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# Proximal pairs and P-KKM mappings

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### Abstract

We introduced a new class of multi-valued mappings, called P-KKM mappings, which generalized the notation of R-KKM maps of Raj and Somasundaram [KKM-type theorem for best proximity points, Applied Mathematics Letters, 25 (2012), 496-499] and appropriate to best proximity theory. An analog version of KKM property and a common fixed point theorem is also proved. 

*Keywords:* P-KKM mapping, proximal pair, normed linear space.
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## 1. Introduction

Let $X$, $Y$ be nonempty subsets of a metric space $(M, d)$ and $d(X, Y) = \inf\{d(x, y) : x \in X, y \in Y\}$. Let $G : X \multimap X$ and $F : X \multimap Y$ be multi-valued maps. $(G(x_0), F(x_0))$ is called a best proximity pair for $F$ with respect to $G$ [3], if

$$d(G(x_0), F(x_0)) = d(X, Y).$$

Since $d(G(x), F(x)) \geq d(X, Y)$, the optimal solution to the problem of minimizing the real-valued function $X \to d(G(x), F(x))$ will be one for which valued $d(X, Y)$ is attained. The best proximity pair theorems in normed linear spaces, has been studied by many authors; see for example([1, 2, 3, 6, 7]).

In 2012, Raj and Somasudaram introduced R-KKM mappings which fit into best proximity point theory [5].

**Definition 1.1** ([5])**.** Let $X$, $Y$ be nonempty subsets of a metric space $(M, d)$. The pair $(X, Y)$ is said to be proximal pair, if for each $(x, y) \in (X, Y)$, there exists $(\tilde{x}, \tilde{y}) \in (X, Y)$ such that $d(x, \tilde{y}) = d(\tilde{x}, y) = d(X, Y)$.

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Let

$$X_0 = \{x \in X : d(x,y) = d(X,Y) \text{ for some } y \in Y\},\ Y_0 = \{y \in Y : d(x,y) = d(X,Y) \text{ for some } x \in X\}.$$

The pair $(X,Y)$ is a proximal pair if and only if $X = X_0$ and $Y = Y_0$. If $X$ and $Y$ are nonempty subset of a normed linear space $E$ such that $d(X,Y) > 0$, then $X_0 \subseteq bd(X)$ and $Y_0 \subseteq bd(Y)$, where $bd(X)$ denotes the boundary of $X$ for any $X \subseteq E$ [7, Proposition 3.1].

**Definition 1.2** ([5])**.** Let $(X,Y)$ is nonempty proximal pair of a normed linear space $E$. A multi-valued mapping $T : X \multimap Y$is said to be a R-KKM map, if for any $\{x_1,...,x_n\}$ of $X$, there exists $\{y_1,...,y_n\}$ of $Y$ with $\|x_i - y_i\| = d(X,Y)$ for all $i = 1,...,n$, such that $co(\{y_1,...,y_n\}) \subseteq \cup_{i=1}^n T(x_i)$.

A subset $C$ of a normed linear space $E$ is said to be finitely closed, if $C \cap L$ is closed for every finite dimensional subspace $L$ of $E$.

**Theorem 1.3** ([5])**.** *Let $(X,Y)$ be a nonempty proximal pair in a normed linear space $E$ and $F : X \multimap Y$ be an R-KKM map such that $F(x)$ is finitely closed, for all $x \in X$. Then the family $\{F(x) : x \in X\}$ has the finite intersection property.*

**Theorem 1.4** ([5])**.** *Let $(X,Y)$ be a nonempty proximal pair in a normed linear space $E$ and $F : X \multimap Y$ be an R-KKM map. If for each $x \in X$, $F(x)$ is closed in $E$ and there exists at least one $x_0 \in X$ such that $F(x_0)$ is compact in $E$, then $\cap\{F(x) : x \in X\}$ is nonempty.*

We recall some definitions and theorems which are used in this paper. Let $X$ be a vector space and $D \subseteq X$. A multi-valued mapping $F : D \multimap X$ is said to be a KKM map, if $co(\{x_1,...,x_n\}) \subseteq \cup_{i=1}^n F(x_i)$ for each finite subset $\{x_1,...,x_n\} \subseteq D$, where $co(\{x_1,...,x_n\})$ denotes the convex hull of $\{x_1,...,x_n\}$.
Let $X$ and $Y$ are nonempty subsets of a topological vector space. Let $F : X \multimap Y$ and $G : Y \multimap Y$ be multi-valued mappings such that for each nonempty finite set $\{x_1,...,x_n\} \subseteq X$, there exists a set $\{y_1,...,y_n\}$ of points of $Y$, not necessarily all different, such that for each subset $\{y_{i_1},...,y_{i_k}\}$ of $\{y_1,...,y_n\}$, we have

