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Convergence of general composite iterative method for infinite family of nonexpansive mappings in Hilbert spaces

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Abstract

In this paper by using W_n-mapping, we introduce a composite iterative method for finding a common fixed point for infinite family of nonexpansive mappings and a solution of a certain variational inequality. Furthermore, the strong convergence of the proposed iterative method is established. Finally, some simulation examples are presented. Our results improve and extend the previous results. ©2016 All rights reserved.

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1. Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let C be a nonempty closed convex subset of H and T is a nonlinear mapping. We use $F(T)$ to denote the set of fixed points of T (i.e., $F(T) = \{x \in H : Tx = x\}$). Recall that a self mapping T of C is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in H$ and is a contraction, if there exists a constant $\alpha \in (0, 1)$ such that $\|Tx - Ty\| \leq \alpha\|x - y\|$ for all $x, y \in C$.
A bounded linear operator A on H is called *strongly positive* with coefficient $\bar{\gamma} > 0$ if,

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \forall x \in H.$$

In 2005, Kim and Xu [4] introduced the following iteration process:

$$x_0 = x \in C \text{ chosen arbitrary },$$

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$$y_n = \beta_n x_n + (1-\beta_n)Tx_n,$$
$$x_{n+1} = \alpha_n u + (1-\alpha_n)y_n. \tag{1.1}$$

They proved in a uniformly smooth Banach space, the sequence $\{x_n\}$ defined by (1.1) converges strongly to a fixed point of T. In 2009 Cho and Qin [2] considered the following composite iterative algorithm:

$$x_0 \in H \text{ chosen arbitrary},$$
$$z_n = \gamma_n x_n + (1-\gamma_n)Tx_n,$$
$$y_n = \beta_n x_n + (1-\beta_n)Tz_n,$$

$$x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)y_n, \ \forall\, n \geq 0. \tag{1.2}$$

In 2009 Wangkeeree and Kamraksa [8] introduced a new iterative scheme:

$$x_0 = x \in C \text{ chosen arbitrary},$$
$$z_n = \gamma_n x_n + (1-\gamma_n)W_n x_n,$$
$$y_n = \beta_n x_n + (1-\beta_n)W_n z_n,$$

$$x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)P_C(y_n - \lambda_n B y_n), \tag{1.3}$$

where the mapping W_n defined by Shimoji and Takahashi [6], as follows:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \gamma_n T_n U_{n,n+1} + (1-\gamma_n)I,\\
U_{n,n-1} &= \gamma_{n-1} T_{n-1} U_{n,n} + (1-\gamma_{n-1})I,\\
&\ \vdots\\
U_{n,k} &= \gamma_k T_k U_{n,k+1} + (1-\gamma_k)I,\\
U_{n,k-1} &= \gamma_{k-1} T_{k-1} U_{n,k} + (1-\gamma_{k-1})I,\\
&\ \vdots\\
U_{n,2} &= \gamma_2 T_2 U_{n,3} + (1-\gamma_2)I,\\
W_n = U_{n,1} &= \gamma_1 T_1 U_{n,2} + (1-\gamma_1)I,
\end{aligned} \tag{1.4}$$

where $\gamma_1, \gamma_2, ...$ are real numbers such that $0 \leq \gamma_n \leq 1, T_1, T_2, ...$ are an infinite family of mappings of H into itself, note that the nonexpansivity of each T_i ensures the nonexpansivity of W_n. In 2010 Singthong and Suantai [7] introduced an iterative method as follows:

$$\begin{aligned}
x_0 &= x \in C \text{ chosen arbitrary},\\
y_n &= \beta_n x_n + (1-\beta_n)K_n x_n,\\
x_{n+1} &= P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n),
\end{aligned} \tag{1.5}$$

where K-mapping defined by Kangtunyakarn and Suantai [3] as follows:

$$\begin{aligned}
U_{n,1} &= \lambda_{n,1} T_1 + (1-\lambda_{n,1})I,\\
U_{n,2} &= \lambda_{n,2} T_2 U_{n,1} + (1-\lambda_{n,2})U_{n,1},\\
U_{n,3} &= \lambda_{n,3} T_3 U_{n,2} + (1-\lambda_{n,3})U_{n,2},
\end{aligned}$$

$$\vdots$$
$$U_{n,N-1} = \lambda_{n,N-1}T_{N-1}U_{n,N-1} + (1-\lambda_{n,N})U_{n,N-1},$$
$$K_n = U_{n,N} = \lambda_{n,N}T_N U_{n,N-1} + (1-\lambda_{n,N})U_{n,N-1}\ , \tag{1.6}$$

where $\{T_i\}_{i=1}^N$ are finite family of nonexpansive mappings and the sequences $\{\lambda_{n,i}\}_i^N$ are in $[0,1]$. The mapping K_n is called the K-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$.

Through out this paper inspired by Singthong and Suantai [7] and Wangkeeree and Kamraksa [8], we introduce a composite iteration method for infinite family of nonexpansive mappings as follows:

$$x_0 = x \in C \ \textit{chosen arbitrary},$$
$$z_n = \gamma_n x_n + (1-\gamma_n)W_n x_n,$$
$$y_n = \beta_n x_n + (1-\beta_n)W_n z_n,$$
$$x_{n+1} = P_C[\alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)y_n], \tag{1.7}$$

where W_n is defined by (1.4), f is a contraction on H, A is a strongly positive linear bounded self-adjoint operator with the coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Then by using this iteration we prove the existence of a common fixed point for infinite family of nonexpansive mappings and the solution of a certain variational inequality. We need the following lemmas for the proof of our main results.

