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Coupled Fixed Point Theorem in Dislocated Quasi b-Metric Spaces

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Abstract

In this paper, we define the notion of a coupled coincidence fixed point and prove a coupled coincidence fixed point theorem in dislocated quasi b-metric space. In order to validate our main result and its corollaries an example is given.

Keywords: complete dislocated quasi b-metric space, Cauchy sequence, self-mapping, fixed point, coupled coincidence point, coupled fixed point.
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1. Introduction

In 1906, Frechet introduced the notion of metric space, which is one of the cornerstones of not only mathematics but also in several quantitative sciences. Due to its importance and application potential, this notion has been extended, improved and generalized in many different ways. An incomplete list of such an attempts are as following: quasi metric space, symmetric space, partial metric space, cone metric space, G-metric space, b-metric space, dislocated metric space, dislocated quasi metric space, partial b-metric space and so on.

The notion of b-metric space was introduced by Czerwik [3] in connection with some problems concerning with the convergence of non-measurable functions with respect to measure. Fixed point theorems regarding b-metric spaces was obtained in [8] and [4] etc. In 2013, Shukla [10] generalized the notion of b-metric spaces and introduced the concept of partial b-metric spaces. Recently, Rahman and Sarwar [5] further generalized the concept of b-metric space and initiated the notion of dislocated quasi b-metric space.

Zeyada, Hassan and Ahmed [11] in 2005, initiated the idea of dislocated quasi-metric space. The most interesting property of this space was that the distance between similar points need not to be zero necessarily. Bhaskar and Lakshmikantham [2] introduced the concept of coupled fixed point in partially ordered metric

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space. Single and coupled fixed point theorem was established by several authors (see [9], [1], [6]). Recently in [7], Rahman, Sarwar and Kumari proved tripled fixed point theorem in dislocated quasi-metric space.

In this note, we introduce the idea of coupled coincidence and coupled fixed point in the frame work of dislocated quasi b-metric space. Also, we have establish a coupled coincidence and coupled fixed point theorem in dislocated quasi b-metric space.

2. Preliminaries

We need the following definitions which may be found in [5].

Definition 2.1. Let X be a non-empty set and $k \geq 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called dislocated quasi b-metric, if $\forall\ x, y, z \in X$,

(d_1) $d(x, y) = d(y, x) = 0$ implies that $x = y$;

(d_2) $d(x, y) \leq k[d(x, z) + d(z, y)]$.

The pair (X, d) is called dislocated quasi b-metric space or shortly (dq b-metric) space.

Remark 2.2. In the definition of dislocated quasi b-metric space if $k = 1$ then it becomes (usual) dislocated quasi metric space. Therefore every dislocated quasi metric space is dislocated quasi b-metric space and every b-metric space is dislocated quasi b-metric space with same coefficient k and zero self distance. However, the converse is not true as clear from the following example.

Example 2.3. Let $X = \mathbb{R}$ and suppose

$$d(x, y) = |2x - y|^2 + |2x + y|^2.$$

Then (X, d) is a dislocated quasi b-metric space with the coefficient $k = 2$. But it is not dislocated quasi-metric space nor b-metric space.

Remark 2.4. Like dislocated quasi metric space in dislocated quasi b-metric space the distance between similar points (same points) need not to be zero necessarily as clear from the above example.

Definition 2.5. A sequence $\{x_n\}$ is called dq $b-$convergent in (X, d) if for $n \in N$ we have $\lim_{n\to\infty} d(x_n, x) = 0$. Then x is called the dq b-limit of the sequence $\{x_n\}$.

Definition 2.6. A sequence $\{x_n\}$ in dq b-metric space (X, d) is called Cauchy sequence if for $\epsilon > 0$ there exists $n_0 \in N$, such that for $m, n \geq n_0$ we have $d(x_m, x_n) < \epsilon$ (OR) $\lim_{m,n\to\infty} d(x_m, x_n) = 0$.

Definition 2.7. A dq b-metric space (X, d) is said to be complete, if every Cauchy sequence in X converges to a point of X.

The following well-known results can be seen in [5].

Lemma 2.8. *Limit of a convergent sequence in dislocated quasi b-metric space is unique.*

Lemma 2.9. *Let (X, d) be a dislocated quasi b-metric space and $\{x_n\}$ be a sequence in dq b-metric space such that*

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n), \tag{2.1}$$

for $n = 1, 2, 3, ...,$ $0 \leq \alpha k < 1, \alpha \in [0, 1)$ and k is defined in dq b-metric space. Then $\{x_n\}$ is a Cauchy sequence in X.

