by $X \times X = \{(x,y) : x,y \in X\}$, is a Banach space under the norm $\|(x,y)\| = \|x\| + \|y\|$.

3. Main Results

In this section, we discuss the existence and uniqueness of solutions to the BVP(4.1).

Theorem 3.1. *The unique solutions of the BVP given by*

$$\begin{cases} \mathcal{D}^\alpha(x(t) - f(t,x(t))) = g(t,y(t),\mathcal{I}^\alpha y(t)), \, t \in J = [0,1], \\ x(0) = 0, \, x(1) = h(x(\eta)), \ 0 < \eta < 1, \end{cases}$$

is

$$x(t) = f(t,x) + \Big(h(x(\eta)) - f(1,0) - f(1,h(x(\eta))\Big)t + \int_0^1 G(t,s)g(s,y(s),\mathcal{I}^\alpha y(s))ds, t \in [0,1],$$

where $G(t,s)$ is defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t-s)^{\alpha-1} - t(1-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ -\, t(1-s)^{\alpha-1}, & 0 \le t \le s \le 1, \end{cases}$$

In view of Theorem 3.1, solution of the coupled system of BVP (4.1) is provided in the form of coupled systems of Fredwholm integral equations as:

$$\begin{cases} x(t) = f(t,x) + \Big(h(x(\eta)) - f(1,0) - f(1,h(x(\eta))\Big)t + \int_0^1 G(t,s)g(s,y(s),\mathcal{I}^\alpha y(s))ds, t \in [0,1], \\ y(t) = f(t,y) + \Big(h(y(\eta)) - f(1,0) - f(1,h(y(\eta))\Big)t + \int_0^1 G(t,s)g(s,x(s),\mathcal{I}^\alpha x(s))ds, t \in [0,1], \end{cases} \tag{3.1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions which satisfy the following conditions:

(i) $|f(t,x) - f(t,y)| < \lambda|x - y|$, where $\mu = (\lambda(1 + K) + K) < 1$, for every $(t,x),(t,y) \in [0,1] \times \mathbb{R}$;

(ii) $|f(t,x)| \le c_1|x(t)|^{q_1} + m_1$ where $c_1, m_1 > 0$, $q_1 \in [0,1)$, for every $(t,x) \in [0,1] \times \mathbb{R}$;

(iii) $|g(t,s,x)| \le c_2|x(t)|^{q_2} + m_2$ where $c_2, m_2 > 0$, $q_2 \in [0,1)$, for every $(t,s,x) \in [0,1] \times [0,1] \times \mathbb{R}$;

(iv) Let there exists $K > 0$, such that $|h(x) - h(\bar{x}| \le K|x - \bar{x}|$, for every $x, \bar{x} \in \mathbb{R}$;

(v) $|hx(t)| \le c_3|x(t)|^{q_3} + m_h$ where $c_3, m_h > 0$, $q_3 \in [0,1)$.

Define operators $\phi : C[0,1] \times C[0,1] \to C[0,1] \times C[0,1]$, by $\phi(x,y)(t) = (\mathcal{F}x(t), \mathcal{F}y(t))$, where $\mathcal{F}x(t) = f(t,x(t)) + \Big(h(x(\eta)) - f(1,0) - f(1,h(x(\eta))\Big)t$, for $t \in [0,1]$ and $\psi : C[0,1] \times C[0,1] \to C[0,1] \times C[0,1]$, by $\psi(x,y)(t) = (Hx(t), Hy(t))$, where $Hx(t) = \int_0^1 G(t,s)g(s,x(s),\mathcal{I}^\alpha x(s))ds$, for $t \in [0,1]$, $T : C[0,1] \times C[0,1] \to C[0,1] \times C[0,1]$, by $T(x,y)(t) = \phi(x,y)(t) + \psi(x,y)(t)$.
Thus the existence of a solution for system of equation (1) is equivalent to the existence of a fixed point for operator T.

Theorem 3.2. *The operator $\phi : C[0,1]\times C[0,1] \to C[0,1]\times C[0,1]$ is Lipschitz with constant μ. Consequently ϕ is α-Lipschitz with the same constant μ and also ϕ satisfies the following growth condition*

$$|\phi(x,y)| \le c_1\|(x,y)\|^{q_1} + c_3\|(x,y)\|^{q_3} + c^*\|(x,y)\|^{q_3} + m^*.$$