Lemma 1.1. *The following inequality holds in a Hilbert space H,*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \forall x, y \in H.$$

Lemma 1.2 ([1]). *Assume $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that $\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n \ \ n \ge 1$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and δ_n is a sequence in $\mathbb{R}$ such that:*

1. $\sum_{n=1}^{\infty} \gamma_n = \infty$,
2. $\limsup_{n\to\infty}(\frac{\delta_n}{\gamma_n}) \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$,

then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.3 ([5]). *Assume that A is a strongly positive linear bounded self-adjoint operator on a Hilbert space H with coefficient $\bar{\gamma}$ and $0 < \rho \le \|A\|^{-1}$, then $\|I - \rho A\| \le 1 - \rho\bar{\gamma}$.*

Lemma 1.4 ([6]). *Let C be nonempty closed convex subset of a Hilbert space, let $T_i : C \longrightarrow C$ be an infinite family of nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let γ_i be a real sequence such that $0 < \gamma_i \le \gamma < 1$ for all $i \ge 1$ then,*

1. *W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{n} F(T_i)$ for each $n \ge 1$.*
2. *For each $x \in C$ and for each positive integer k, the $\lim_{n\to\infty} U_{n,k}$ exists.*
3. *The mapping $W : C \longrightarrow C$ defined by,*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x \quad x \in C,$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$ and is called the W-mapping generated by $T_1, T_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$.

Lemma 1.5 ([6]). *Let C be a nonempty closed convex subset of a Hilbert space H, let $T_i : C \longrightarrow C$ be an infinite family of nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and let γ_i be a real sequence such that $0 < \gamma_i \le \gamma < 1$ for all $i \ge 1$, if K is any bounded subset of C then,*

$$\limsup_{n\to\infty} \|Wx - W_n x\| = 0 \quad x \in K.$$

Lemma 1.6 ([5]). *Let H be a Hilbert space, let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, let T be a nonexpansive mapping with a fixed point x_t of the contraction,*

$$x \longmapsto t\gamma f(x) + (I - tA)Tx.$$

Then x_t converges strongly as $t \to 0$ to a fixed point $\bar{x}$ of T which solves the variational inequality $\langle (A - \gamma f)\bar{x}, \bar{x} - z\rangle \leq 0$ $\forall z \in F(T)$.

Lemma 1.7 ([3]). *Let C be a nonempty closed convex subset of strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\lambda_1, \ldots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, \ldots, N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping of C into itself generated by $T_1, \ldots, T_N$ and $\lambda_1, \ldots, \lambda_N$. Then,*

$$F(K) = \bigcap_{i=1}^{N} F(T_i). \tag{1.8}$$

Lemma 1.8 ([7]). *Let C be a nonempty closed convex subset of a Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\{\lambda_{n,i}\}_{i=1}^N$ sequences in $[0,1]$ such that $\lambda_{n,i} \to \lambda_i$, as $n \to \infty$, $(i = 1, 2, \ldots, N)$. Moreover, for every $n \in \mathbb{N}$, K and K_n be the $K-$mapping generated by $T_1, \ldots, T_N$ and $\lambda_1, \ldots, \lambda_N$ and $T_1, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$, respectively. Then, for every bounded sequence $x_n \in C$, we have $\lim_{n\to\infty} \|K_n x_n - K x_n\| = 0$.*

2. Main Results

In this section, we prove strong convergence of the sequences $\{x_n\}$ defined by the iteration scheme (1.7), for finding a common fixed point of infinite family of nonexpansive mappings which solves the variational inequality.

Theorem 2.1. *Let C be a closed convex subset of a real Hilbert space H. Let f be a contraction of C into itself, let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ and $\{T_i : C \longrightarrow C\}$ be an infinite family of nonexpansive mappings. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $F = \bigcap_{i=1}^\infty F(T_i) \neq \emptyset$. Let $x_0 \in C$, given that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $[0,1]$ satisfying the following conditions:*

(C_1) $\lim_{n\to\infty} \alpha_n = 0$ $\sum_{n=1}^\infty \alpha_n = \infty$,

(C_2) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$,

(C_3) $\sum_{n=1}^\infty |\gamma_n - \gamma_{n-1}| < \infty$,

(C_4) $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$,

(C_5) $\sum_{n=1}^\infty |\beta_n - \beta_{n-1}| < \infty$,

(C_6) $(1 + \beta_n)\gamma_n - 2\beta_n > d$ *for some* $d \in (0,1)$,

then the sequence $\{x_n\}$ defined by (1.7) converges strongly to $q \in F$ which solves the variational inequality $\langle \gamma f(q) - Aq, p - q\rangle \leq 0$, $\forall p \in F$.

Proof. Since $\alpha_n \to 0$ as $n \to \infty$ without loss of generality we have $\alpha_n < (1 - \delta_n)\|A\|^{-1}$ $\forall n \geq 0$, noticing that A is a bounded linear self-adjoint operator with,

$$\|A\| = \sup\{| < Ax, x > | \ : x \in H, \|x\| = 1\},$$

we have,

$$\begin{aligned} < ((1-\delta_n)I - \alpha_n A)x, x > &= (1-\delta_n) < x, x > -\alpha_n < Ax, x > \\ &\geq (1-\delta_n) - \alpha_n \|A\| \geq 0, \end{aligned}$$

then $(1-\delta_n)I - \alpha_n A$ is positive. Also,

$$\begin{aligned} \|(1-\delta_n)I - \alpha_n A\| &= \sup\{| < ((1-\delta_n)I - \alpha_n A))x, x > |, x \in H, \|x\| = 1\} \\ &= \sup\{1 - \delta_n - \alpha_n < Ax, x >, x \in H, \|x\| = 1\} \\ &\leq 1 - \delta_n - \alpha_n \bar{\gamma}. \end{aligned} \tag{2.1}$$

Next we prove that $\{x_n\}$ is bounded. We pick $p \in F = \bigcap_{i=1}^{\infty} F(T_i) = F(W) = F(W_n)$,

$$\begin{aligned} \|z_n - p\| &= \|\gamma_n x_n + (1-\gamma_n) W_n x_n - p\| \\ &= \|\gamma_n (x_n - p) + (1-\gamma_n)(W_n x_n - W_n p)\| \\ &\leq \gamma_n \|(x_n - p)\| + (1-\gamma_n)\|(x_n - p)\| \\ &= \|x_n - p\|, \end{aligned}$$

and we have,

$$\begin{aligned} \|y_n - p\| &= \|\beta_n x_n + (1-\beta_n) W_n z_n - p\| \\ &= \|\beta_n (x_n - p) + (1-\beta_n)(W_n z_n - W_n p)\| \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)\|z_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned}$$