Theorem 2.10. *Let (X, d) be a complete dislocated quasi b-metric space. Let $T : X \to X$ be a continuous contraction with $\alpha \in [0, 1)$ and $0 \leq k\alpha < 1$ where $k \geq 1$. Then T has a unique fixed point in X.*

Remark 2.11. Like a b-metric space, a dislocated quasi b-metric space is also continuous on its two variables.

Definition 2.12. An element $(x, y) \in X \times X$ is called coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.13. An element $(x, y) \in X \times X$ is called coupled coincidence point of the mapping $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 2.14. Suppose X is a non-empty set. Then the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if $gF(x, y) = F(gx, gy)$ for all $x, y \in X$.

3. Main Result

Lemma 3.1. *Let (X, d) be a dq b-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that*

$$d(F(x, y), F(u, v)) \leq \alpha[d(gx, gu) + d(gy, gv)], \tag{3.1}$$

for all $x, y, u, v \in X$. Assume that (x, y) is a coupled coincidence point of the mappings F and g. If $2\alpha k < 1$, with $\alpha \in [0, 1)$ and $k \geq 1$. Then

$$F(x, y) = gx = gy = F(y, x).$$

Proof. Since (x, y) is a coupled coincidence point of the mappings F and g, we have $F(x, y) = gx$ and $F(y, x) = gy$. Assume that $gx \neq gy$. Then by (3.1), we get

$$d(gx, gy) = d(F(x, y), F(y, x)) \leq \alpha[d(gx, gy) + d(gy, gx)].$$

Also by (3.1), we have

$$d(gy, gx) = d(F(y, x), F(x, y)) \leq \alpha[d(gy, gx) + d(gx, gy)].$$

Therefore

$$d(gx, gy) + d(gy, gx) \leq 2\alpha[d(gx, gy) + d(gy, gx)].$$

Since $2\alpha k < 1$ so $2\alpha < 1$, we get

$$d(gx, gy) + d(gy, gx) < d(gx, gy) + d(gy, gx),$$

which is a contradiction. So $gx = gy$ and hence

$$F(x, y) = gx = gy = F(y, x).$$

□

Theorem 3.2. *Let (X, d) be a dq b-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two continuous mappings such that*

$$d(F(x, y), F(u, v)) \leq \alpha[d(gx, gu) + d(gy, gv)], \tag{3.2}$$

for all $x, y, u, v \in X$. Assume that F and g satisfy the following conditions:

1. *$F(X \times X) \subseteq g(X)$.*
2. *$g(X)$ is a complete dq b-metric space.*
3. *g commutes with F.*

If $2\alpha k < 1$, with $\alpha \in [0, 1)$ and $k \geq 1$, then there is a unique x in X such that $gx = F(x, x) = x$.

Proof. Let $x_0, y_0 \in X$. By condition (1), Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Continuing this process we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$. For $n \in N$, by Eq. 3.2, we have

$$d(gx_n, gx_{n+1}) = d(F(x_{n-1}, y_{n-1}, F(x_n, y_n)) \leq \alpha[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)].$$

We also have

$$d(gx_{n-1}, gx_n) = d(F(x_{n-2}, y_{n-2}, F(x_{n-1}, y_{n-1})) \leq \alpha[d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})]$$

and

$$d(gy_{n-1}, gy_n) = d(F(y_{n-2}, x_{n-2}), F(y_{n-1}, x_{n-1})) \leq \alpha[d(gy_{n-2}, gy_{n-1}) + d(gx_{n-2}, gx_{n-1})].$$

Adding the last two in-equalities we have

$$[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)] \leq 2\alpha[d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})].$$

Since this holds for all $n \in N$, thus we get

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq \alpha[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)] \\ &\leq 2\alpha^2[d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})]. \end{aligned}$$

Continuing the process we get

$$d(gx_n, gx_{n+1}) \leq \frac{1}{2}(2\alpha)^n[d(gx_0, gx_1) + d(gy_0, gy_1)]. \tag{3.3}$$

Now for $m, n \in N$ with $m > n$ and using the triangular inequality in the definition of dq b-metric space we have

$$d(gx_n, gx_m) \leq kd(gx_n, gx_{n+1}) + k^2 d(gx_{n+1}, gx_{n+2}) + \dots .$$

Since $2\alpha k < 1$ and using (3.3) we get

$$d(gx_n, gx_m) \leq \frac{1}{2}\left(\sum_{i=n}^{\infty}(2\alpha k)^i\right)[d(gx_0, gx_1) + d(gy_0, gy_1)].$$

$$d(gx_n, gx_m) \leq \frac{(2\alpha k)^n}{2(1 - 2\alpha k)}[d(gx_0, gx_1) + d(gy_0, gy_1)].$$