$$G(co(\{y_{i_1},...,y_{i_k}\})) \subseteq \cup_{j=1}^k F(x_{i_j}).$$

Then $F$ is called a generalized KKM mapping with respect to $G$. If the multi-valued mapping $G : Y \multimap Y$ satisfies the requirement that for any generalized KKM mapping $F : X \multimap Y$ with respect to $G$ the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property, then $G$ is said to have the KKM property. We denote

$$KKM(X) = \{G : X \multimap X : \text{ G has the KKM Property }\}.$$

We say that the multi-valued map $F : X \multimap Y$ has a continuous selection, if there exist a continuous function $f : X \to Y$ such that $f(x) \in F(x)$ for each $x \in X$. We denote

$$\mathcal{S}(X,Y) = \{F : X \multimap Y : \text{ F has a continuous selection }\}.$$

The continuous functions have the KKM property. Thus if a multi-valued mapping $G$ has a continuous selection, then $G$ has trivially KKM property [3].
In this paper, we introduced a generalization of R-KKM mappings (we called P-KKM mappings) which is suitable for best proximity theory and proved an analog version of KKM property. Then we find a best proximity pair for $F$ with respect to $G$. Finally we give a common fixed point theorem for P-KKM mappings.

## 2. P-KKM mappings

This section is devoted to formulate a new generalized version of KKM mappings.

**Definition 2.1.** Let $(X, Y)$ be a nonempty proximal pair of a normed linear space $E$. A multi-valued mapping $F : X \multimap Y$ is said to be a P-KKM map with respect to $G : X \multimap X$, if for any finite subset $\{x_1, ..., x_n\}$ of $X$, there exists a finite subset $\{y_1, ..., y_n\}$ of $Y$ with $co(\{y_1, ..., y_n\}) \subseteq Y$ and $\|x_i - y_i\| = d(X, Y)$ and there exists a bijective continuous map $r : Y \to X$ by $r(y_i) = x_i$ for all $i = 1, ..., n$ such that,

$$r^{-1} \circ G \circ r(co\{y_i : i \in I\}) \subseteq \cup_{i \in I} F(x_i),$$

for each nonempty subset $I$ of $\{1, ..., n\}$.

*Remark* 2.2. Note that $F$ is a generalized KKM map with respect to $r^{-1} \circ G \circ r$. If $G = I_X$, then $F$ is an R-KKM map. When $X = Y$ and $r$ is identity map then $F$ is a KKM map with respect to $G$. If $F$ is a P-KKM map with respect to $G$, then $r^{-1} \circ G \circ r(y_i) \subseteq F(x_i)$ for each $i = 1, ..., n$. In this case, when $y_i \in r^{-1}(G(x_i))$ we have,

$$d(G(x), F(x)) = d(X, Y),$$

for each $x \in X$.

**Proposition 2.3.** *Let $(X, Y)$ be a nonempty proximal pair in a normed linear space $E$ and let $F : X \multimap Y$ be a P-KKM map with respect to $G : X \multimap X$. If $G \in \mathcal{S}(X, X)$, then the family $\{F(x) : x \in X\}$ has the finite intersection property.*

*Proof.* Since $G \in \mathcal{S}(X, X)$ and $r$ is a bijective continuous map, then $r^{-1} \circ G \circ r \in \mathcal{S}(Y, Y)$ and so $\{r^{-1} \circ G \circ r(y) : y \in Y\}$ has the KKM property. □

## 3. Main results

Here, we proved KKM property using Brouwer's fixed point theorem which states that, every continuous function on a closed, bounded and convex subset $K$ of $\mathbb{R}^n$, has at least one fixed point in $K$.

**Theorem 3.1.** *Let $(X, Y)$ be a nonempty proximal pair in a normed linear space $E$ and let $F : X \multimap Y$ be a P-KKM map with respect to $G : X \multimap X$ such that $F(x)$ is finitely closed, for all $x \in X$. Then the family $\{F(x) : x \in X\}$ has the finite intersection property.*