It follows that,

$$\begin{aligned} \|x_{n+1} - p\| &= \|P_C[\alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha A) y_n] - P_C(p)\| \\ &\leq \|\alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A) y_n - p\| \\ &= \|\alpha_n (\gamma f(x_n - Ap) + \delta_n (x_n - p) + ((1-\delta_n)I - \alpha_n A)(y_n - p)\|, \end{aligned}$$

by (2.1) we have,

$$\begin{aligned} &\leq \alpha_n \|\gamma f(x_n) - Ap\| + \delta_n \|x_n - p\| + (1 - \delta_n - \alpha_n \bar{\gamma})\|y_n - p\| \\ &\leq \alpha_n \gamma \|f(x_n) - f(p)\| + \alpha_n \|\gamma f(p) - Ap\| + \delta_n \|x_n - p\| \\ &\quad + (1 - \delta_n - \alpha_n \bar{\gamma})\|x_n - p\| \\ &\leq \alpha_n \gamma \alpha \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\| + (1 - \alpha_n \bar{\gamma})\|x_n - p\| \\ &= [1 - \alpha_n (\bar{\gamma} - \gamma \alpha)]\|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\|. \end{aligned}$$

By simple induction we have $\|x_n - p\| \leq max\{\|x_0 - p\|, \frac{\|Ap - \gamma f(p)\|}{\bar{\gamma} - \gamma \alpha}\}$, which gives that the sequence $\{x_n\}$ is bounded so are $\{y_n\}$ and $\{z_n\}$. Next we claim that, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. We know that,

$$\begin{aligned} z_n &= \gamma_n x_n + (1-\gamma_n) W_n x_n \ , \\ z_{n-1} &= \gamma_{n-1} x_{n-1} + (1-\gamma_{n-1}) W_{n-1} x_{n-1}. \end{aligned}$$

So we obtain,

$$\begin{aligned} z_n - z_{n-1} = &(1-\gamma_n)(W_n x_n - W_{n-1} x_{n-1}) + \gamma_n (x_n - x_{n-1}) \\ &+ (\gamma_n - \gamma_{n-1})(x_{n-1} - W_{n-1} x_{n-1}). \end{aligned}$$

This implies that,

$$\begin{aligned}\|z_n - z_{n-1}\| \leq &(1-\gamma_n)\|W_n x_n - W_{n-1}x_{n-1}\| + \gamma_n\|x_n - x_{n-1}\| \\ &+ |\gamma_n - \gamma_{n-1}|\|x_{n-1} - W_{n-1}x_{n-1}\| \\ =&(1-\gamma_n)\|W_n x_n - W_n x_{n-1} + W_n x_{n-1} - W_{n-1}x_{n-1}\| \\ &+ \gamma_n\|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|x_{n-1} - W_{n-1}x_{n-1}\| \\ \leq&(1-\gamma_n)\|W_n x_n - W_n x_{n-1}\| \\ &+ (1-\gamma_n)\|W_n x_{n-1} - W_{n-1}x_{n-1}\| \\ &+ \gamma_n\|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|x_{n-1} - W_{n-1}x_{n-1}\|.\end{aligned}$$

On the other hand we have,

$$\begin{aligned}\|W_n x_{n-1} - W_{n-1}x_{n-1}\| =&\|\gamma_1 T_1 U_{n,2}x_{n-1} - \gamma_1 T_1 U_{n-1,2}x_{n-1}\| \\ \leq&\gamma_1\|U_{n,2}x_{n-1} - U_{n-1,2}x_{n-1}\| \\ =&\gamma_1\|\gamma_2 T_2 U_{n,3}x_{n-1} - \gamma_2 T_2 U_{n-1,3}x_{n-1}\| \\ \leq&\gamma_1\gamma_2\|U_{n,3}x_{n-1} - U_{n-1,3}x_{n-1}\| \\ &\vdots \\ \leq&\gamma_1\gamma_2\ldots\gamma_{n-1}\|U_{n,n}x_{n-1} - U_{n-1,n}x_{n-1}\| \\ \leq&M_1\prod_{i=1}^{n-1}\gamma_i,\end{aligned} \tag{2.2}$$

where $M_1 \geq 0$ is an appropriate constant such that,

$$\|U_{n,n}x_{n-1} - U_{n-1,n}x_{n-1}\| \leq M_1 \quad \forall\ n \geq 0.$$

Note that the boundedness of x_n and the nonexpansivity of T_n ensure the existence of M_1. So we have,

$$\begin{aligned}\|z_n - z_{n-1}\| \leq&\gamma_n\|x_n - x_{n-1}\| + (1-\gamma_n)M_1\prod_{i=1}^{n-1}\gamma_i \\ &+ (1-\gamma_n)\|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|x_{n-1} - W_{n-1}x_{n-1}\| \\ =&\|x_n - x_{n-1}\| \\ &+ (1-\gamma_n)M_1\prod_{i=1}^{n-1}\gamma_i + |\gamma_n - \gamma_{n-1}|\|x_{n-1} - W_{n-1}x_{n-1}\|.\end{aligned}$$