Taking $m, n \to \infty$ we get

$$\lim_{m,n\to\infty} d(gx_n, gx_m) = 0.$$

Thus $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Similarly, we may show that $\{gy_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete dq b-metric, we get that $\{gx_n\}$ and $\{gy_n\}$ are dq b-convergent to some $x \in X$ and $y \in X$ respectively. Since F and g are continuous and also F and g are commutative we have

$$ggx_{n+1} = gF(x_n, y_n) = F(gx_n, gy_n)$$

and

$$ggy_{n+1} = gF(y_n, x_n) = F(gy_n, gx_n).$$

Taking limit $n \to \infty$ in the above two equations we get

$$gx = \lim_{n\to\infty} ggx_{n+1} = g \lim_{n\to\infty} gx_{n+1} = F(\lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n) = F(x, y).$$

Similarly we can show that $gy = F(y, x)$. So by Lemma 3.1, (x, y) is a coupled coincidence point of the mappings F and g. Thus

$$F(x, y) = gx = gy = F(y, x).$$

Since $\{gx_{n+1}\}$ is a sub-sequence of $\{gx_n\}$ so it also converges to x. Thus

$$d(gx_{n+1}, gx) = d(F(x_n, y_n), F(x, y)) \leq \alpha[d(gx_n, gx) + d(gy_n, gy)].$$

Taking limit $n \to \infty$ and using the fact that dq b-metric is continuous on its variables we have

$$d(x, gx) \leq \alpha[d(x, gx) + d(y, gy)].$$

Similarly, we can show that

$$d(y, gy) \leq \alpha[d(x, gx) + d(y, gy)].$$

Adding the last to inequalities we have

$$d(x, gx) + d(y, gy) \leq 2\alpha[d(x, gx) + d(y, gy)].$$

Since $2\alpha < 1$ so the last inequality is possible only if $d(x, gx) = 0$ and $d(y, gy) = 0$. Similarly we can show that $d(gx, x) = 0$ and $d(gy, y) = 0$. So by (d_1) of Definition 2.1, we get that $gx = x$ and $gy = y$. Thus we have

$$gx = F(x, x) = x.$$

Uniqueness. To prove the uniqueness, let $z \in X$ with $z \neq x$ such that $z = gz = F(z, z)$. Then

$$d(x, z) = d(F(x, x), F(z, z)) \leq \alpha[d(gx, gz) + d(gx, gz)] = 2\alpha d(gx, gz).$$

Since $2\alpha < 1$, we get $d(x, z) < d(x, z)$ which is a contradiction. Thus F and g have a unique common coupled fixed point. □

The following corollary may be deduced from the above Theorem.

Corollary 3.3. *Let (X, d) be a complete dq b-metric spaces. Let $F : X \times X \to X$ be a continuous mappings such that*

$$d(F(x, y), F(u, v)) \leq \alpha[d(x, u) + d(y, v)],$$

for all $x, y, u, v \in X$ and $2\alpha < 1$. Then there is a unique x in X such that $F(x, x) = x$.

Example 3.4. Let $X = [0, 1]$. Define $d : X \times X \to \mathbb{R}^+$ by

$$d(x, y) = |2x + y|^2 + |2x - y|^2,$$

for all $x, y \in X$. Define $F(x, y) = \frac{xy}{6}$. Since

$$|2xy + uv|^2 \leq |2x + u|^2 + |2y + v|^2 \text{ and } |2xy - uv|^2 \leq |2x - u|^2 + |2y - v|^2$$

hold for all $x, y, u, v \in X$. We have

$$\begin{aligned} d(F(x, y), F(u, v)) &= |\frac{2xy}{6} + \frac{uv}{6}|^2 + |\frac{2xy}{6} - \frac{uv}{6}|^2 \\ &= \frac{1}{36}(|2xy + uv|^2 + |2xy - uv|^2 \\ &\leq \frac{1}{36}|2x + u|^2 + |2y + v|^2 + |2x - u|^2 + |2y - v|^2 \\ &\leq \frac{1}{18}[d(x, u) + d(y, v)] \end{aligned}$$

hold for all $x, y, u, v \in X$. Also F satisfy all the conditions of Corollary 3.3. Thus F have a coupled fixed point which is in the form of $F(0, 0) = 0$.

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