Proof. Consider

$$\begin{aligned}
\Big\|\mathcal{F}x(t) - \mathcal{F}\overline{x}(t)\Big\| &= \Big|f(t,x(t)) + \Big(h(x(\eta)) - f(1,0) - f(1,h(x(\eta))\Big)t \\
&- f(t,\overline{x}(t)) - \Big(h(\overline{x}(\eta)) - f(1,0) - f(1,h(\overline{x}(\eta))\Big)t\Big| \\
&\le \Big|f(t,x(t)) - f(t,\overline{x}(t))\Big| + \Big|h(x(\eta)) - h(\overline{x}(\eta))\Big| + \Big|f(1,h(x(\eta)) - f(1,h(\overline{x}(\eta))\Big| \\
&\le \lambda\Big|x - \overline{x}\Big| + K\Big|x(\eta) - \overline{x}(\eta)\Big| + \lambda\Big|h(x(\eta)) - h(\overline{x}(\eta))\Big| \\
&\le \lambda\|x - \overline{x}\| + K\|x - \overline{x}\| + \lambda K\|x - x\| \\
&= (\lambda + K + \lambda K)\|x - \bar{x}\| \\
&= (\lambda(1 + K) + K)\|x - \bar{x}\| \\
&= \mu\|x - \bar{x}\|.
\end{aligned}$$

Similarly

$$\Big\|\mathcal{F}y(t) - \mathcal{F}\overline{y}(t)\Big\| \le \mu\|y - \overline{y}\|, \ \textit{for every } y, \bar{y} \in C[0,1],$$

This implies

$$\begin{aligned}
\Big\|\phi(x,y)(t)-\phi(\overline{x},\overline{y})(t)\Big\| &= \Big\|(\mathcal{F}x(t),\mathcal{F}y(t))-(\mathcal{F}\overline{x}(t),\mathcal{F}\overline{y}(t))\Big\| \\
&= \Big\|\mathcal{F}x(t)-\mathcal{F}\overline{x}(t),\mathcal{F}y(t)-\mathcal{F}\overline{y}(t)\Big\| \\
&= \Big\|\mathcal{F}x(t)-\mathcal{F}\overline{x}(t)\Big\| + \Big\|\mathcal{F}y(t)-\mathcal{F}\overline{y}(t)\Big\| \\
&\le \mu\|x-\overline{x}\| + \mu\|y-\overline{y}\| \\
&= \mu\Big(\|x-\overline{x}\|+\|y-\overline{y}\|\Big) \\
&= \mu\Big(\|(x,y)-(\overline{x},\overline{y})\|\Big).
\end{aligned}$$

By proposition (2.4) ϕ is α-Lipschitz with constant μ. Now for the growth condition using (ii), we get

$$|\phi(x,y)(t)| = |(\mathcal{F}x(t),\mathcal{F}y(t))| = |\mathcal{F}x(t)|+|\mathcal{F}y(t)|.$$

$$\begin{aligned}
|(\mathcal{F}x)(t)| &= \Big|f(t,x(t))+\Big(h(x(\eta))-f(1,0)-f(1,h(x(\eta)))\Big)t\Big|, \quad t\in[0,1] \\
&\le \Big|f(t,x(t))|+|\Big(h(x(\eta))-f(1,0)-f(1,h(x(\eta)))\Big)t\Big| \\
&\le \Big|f(t,x(t))|+|(h(x(\eta))|+|f(1,0)+f(1,h(x(\eta))\Big| \\
&\le |f(t,x(t))|+\|(h(x(\eta))\| + c_1\|h(x(\eta))\|^{q_1}+2m_1 \\
&\le c_1\|x\|^{q_1}+m_1+c_3\|x\|^{q_3}+c_1(c_3\|x\|^{q_3})^{q_1}+2m_1 \\
&= c_1\|x\|^{q_1}+c_3\|x\|^{q_3}+c_1c_3^{q_1}\|x\|^{q_3q_1}+3m_1, \text{ using } q_1q_3<q_3 \\
&= c_1\|x\|^{q_1}+c_3\|x\|^{q_3}+c^*\|x\|^{q_3}+m^*.
\end{aligned}$$

Similarly

$$|(\mathcal{F}y)(t)| \le c_1\|y\|^{q_1}+c_3\|y\|^{q_3}+c^*\|y\|^{q_3}+m^*.$$

This implies

$$|\phi(x,y)| \le c_1\|(x,y)\|^{q_1}+c_3\|(x,y)\|^{q_3}+c^*\|(x,y)\|^{q_3}+m^*.$$

□

Proposition 3.3. The operator $\psi : C[0,1]\times C[0,1] \to C[0,1]\times C[0,1]$ is compact. Consequently ψ is α-Lipschitz with zero constant and under the hypothesis (h_3) satisfies the growth condition

$$\|\psi(x,y)\| \le \frac{1}{\Gamma(\alpha+1)}\Big(c_2\|(x,y)\|^{q_2}+m_2\Big).$$

Proof. Let (x_n,y_n) be a sequence in $X\times X$, $(x,y)\in X\times X$ such that $(x_n,y_n)\to(x,y)$.This implies that