Similar to (2.2), we have,

$$\|U_{n,n}z_{n-1} - U_{n-1,n}z_{n-1}\| \leq M_2.$$

So,

$$\begin{aligned}\|y_n - y_{n-1}\| =&\|\beta_n x_n + (1-\beta_n)W_n z_n - \beta_{n-1}x_{n-1} - (1-\beta_{n-1})W_{n-1}z_{n-1}\| \\ =&\|\beta_n x_n - \beta_n x_{n-1} + \beta_n x_{n-1} - \beta_{n-1}x_{n-1} \\ &+ (1-\beta_n)(W_n z_n - W_n z_{n-1}) \\ &+ (1-\beta_n)(W_n z_{n-1} - W_{n-1}z_{n-1}) \\ &+ (1-\beta_n)W_{n-1}z_{n-1} - (1-\beta_{n-1})W_{n-1}z_{n-1}\| \\ \leq&\|\beta_n(x_n - x_{n-1}) + (\beta_n - \beta_{n-1})x_{n-1} \\ &+ (1-\beta_n)(W_n z_n - W_n z_{n-1})\end{aligned}$$

$$
\begin{aligned}
&+ (1-\beta_n)(W_n z_{n-1} - W_{n-1}z_{n-1}) \\
&+ (1-\beta_n)W_{n-1}z_{n-1} - (1-\beta_{n-1})W_{n-1}z_{n-1}\| \\
\leq& \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\|z_n - z_{n-1}\| \\
&+ (1-\beta_n)M_2\prod_{i=1}^{n-1}\gamma_i \\
&+ |\beta_n - \beta_{n-1}|\|x_{n-1} - W_{n-1}z_{n-1}\| \\
\leq& \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\|x_n - x_{n-1}\| \\
&+ (1-\beta_n)|\gamma_n - \gamma_{n-1}|\|x_{n-1} - W_{n-1}x_{n-1}\| \\
&+ (1-\beta_n)(1-\gamma_n)M_1\prod_{i=1}^{n-1}\gamma_i + (1-\beta_n)M_2\prod_{i=1}^{n-1}\gamma_i \\
&+ |\beta_n - \beta_{n-1}|\|x_{n-1} - W_{n-1}z_{n-1}\| \\
=& \|x_n - x_{n-1}\| + (1-\beta_n)|\gamma_n - \gamma_{n-1}|\|x_{n-1} - W_{n-1}x_{n-1}\| \\
&+ (1-\beta_n)(1-\gamma_n)M_1\prod_{i=1}^{n-1}\gamma_i + (1-\beta_n)M_2\prod_{i=1}^{n-1}\gamma_i \\
&+ |\beta_n - \beta_{n-1}|\|x_{n-1} - W_{n-1}z_{n-1}\|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|x_{n+1} - x_n\| =& \|P_C[\alpha_n\gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)y_n] \\
&- P_C[\alpha_{n-1}\gamma f(x_{n-1}) \\
&+ \delta_{n-1}x_{n-1} + ((1-\delta_{n-1})I - \alpha_{n-1}A)y_{n-1}]\| \\
\leq& \|\alpha_n\gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)y_n \\
&- \alpha_{n-1}\gamma f(x_{n-1}) - \delta_{n-1}x_{n-1} - ((1-\delta_{n-1})I - \alpha_{n-1}A)y_{n-1}\| \\
\leq& \|((1-\delta_n)I - \alpha_n A)(y_n - y_{n-1}) \\
&- ((\delta_n - \delta_{n-1})y_{n-1} + (\alpha_{n-1} - \alpha_n)Ay_{n-1}) \\
&+ \gamma\alpha_n(f(x_n) - f(x_{n-1})) + \gamma(\alpha_n - \alpha_{n-1})f(x_{n-1}) \\
&+ \delta_n x_n - \delta_n x_{n-1} + \delta_n x_{n-1} - \delta_{n-1}x_{n-1}\| \\
\leq& (1-\delta_n - \alpha_n\bar{\gamma})\|y_n - y_{n-1}\| + |\delta_n - \delta_{n-1}|\|y_{n-1}\| \\
&+ |\alpha_n - \alpha_{n-1}|\|Ay_{n-1}\| + \gamma\alpha_n\alpha\|x_n - x_{n-1}\| \\
&+ \gamma|\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + \delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| \\
\leq& (1-\delta_n - \alpha_n\bar{\gamma})[\|x_n - x_{n-1}\| + (1-\beta_n)|\gamma_n - \gamma_{n-1}|\|x_{n-1} - W_{n-1}x_{n-1}\| \\
&+ (1-\beta_n)(1-\gamma_n)M_1\prod_{i=1}^{n-1}\gamma_i + (1-\beta_n)M_2\prod_{i=1}^{n-1}\gamma_i \\
&+ |\beta_n - \beta_{n-1}|\|x_{n-1} - W_{n-1}z_{n-1}\|] + |\delta_n - \delta_{n-1}|\|y_{n-1}\| \\
&+ |\alpha_n - \alpha_{n-1}|\|Ay_{n-1}\| \\
&+ \gamma\alpha_n\alpha\|x_n - x_{n-1}\| + \gamma|\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + \delta_n\|x_n - x_{n-1}\| \\
&+ |\delta_n - \delta_{n-1}|\|x_{n-1}\| \\
=& (1-\alpha_n\bar{\gamma})\|x_n - x_{n-1}\| \\
&+ (1-\delta_n - \alpha_n\bar{\gamma})[(1-\beta_n)|\gamma_n - \gamma_{n-1}|\|x_{n-1} - W_{n-1}x_{n-1}\| \\
&+ (1-\beta_n)(1-\gamma_n)M_1\prod_{i=1}^{n-1}\gamma_i + (1-\beta_n)M_2\prod_{i=1}^{n-1}\gamma_i
\end{aligned}
$$

$$\begin{aligned}
&+ |\beta_n - \beta_{n-1}|\|x_{n-1} - W_{n-1}z_{n-1}\|] + |\alpha_n - \alpha_{n-1}|\|Ay_{n-1}\| \\
&+ \gamma\alpha_n\alpha\|x_n - x_{n-1}\| \\
&+ \gamma|\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| + |\delta_n - \delta_{n-1}|\|y_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| \\
\leq&(1 - \alpha_n(\bar{\gamma} - \gamma\alpha))\|x_n - x_{n-1}\| \\
&+ (1 - \delta_n - \alpha_n\bar{\gamma})[(1 - \beta_n)|\gamma_n - \gamma_{n-1}| \sup\{\|x_{n-1} + \|W_{n-1}x_{n-1}\|\} \\
&+ (1 - \beta_n)\left((1 - \gamma_n)M_1 \prod_{i=1}^{n-1} \gamma_i + M_2 \prod_{i=1}^{n-1} \gamma_i\right)] \\
&+ |\alpha_n - \alpha_{n-1}| \sup\{\|Ay_{n-1}\| + \gamma f(x_{n-1})\} + |\delta_n - \delta_{n-1}| \sup\{\|y_{n-1}\| \\
&+ \|x_{n-1}\|\} + |\beta_n - \beta_{n-1}| \sup\{\|x_{n-1}\| + \|W_{n-1}z_{n-1}\|\}.
\end{aligned}$$