*Proof.* Suppose there exists a finite subset $\{x_1, ...x_n\}$ of $X$ such that $\cap_{i=1}^n F(x_i) = \emptyset$. Since $F$ is a P-KKM map with respect to $G$, for this finite subset $\{x_1, ...x_n\}$, there exists a finite subset $\{y_1, ...y_n\}$ of $Y$ with $co(\{y_1, ..., y_n\}) \subseteq Y$ and $\|x_i - y_i\| = d(X, Y)$ and there exists a bijective continuous map $r : Y \to X$ by $r(y_i) = x_i$ for all $i = 1, ..., n$, such that,

$$r^{-1} \circ G \circ r(co(\{y_i : i \in I\})) \subseteq \cup_{i \in I} F(x_i),$$

for each nonempty subset $I$ of $\{1, ..., n\}$. Let $L = span\{y_1, ...y_n\}$ of $E$ and fix $K = co(\{y_1, ...y_n\}) \subseteq L$. Suppose $y \in K$. For each $z \in r^{-1} \circ G \circ r(y) \cap L$, since $L \cap (\cap_{i=1}^{i=n} F(x_i)) = \emptyset$, there exists $i_0$ such that $z \notin F(x_{i_0}) \cap L$. But $F(x_i) \cap L$ is closed for each $i \in \{1, ..., n\}$, thus $d(z, F(x_{i_0})) > 0$. Let

$$s_i(y) = \{z \in r^{-1} \circ G \circ r(y) \cap L : d(z, F(x_i) \cap L) > 0\}.$$

Clearly, $s_i(y) \neq \emptyset$ for some $i \in \{1, 2, ..., n\}$, $s_i(y) \subset r^{-1} \circ G \circ r(y)$ and $s_i(y) \cap (F(x_i) \cap L) = \emptyset$ for each $i \in \{1, 2, ..., n\}$. When $s_i(y) \neq \emptyset$, we define $\alpha_i(y) = z_y$ where $z_y \in s_i(y)$ and $d(z_y, F(x_i) \cap L) \geq d(z, F(x_i) \cap L)$ for each $z \in s_i(y)$. Therefore $d(\alpha_i(y), F(x_i) \cap L) > 0$, for some $i \in \{1, ..., n\}$. We use the function $\Gamma : K \to \mathbb{R}$ by

$$\Gamma(y) = \sum_{i=1}^{n} d(\alpha_i(y), F(x_i) \cap L),$$

to define the map $f : K \to K$ by

$$f(y) = \frac{1}{\Gamma(y)} \sum_{i=1}^{n} d(\alpha_i(y), F(x_i) \cap L) \cdot y_i.$$

Clearly, $f$ is well-defined and continuous map on closed bounded convex subset of finite dimensional space $L$. So by Brouwer's fixed point theorem, there exists $\tilde{y} \in K$ such that $f(\tilde{y}) = \tilde{y}$.
Let $I_K = \{i \in \{1, ..., n\} : d(\alpha_i(\tilde{y}), F(x_i) \cap L) > 0\}$. Then $\alpha_i(\tilde{y}) \notin \cup_{i \in I_K} F(x_i)$ and $\alpha_i(\tilde{y}) \in r^{-1} \circ G \circ r(co(\{y_i : i \in I_K\}))$. But $\tilde{y} = f(\tilde{y}) \in co(\{y_i : i \in I_K\})$ and so,

$$\alpha_i(\tilde{y}) \in r^{-1} \circ G \circ r(\tilde{y}) \subseteq r^{-1} \circ G \circ r(co(\{y_i : i \in I_K\})) \subseteq \cup_{i \in I_K} F(x_i),$$

which is a contradiction. □

**Theorem 3.2.** *Let $(X, Y)$ be a nonempty proximal pair of a normed linear space $E$ and let $F : X \multimap Y$ be a P-KKM map with respect to $G : X \multimap X$. If for each $x \in X$, $F(x)$ is closed in $E$ and there exists a nonempty finite subset $D$ of $X$ such that $\cap_{x \in D} F(x)$ is a compact set, then $\cap_{x \in X} F(x) \neq \emptyset$.*

*Proof.* Let $T : X \multimap Y$ be defined by $T(x) = F(x) \cap (\cap_{z \in D} F(z))$ for each $x \in X$. By Theorem 3.1 the family $\{F(x) : x \in X\}$ has the finite intersection property. So $T$ has nonempty compact values and $\{T(x) : x \in X\}$ has the finite intersection property. Hence $\cap_{x \in X} T(x) \neq \emptyset$ and this implies that $\cap_{x \in X} F(x) \neq \emptyset$. □

*Remark* 3.3. If $G = I_X$, Theorem 3.1 reduces to Theorem 1.3 and Theorem 1.4 is a special case of Theorem 3.2.