$x_n \to x$ and $y_n \to y$. We need to prove that $\|\psi(x_n, y_n) - \psi(x, y)\| \to 0$, as $n \to \infty$. Consider

$$\begin{aligned}
\|\psi(x_n, y_n) - \psi(x, y)\| =& \|(Hx_n, Hy_n) - (Hx, Hy)\| \\
=& \|(Hx_n - Hx, Hy_n - Hy)\| \\
\|Hx_n - Hx\| =& \Big| \int_0^1 G(t,s) g(s, x_n(s), \mathcal{I}^{\alpha} x_n(s)) - \int_0^1 G(t,s) g(s, x(s), \mathcal{I}^{\alpha} x(s)) \Big| \\
=& \Big| \int_0^1 G(t,s) \Big(g(s, x_n(s), \mathcal{I}^{\alpha} x_n(s)) - g(s, x(s), \mathcal{I}^{\alpha} x(s)) \Big) ds \Big| \\
\leq& \int_0^1 \Big| G(t,s) \Big| \Big| g(s, x_n(s), \mathcal{I}^{\alpha} x_n(s)) - g(s, x(s), \mathcal{I}^{\alpha} x(s)) \Big| ds \\
\leq& \int_0^1 max_{C[0,1]} \Big| G(t,s) \Big| \Big| g(s, x_n(s), \mathcal{I}^{\alpha} x_n(s)) - g(s, x(s), \mathcal{I}^{\alpha} x(s)) \Big| ds \\
=& \int_0^1 \Big| \frac{(1-s)^{\alpha-1}}{\Gamma\alpha} \Big| \Big| g(s, x_n(s), \mathcal{I}^{\alpha} x_n(s)) - g(s, x(s), \mathcal{I}^{\alpha} x(s)) \Big| ds.
\end{aligned}$$

Using the continuity of g, we have

$$|g(s, x_n(s), \mathcal{I}^{\alpha} x_n(s)) - g(s, x(s), \mathcal{I}^{\alpha} x(s))| \to 0,$$

implies that $\|Hx_n - Hx\| \to 0$. Similarly $\|Hy_n - Hy\| \to 0$. This implies

$$\|\psi(x_n, y_n) - \psi(x, y)\| \to 0.$$

The continuity of ψ is proved. Moreover, ψ satisfies the following growth condition

$$\begin{aligned}
\|\psi(x, y)\| =& \|(Hx, Hy)\| = \|Hx\| + \|Hy\| \\
\|Hx\| \leq& \frac{1}{\Gamma(\alpha+1)} \Big(c_2 \|x\|^{q_2} + m_2 \Big).
\end{aligned}$$

Similarly, one has

$$\|Hy\| \leq \frac{1}{\Gamma(\alpha+1)} \Big(c_2 \|y\|^{q_2} + m_2 \Big).$$

From, which we have

$$\|\psi(x, y)\| \leq \frac{1}{\Gamma(\alpha+1)} \Big(c_2 \|(x, y)\|^{q_2} + m_2 \Big), \tag{3.2}$$

for every $(x, y) \in C[0,1] \times C[0.1]$.
In order to prove compactness of ψ, we consider a bounded set $B \in X \times X$ and a sequence (x_n, y_n) in B, then using (3), we have

$$\|\psi(x_n, y_n)\| \leq \frac{1}{\Gamma(\alpha+1)} \Big(c_2 \|(x_n, y_n)\|^{q_2} + m_2 \Big),$$

for every $(x, y) \in X \times X$, which implies that $\psi(B)$ is bounded.

Now, for equi-continuity, choose $0 \leq t_1 \leq t_2 \leq 1$. Then we have

$$\begin{aligned}
|\psi(x,y)(t_1) - \psi(x,y)(t_2)| =& |(Hx(t_1), Hy(t_1)) - (Hx(t_2), Hy(t_2))| \\
=& |Hx(t_1) - Hx(t_2), Hy(t_1) - Hy(t_2)| \\
|Hx(t_1) - Hx(t_2)| =& |\int_0^1 G(t_1,s)g(s,x(s),\mathcal{I}^\alpha x(s))ds - \int_0^1 G(t_2,s)g(s,x(s),\mathcal{I}^\alpha x(s))ds| \\
=& |\int_0^1 \Big(G(t_1,s) - G(t_2,s)\Big) g(s,x(s),\mathcal{I}^\alpha x(s))ds| \\
\leq& \int_0^1 |G(t_1,s) - G(t_2,s)||g(s,x(s),\mathcal{I}^\alpha x(s))|ds \\
\leq& \int_0^1 |G(t_1,s) - G(t_2,s)|(c_2\|x\|^{q_2} + m_2) \\
\leq& (c_2\|x\|^{q_2} + m_2)\Big(\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&- \frac{(t_2-t_1)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}ds\Big) \\
=& (c_2\|x\|^{q_2} + m_2)\Big(\frac{t_1^\alpha}{\Gamma(\alpha+1)} - \frac{t_2^\alpha}{\Gamma(\alpha+1)} - \frac{t_2-t_1}{\Gamma(\alpha+1)}\Big) \\
=& (c_2\|x\|^{q_2} + m_2)\Big(t_1^\alpha - t_2^\alpha - (t_2-t_1)\Big).
\end{aligned}$$