Now by Lemma 1.2 and C_3, C_4, C_5 we have $\|x_n - x_{n-1}\| \to 0$. On the other hand,

$$\begin{aligned}
\|x_{n+1} - y_n\| =&\|P_C[\alpha_n\gamma f(x_n) + \delta_n x_n + ((1 - \delta_n)I - \alpha_n A)y_n] - P_C(y_n)\| \\
\leq&\|\alpha_n\gamma f(x_n) + \delta_n x_n + ((1 - \delta_n)I - \alpha_n A)y_n - y_n\| \\
=&\|\alpha_n\gamma f(x_n) + \delta_n x_n - \delta_n x_{n+1} + \delta_n x_{n+1} \\
&+ y_n - \delta_n y_n - y_n - \alpha_n Ay_n\| \\
=&\|\alpha_n\gamma f(x_n) + \delta_n(x_n - x_{n+1}) + \delta_n(x_{n+1} - y_n) - \alpha_n Ay_n\| \\
\leq&\alpha_n\|\gamma f(x_n) - Ay_n\| + \delta_n\|x_n - x_{n+1}\| + \delta_n\|x_{n+1} - y_n\|.
\end{aligned}$$

So, $\|x_{n+1} - y_n\| \leq \frac{\alpha_n}{(1-\delta_n)}\|\gamma f(x_n) - Ay_n\| + \frac{\delta_n}{(1-\delta_n)}\|x_n - x_{n+1}\|$, which implies, $\|x_{n+1} - y_n\| \to 0$. Also we have $\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|$, which implies $\|x_n - y_n\| \to 0$. Notice that,

$$\|z_n - x_n\| = \|\gamma_n x_n + (1 - \gamma_n W_n x_n - x_n\| = \|(\gamma_n - 1)x_n + (1 - \gamma_n)W_n x_n\|$$

and

$$\|y_n - W_n z_n\| = \|\beta_n x_n + (1 - \beta_n)W_n z_n - W_n z_n\| = \beta_n\|x_n - W_n z_n\|.$$

By two above equalities we have,

$$\begin{aligned}
\|W_n x_n - x_n\| \leq&\|x_n - y_n\| + \|y_n - W_n x_n\| \\
\leq&\|x_n - y_n\| + \|y_n - W_n z_n\| + \|W_n z_n - W_n x_n\| \\
\leq&\|x_n - y_n\| + \beta_n\|x_n - W_n x_n\| + \beta_n\|W_n x_n - W_n z_n\| \\
&+ \|z_n - x_n\| \\
\leq&\|x_n - y_n\| + \beta_n\|x_n - W_n x_n\| + (1 + \beta_n)\|z_n - x_n\| \\
\leq&\|x_n - y_n\| + \beta_n\|x_n - W_n x_n\| \\
&+ (1 - \gamma_n)(1 + \beta_n)\|W_n x_n - x_n\|.
\end{aligned}$$

Therefore,

$$[(1 + \beta_n)\gamma_n - 2\beta_n]\|W_n x_n - x_n\| \leq \|x_n - y_n\| \to 0,$$

so $\lim_{n\to\infty} \|W_n x_n - x_n\| = 0$.
Furthermore we have,

$$\|Wx_n - x_n\| \leq \|Wx_n - W_n x_n\| + \|W_n x_n - x_n\|,$$

hence $\lim_{n\to\infty} \|Wx_n - x_n\| = 0$.
We show that $\limsup_{n\to\infty}\langle\gamma f(q) - Aq, x_n - q\rangle \leq 0$, where $q = \lim_{t\to 0} x_t$ and x_t is the fixed point of the

contraction $x \mapsto t\gamma f(x) + (I - tA)Wx$. We have, $\|x_t - x_n\| = \|(I - tA)(Wx_t - x_n) + t(\gamma f(x_t) - Ax_n)\|$ and by Lemma 1.1,

$$\begin{aligned} \|x_t - x_n\|^2 =& \|(I - tA)(Wx_t - x_n) + t(\gamma f(x_t) - Ax_n)\|^2 \\ \leq& (1 - t\bar{\gamma})^2 \|Wx_t - x_n\|^2 + 2t\langle \gamma f(x_t) - Ax_n, x_t - x_n\rangle \\ \leq& (1 - 2\bar{\gamma}t + (\bar{\gamma}t)^2)\|x_t - x_n\|^2 + f_n(t) + 2t\langle \gamma f(x_t) - Ax_t, x_t - x_n\rangle \\ &+ 2t\langle Ax_t - Ax_n, x_t - x_n\rangle \end{aligned} \tag{2.3}$$

where $f_n(t) = (2\|x_t - x_n\| + \|x_n - Wx_n\|)\, \|x_n - Wx_n\| \to 0\, (as\ n \to \infty)$. Since A is strongly positive linear mapping, so we have,

$$\langle Ax_t - Ax_n, x_t - x_n\rangle = \langle A(x_t - x_n), x_t - x_n\rangle \geq \bar{\gamma}\|x_t - x_n\|^2.$$