**Theorem 3.4.** *Let $(X, Y)$ be a nonempty proximal pair of a normed linear space $E$ and let $F : X \multimap Y$ be a P-KKM map with respect to $G : X \multimap X$. If there exists $x_0 \in X$ such that $x_0 \in G(x_0)$, then,*

$$d(G(x_0), F(x_0)) = d(X, Y).$$

*Proof.* Since $x_0 \in X$ and $(X, Y)$ is a nonempty proximal pair, there exists $y \in Y$ such that $\|x_0 - y\| = d(X, Y)$. By the hypothesis there exists a bijective map $r : Y \to X$ such that $r(y) = x_0$ and $r^{-1} \circ G \circ r(y) \subseteq F(x_0)$. But $x_0 \in G(x_0) = G \circ r(y)$ and so $y = r^{-1}(x_0) \in r^{-1} \circ G \circ r(y)$. Hence $y \in F(x_0)$. Then,

$$d(X, Y) \leq d(G(x_0), F(x_0)) \leq \|x_0 - y\| = d(X, Y).$$

□

By the above theorem when $X = Y$, we have the following coincidence theorem.

**Theorem 3.5.** *Let $X$ be a nonempty subset of a normed linear space $E$ and $F : X \multimap X$ be a KKM map with respect to $G : X \multimap X$. If there exists $x_0 \in X$ such that $x_0 \in G(x_0)$, $G(x_0)$ is compact and $F(x_0)$ is closed, then,*

$$G(x_0) \cap F(x_0) \neq \emptyset.$$

**Example 3.6.** Consider the space $\mathbb{R}^2$ with Euclidean norm. Each two parallel lines is a proximal pair. Let $X = \{(1, x) : 0 \leq x \leq 1\}$ and $Y = \{(-1, x) : 0 \leq x \leq 1\}$. Clearly $(X, Y)$ is a proximal pair in $\mathbb{R}^2$. Define a multi-valued map $F : X \multimap Y$ by $F(1, x) = \{(-1, y) : 0 \leq y \leq 1 - x\}$.
We have $F(1, 0) = Y$, $F(1, \frac{1}{2}) = \{(-1, x) : 0 \leq x \leq \frac{1}{2}\}$ and $F(1, 1) = \{(-1, 0)\}$. If $G : X \multimap X$ defined by $G(1, x) = \{(1, y) : 0 \leq y \leq \frac{x}{2}\}$, then $F$ is a P-KKM map with respect to $G$. We can assume $r : Y \to X$ by $r(-1, x) = (1, x)$. Hence $\cap_{x \in X} F(x) = \{(-1, 0)\}$ and Theorems 3.1 and Theorems 3.2 are hold. The pair $(F(1, 0), G(1, 0))$ is a best proximity pair for $F$ with respect to $G$.

## 4. A Common fixed point theorem

The following theorem is a generalization of [4, Theorem 4.3].

**Theorem 4.1.** *Let $(X, Y)$ be a nonempty proximal pair of a normed linear space $E$ and let $F : Y \times X \multimap Y$, $G : X \multimap X$ be multi-valued mappings with nonempty values. Assume that:*

*(i) for each $x \in X$, $\{y \in Y : y \in F(y, x)\}$ is closed in $Y$,*

*(ii) $F(y, .)$ is a P-KKM map with respect to $G$ on $X$, for each $y \in Y$,*

*(iii) there exists $y_0 \in Y$ such that $F(y_0, X)$ is contained in a compact subset of $Y$.*

*Then there exists $\bar{y} \in Y$ such that $\bar{y} \in F(\bar{y}, x)$ for each $x \in X$.*

*Proof.* For each $x \in X$, let $T : X \multimap Y$ by $T(x) = \{y \in Y : y \notin F(y, x)\}$. By (i), $T(x)$ is an open subset of $Y$. Suppose that for each $y \in Y$, there exists $x = x(y) \in X$ such that $y \notin F(y, x)$. Then $Y = \cup_{x\in X} T(x)$. By (iii) $\overline{F(y_0, X)}$ is a compact set, so there exists $\{x_1, ...x_n\}$ such that $F(y_0, X) \subseteq \cup_{i=1}^{n} T(x_i)$. For this finite subset, by (ii) there exists a finite subset $\{z_1, ...z_n\}$ of $Y$ with $co(\{z_1, ...z_n\}) \subseteq Y$ and $\|z_i - x_i\| = d(X, Y)$ and there exists a bijective continuous map $r : Y \to X$ by $r(z_i) = x_i$ for all $i \in \{1, ..., n\}$, such that $r^{-1} \circ G \circ r(co(\{z_i : i \in I\})) \subseteq \cup_{i\in I} F(y, x_i)$ for each nonempty subset $I$ of $\{1, ..., n\}$ and each $y \in Y$.
Set $L = span\{z_1, ...z_n\}$ of $E$. If $A = co(\{z_1, ...z_n\}) \subseteq L$, then,

$$r^{-1} \circ G \circ r(A) \subseteq \cup_{i\in I} F(y_0, x_i) \subseteq F(y_0, X) \subseteq \cup_{i\in I} T(x_i).$$

The rest of the proof is similar to the proof of Theorem 3.1. □

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