It follows $|Hx(t_1) - Hx(t_2)| \to 0$ and $|Hy(t_1) - Hy(t_2)| \to 0$ as $t_1 \to t_2$, which implies that $\psi(x,y)$ is equi-continues. For every $(x,y) \in B$, the set $\psi(B) \in X \times Y$ satisfies the hypothesis of Arzela-Ascoli theorem, $\psi(B)$ is relatively compact in $X \times Y$. Hence ψ is α-Lipschitz with constant 0. □

Theorem 3.4. *If the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfy the condition (i), (ii), (iii), then the system of equations*

$$\begin{cases}
x(t) = f(t,x) + \Big(h(x(\eta)) - f(1,0) - f(1,h(x(\eta)))\Big)t + \int_0^1 G(t,s)g(s)ds, t \in [0,1], \\
y(t) = f(t,y) + \Big(h(y(\eta)) - f(1,0) - f(1,h(y(\eta)))\Big)t + \int_0^1 G(t,s)g(s)ds, t \in [0,1],
\end{cases}$$

has at least one solution $(x,y) \in C[0,1] \times C[0,1]$ and the set of the solutions of equation (1) is bounded in C [0,1].

Proof. Let ϕ, ψ, T be the operators defined in the beginning of this section. They are continuous and bounded. Moreover, ϕ is α-Lipschitz with constant μ [0,1) and ψ is α -Lipschitz with zero constant (see Propositions (2.7) and (2.6)). Proposition (2.7) shows us that T is a strict α-contraction with constant μ. Set

$$S = \{(x,y) \in C[0,1] \times C[0,1] : \textit{for all } \lambda \in [0,1] \textit{ such that } (x,y) = \lambda T(x,y)\}.$$

Next, we prove that S is bounded in C[0,1]. Consider $(x,y) \in S$ and $\lambda \in [0,1]$ such that $(x,y) = \lambda T(x,y)$.

$$\begin{aligned}
\|(x,y)\| = \lambda\|T(x,y)\| &\leq \lambda\Big(\|\phi(x,y)\| + \|\psi(x,y)\|\Big) \\
&\leq \lambda\Big(c_1\|(x,y)\|^{q_1} + c_3\|(x,y)\|^{q_3} + c^*\|(x,y)\|^{q_3} + m^* + \frac{1}{\Gamma(\alpha+1)}(c_2\|(x,y)\|^{q_2} + m_2)\Big).
\end{aligned}$$

This inequality, together with $q_1 < 1$, $q_2 < 1$ and $q_3 < 1$ shows us that S is bounded in C[0,1].
Consequently, by Theorem 2.6 we deduce that T has at least one fixed point and the set of the fixed points of T is bounded in C [0,1]. □

Remark 3.5. The results of Theorem 3.4 also valid for using $q_1 = q_2 = q_3 = 1$.

4. Example

Example 4.1.

$$\begin{cases} \mathcal{D}^{1.5}\left[x(t) - \dfrac{\exp(-t)\sin(x(t))}{50+t^2}\right] = \dfrac{\sin(y(t)) + I^{1.5}\sin(y(t))}{40+\exp(\pi t)}, \ t \in [0,1], \\ \mathcal{D}^{1.5}\left[(y(t) - \dfrac{\exp(-t)\sin(y(t))}{50+t^2}\right] = \dfrac{\sin(x(t)) + I^{1.5}\sin(x(t))}{40+\exp(\pi t)}, \ t \in [0,1], \\ x(0) = 0, \ x(1) = \dfrac{1}{10}\sin(x(0.5)), \\ y(0) = 0, \ y(1) = \dfrac{1}{10}\sin(y(0.5)). \end{cases} \tag{4.1}$$

From the given system (4.1), we have $c_1 = \frac{1}{50}$, $c_2 = \frac{1}{41}$, $m_1 = m_2 = 0, q_1 = q_2 = q_3 = 1$, $K = \frac{1}{10}$, $c_3 = \frac{1}{10}$, $m_h = 0$, $m^* = 0$. Further $\lambda = \frac{1}{50}$. Then with the help of these values one can easily prove that the conditions of Theorem 3.4 are satisfied. Hence the given system of boundary value problem has at least one solution.

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