From (2.3) we have,

$$\begin{aligned} 2t\langle Ax_t - \gamma f(x_t), x_t - x_n\rangle \leq& (\bar{\gamma}^2 t^2 - 2\bar{\gamma}t)\|x_t - x_n\|^2 + f_n(t) \\ &+ 2t\langle Ax_t - Ax_n, x_t - x_n\rangle \\ \leq& \left(\bar{\gamma}t^2\right) \langle A(x_t - x_n), x_t - x_n\rangle + f_n(t) \\ &+ 2t\langle A(x_t - x_n), x_t - x_n\rangle \\ =& \bar{\gamma}t^2 \langle A(x_t - x_n), x_t - x_n\rangle + f_n(t), \end{aligned}$$

which implies, $\langle Ax_t - \gamma f(x_t), x_t - x_n\rangle \leq \frac{\bar{\gamma}t}{2}\langle A(x_t) - A(x_n), x_t - x_n\rangle + \frac{f_n(t)}{2t}$.
Letting $n \to \infty$,

$$\limsup \langle Ax_t - \gamma f(x_t), x_t - x_n\rangle \leq \frac{t}{2} M_3, \tag{2.4}$$

where M_3 is a constant such that, $\bar{\gamma}\langle Ax_t - Ax_n, x_t - x_n\rangle \leq M_3, \forall t \in (0, \min\{\|A\|^{-1}, 1\})$ and $n \geq 1$, taking $t \to 0$, from (2.4) we have,

$$\limsup_{t\to 0} \limsup_{n\to\infty} \langle Ax_t - \gamma f(x_t), x_t - x_n\rangle \leq 0. \tag{2.5}$$

On the other hand we have,

$$\begin{aligned} \langle \gamma f(q) - Aq, x_n - q\rangle =& \langle \gamma f(q) - Aq, x_n - q\rangle \\ &- \langle \gamma f(q) - Aq, x_n - x_t\rangle + \langle \gamma f(q) - Aq, x_n - x_t\rangle \\ &- \langle \gamma f(q) - Ax_t, x_n - x_t\rangle + \langle \gamma f(q) - Ax_t, x_n - x_t\rangle \\ &- \langle \gamma f(x_t) - Ax_t, x_n - x_t\rangle + \langle \gamma f(x_t) - Ax_t, x_n - x_t\rangle. \end{aligned}$$

So,

$$\langle \gamma f(q) - Aq, x_n - q\rangle = \langle \gamma f(q) - Aq, x_t - q\rangle + \langle Ax_t - Aq, x_n - x_t\rangle + \langle \gamma f(q) - \gamma f(x_t), x_n - x_t\rangle + \langle \gamma f(x_t) - Ax_t, x_n - x_t\rangle.$$

Hence,

$$\begin{aligned} \limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q\rangle \leq& \|\gamma f(q) - Aq\| \|x_t - q\| + \|A\| \|x_t - q\| \limsup_{n\to\infty} \|x_n - x_t\| \\ &+ \alpha\gamma \|q - x_t\| \limsup_{n\to\infty} \|x_n - x_t\| + \limsup_{n\to\infty} \langle \gamma f(x_t) - Ax_t, x_n - x_t\rangle. \end{aligned}$$

Therefore from (2.5) we have,

$$\begin{aligned}\limsup_{n\to\infty}\langle\gamma f(q)-Aq,x_n-q\rangle =&\limsup_{t\to 0}\limsup_{n\to\infty}\langle\gamma f(q)-Aq,x_n-q\rangle\\ \le&\limsup_{t\to 0}\|\gamma f(q)-Aq\|\|x_t-q\|\\ &+\limsup_{t\to 0}\|A\|\|x_t-q\|\limsup_{n\to\infty}\|x_n-x_t\|\\ &+\limsup_{t\to 0}\gamma\alpha\|q-x_t\|\limsup_{n\to\infty}\|x_n-x_t\|\\ &+\limsup_{t\to 0}\limsup_{n\to\infty}\langle\gamma f(x_t)-Ax_t,x_n-x_t\rangle\le 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle\gamma f(q)-Aq,y_n-q\rangle =&\langle\gamma f(q)-Aq,y_n-x_n\rangle+\langle\gamma f(q)-Aq,x_n-q\rangle\\ \le&\|\gamma f(q)-Aq\|\|y_n-x_n\|+\langle\gamma f(q)-Aq,x_n-q\rangle,\end{aligned}$$

then, $\limsup_{n\to\infty}\langle\gamma f(q)-Aq,y_n-q\rangle\le 0$. Finally we prove that $x_n\to q$.

$$\begin{aligned}\|x_{n+1}-q\|^2 =&\|P_C[\alpha_n\gamma f(x_n)+\delta_n x_n+((1-\delta_n)I-\alpha_n A)y_n]-P_C(q)\|^2\\ \le&\|\alpha_n\gamma f(x_n)+\delta_n x_n+((1-\delta_n)I-\alpha_n A)y_n-q\|^2\\ =&\|\alpha_n(\gamma f(x_n)-Aq)+\delta_n(x_n-q)+((1-\delta_n)I-\alpha_n A)(y_n-q)\|^2\\ =&\|((1-\delta_n)I-\alpha_n A)(y_n-q)+\delta_n(x_n-q)+\alpha_n(\gamma f(x_n)-Aq)\|^2\\ =&\|((1-\delta_n)I-\alpha_n A)(y_n-q)+\delta_n(x_n-q)\|^2\\ &+\alpha_n^2\|\gamma f(x_n)-Aq\|^2+2\delta_n\alpha_n\langle x_n-q,\gamma f(x_n)-Aq\rangle\\ &+2\alpha_n\langle((1-\delta_n)I-\alpha_n A)(y_n-q),\gamma f(x_n)-Aq\rangle\\ \le&[((1-\delta_n)-\alpha_n\bar\gamma)\|y_n-q\|+\delta_n\|x_n-q\|]^2\\ &+\alpha_n^2\|\gamma f(x_n)-Aq\|^2+2\delta_n\alpha_n\langle x_n-q,\gamma f(x_n)-Aq\rangle\\ &+2\alpha_n\langle((1-\delta_n)I-\alpha_n A)(y_n-q),\gamma f(x_n)-Aq\rangle\\ =&[((1-\delta_n)-\alpha_n\bar\gamma)\|y_n-q\|+\delta_n\|x_n-q\|]^2\\ &+\alpha_n^2\|\gamma f(x_n)-Aq\|^2+2\delta_n\alpha_n\gamma\langle x_n-q,f(x_n)-f(q)\rangle\\ &+2\delta_n\alpha_n\langle x_n-q,\gamma f(q)-Aq\rangle+2(1-\delta_n)\gamma\alpha_n\langle y_n-q,f(x_n)-f(q)\rangle\\ &+2(1-\delta_n)\alpha_n\langle y_n-q,\gamma f(q)-Aq\rangle-2\alpha_n^2\langle A(y_n-q),\gamma f(q)-Aq\rangle\\ \le&[((1-\delta_n)-\alpha_n\bar\gamma)\|x_n-q\|+\delta_n\|x_n-q\|]^2\\ &+\alpha_n^2\|\gamma f(x_n)-Aq\|^2+2\delta_n\alpha_n\gamma\alpha\|x_n-q\|^2\\ &+2\delta_n\alpha_n\langle x_n-q,\gamma f(q)-Aq\rangle+2(1-\delta_n)\gamma\alpha_n\alpha\|x_n-q\|^2\\ &+2(1-\delta_n)\alpha_n\langle y_n-q,\gamma f(q)-Aq\rangle-2\alpha_n^2\langle A(y_n-q),\gamma f(q)-Aq\rangle\\ =&[(1-\alpha_n\bar\gamma)^2+2\delta_n\alpha_n\gamma\alpha+2(1-\delta_n)\gamma\alpha_n\alpha]\|x_n-q\|^2\\ &+\alpha_n^2\|\gamma f(x_n)-Aq\|^2+2\delta_n\alpha_n\langle x_n-q,\gamma f(q)-Aq\rangle\\ &+2(1-\delta_n)\alpha_n\langle y_n-q,\gamma f(q)-Aq\rangle-2\alpha_n^2\langle A(y_n-q),\gamma f(q)-Aq\rangle\\ \le&[1-2(\bar\gamma-\alpha\gamma)\alpha_n]\|x_n-q\|^2+\bar\gamma^2\alpha_n^2\|x_n-q\|^2\\ &+\alpha_n^2\|\gamma f(x_n)-Aq\|^2+2\delta_n\alpha_n\langle x_n-q,\gamma f(q)-Aq\rangle\\ &+2(1-\delta_n)\alpha_n\langle y_n-q,\gamma f(q)-Aq\rangle+2\alpha_n^2\|A(y_n-q)\|\|\gamma f(q)-Aq\|\\ =&[1-2(\bar\gamma-\alpha\gamma)\alpha_n]\|x_n-q\|^2+\alpha_n\{\alpha_n[\bar\gamma^2\|x_n-q\|^2\\ &+\|\gamma f(x_n)-Aq\|^2+2\|A(y_n-q)\|\|\gamma f(q)-Aq\|]+2\delta_n\langle x_n-q,\gamma f(q)-Aq\rangle\\ &+2(1-\delta_n)\langle y_n-q,\gamma f(q)-Aq\rangle\}.\end{aligned}$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\|y_n - p\|$ are bounded, we can take a constant $M_4 > 0$ such that,

$$\bar{\gamma}^2\|x_n - q\|^2 + \|\gamma f(x_n) - Aq\|^2 + 2\|A(y_n - q)\|\|\gamma f(q) - Aq\| \le M_4, \ \forall\ n \ge\ 0,$$

then it follows that, $\|x_{n+1} - q\|^2 \le [1 - 2(\bar{\gamma} - \alpha\gamma)\alpha_n]\|x_n - q\|^2 + \alpha_n\sigma_n$, where,

$$\sigma_n = 2\delta_n\langle x_n - q, \gamma f(q) - Aq\rangle + 2(1 - \delta_n)\langle y_n - q, \gamma f(q) - Aq\rangle + \alpha_n M_4.$$

Finally, we have $\limsup_{n\to\infty} \sigma_n \le 0$ and by Lemma 1.2 $x_n \to q$. □

Similar proof shows that the followings composite iteration converges to $q \in F$, which solves variational inequality,

$$\begin{aligned} x_0 =& x \in C \ \textit{chosen arbitrary}, \\ z_n =& \lambda_n x_n + (1 - \lambda_n)K_n x_n, \\ y_n =& \beta_n x_n + (1 - \beta_n)K_n z_n, \\ x_{n+1} =& P_C[\alpha_n\gamma f(x_n) + \delta_n x_n + ((1 - \delta_n)I - \alpha_n A)y_n]. \end{aligned} \tag{2.6}$$

Corollary 2.2. *Let C be a closed convex subset of a real Hilbert space H. Let f be a contraction of C into itself, let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ and $\{T_i : C \longrightarrow C\}$ be a finite family of nonexpansive mappings. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $x_0 \in C$, given that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ be sequences in $[0, 1]$ satisfying the following conditions:*

(C_1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C_2) $0 < \liminf_{n\to\infty} \delta_n \le \limsup_{n\to\infty} \delta_n < 1$,

(C_3) $\sum_{n=1}^{\infty} |\lambda_{n,i} - \lambda_{n-1,i}| < \infty$, *for all* $i = 1, 2, \dots, N$,

(C_4) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$,

(C_5) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$,

(C_6) $(1 + \beta_n)\gamma_n - 2\beta_n > d$, *for some* $d \in (0, 1)$.

If $\{x_n\}_{n=1}^{\infty}$ is the composite process defined by (2.6), then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to $q \in F$, which solves variational inequality $\langle \gamma f(q) - Aq, p - q\rangle \le 0$,$\forall p \in F$.

If $\lambda_n = 1$ and $\delta_n = 0$ in Corollary 2.2, then we get the result of Singthong and Suantai [7].

Corollary 2.3. *Let H be a Hilbert space, C a closed convex subset of H. Let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} \ge 0$, and f is a contraction. Let $\{T_i\}_i^N$ be a finite family of nonexpansive mappings of C into itself and let K_n be defined by (1.6). Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $x_1 \in C$, given that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $(0, 1)$, and suppose that the following conditions are satisfied:*

(C_1) $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(C_2) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,
(C_3) $\sum_{n=1}^{\infty} |\gamma_{n,i} - \gamma_{n-1,i}| < \infty$ *for all* $i = 1, 2, \dots, N$,
(C_4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(C_5) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

If $\{x_n\}_{n=1}^{\infty}$ is the composite process defined by (1.5), then the sequence $\{x_n\}$ converges strongly to $q \in F$, which solves the variational inequality $\langle \gamma f(q) - Aq, p - q\rangle \le 0$,$\forall p \in F$.

3. Simulation examples

In this section, we give three numerical examples to support the theoretical results. The iterations havebeen carried out on MATLAB 7.12. Here we recall $r(n) = log_{10}\|x_{n+1} - x_n\|$ and $\delta(n) = log_{10}\frac{\|x_n - x^*\|}{\|x^*\|}$

(i.e. $\delta(n)$ is relative error), where x^* is a fixed point of W_n-mapping or K-mapping. In the following, we assume $\gamma_1 = \frac{1}{2}$, $\gamma_2 = \frac{1}{3}$, $\gamma_3 = \frac{1}{4}$, and $x_0 = 3$.

	x^*	iteration	$T_1(x^*)$	$T_2(x^*)$
W_n mapping	0.75290	25	0.6837577884	0.7297090424
K mapping	0.71491	19	0.6555494556	0.7551522437

Table 1: $T_1(x) = sin(x)$ and $T_2(x) = cos(x)$.

	x^*	iteration	$T_1(x^*)$	$T_3(x^*)$
W_n mapping	0.0089628	44834	0.0089626800	0.0089625600
K mapping	0.0080118	40066	0.0080117142	0.0080116285

Table 2: $T_1(x) = sin(x)$ and $T_3(x) = tan^{-1}(x)$.

	x^*	iteration	$T_1(x^*)$	$T_2(x^*)$	$T_3(x^*)$
W_n mapping	0.59403	85	0.5597051868	0.8286918026	0.5360182305
K mapping	0.67735	18	0.6267302508	0.7792362880	0.5953623347

Table 3: $T_1(x) = sin(x)$, $T_2(x) = cos(x)$ and $T_3(x) = tan^{-1}(x)$.

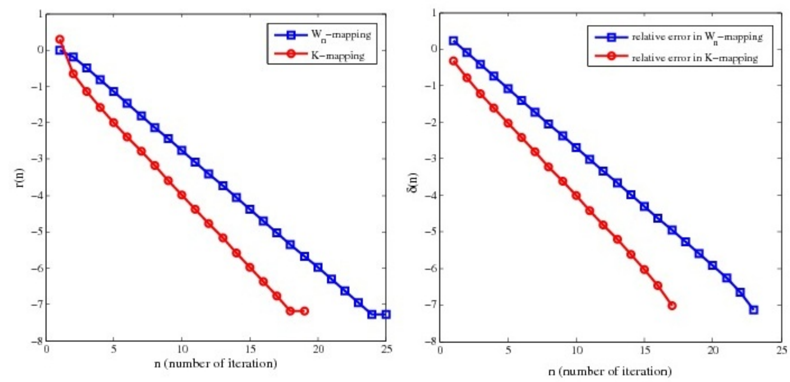

Figure 1: The results obtained for T_1 and T_2.

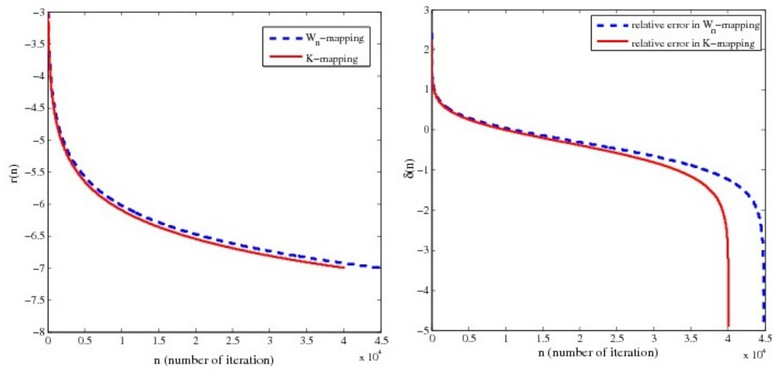

Figure 2: The results obtained for T_1 and T_3.

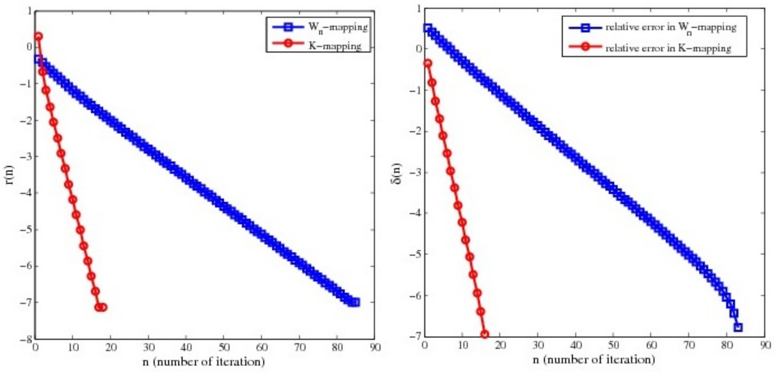

Figure 3: The results obtained for T_1, T_2 and T_3.

4. Conclusion

Finding the fixed point of nonexpansive mappings and variational inequalities is so important in many fields. In this paper, we have constructed an iterative algorithm for finding a common fixed point of an infinite family of nonexpansive mappings and a solution of certain variational inequality. Finally, some numerical examples were presented to support the theoretical results of this paper. Moreover, these examples compare the error and speed of convergence of W_n-mapping and K-mapping.

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