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Fixed points of multivalued θ-contractions on closed ball

Eskandar Ameer[a,*], Muhammad Arshad[b]

[a]*Department of Mathematics, Taiz University, Taiz, Yemen.*
[a]*Department of Mathematics, International Islamic University, H-10, Islamabad - 44000, Pakistan.*
[b]*Department of Mathematics, International Islamic University, H-10, Islamabad - 44000, Pakistan.*

Abstract

We introduce the notion of multivalued θ-contractions on closed ball and we obtain some new fixed point results for such contractions. An example is given here to illustrate the usability of the obtained results.
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1. Introduction and preliminaries

We recollect some essential notations, required definitions, and primary results coherent with the literature. For a nonempty set X, we denote by $N(X)$ the class of all nonempty subsets of X. Let (X, d) be a metric space. For $x \in X$ and $\varepsilon > 0$, $\overline{B(x,\varepsilon)} = \{y \in X : d_l(x, y) \leq \varepsilon\}$ is a closed ball in (X, d_l). For $x \in X$ and $A \subseteq X$, we denote $D(x, A) = \inf\{d(x, y) : y \in A\}$. We denote by $CL(X)$ the class of all nonempty closed subsets of X, by $CB(X)$ the class of all nonempty closed and bounded subsets of X and by $CO(X)$ the class of all compact subsets of X, Let H be the Hausdorff metric induced by the metric d on X, that is

$$H(A, B) = \max\left\{\sup_{x\in A} D(x, B),\ \sup_{y\in B} D(y, A)\right\},$$

for every $A, B \in CB(X)$. If $T : X \to CB(X)$ be a multi-valued. A point $q \in X$ is said to be a fixed point of T if $q \in Tq$.

In 1969, Nadler [6] extended the famous Banach contraction principle to multivalued mappings and afterwards proved the following result:

*Corresponding author
Email addresses: eskandarameer@gmail.com (Eskandar Ameer), marshadzia@iiu.edu.pk (Muhammad Arshad)

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Theorem 1.1 ([6]). *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued mapping such that for all $x, y \in X$*

$$H(T(x), T(y)) \leq \lambda d(x, y),$$

where $0 < \lambda < 1$. Then T has a fixed point.

Reich [7] proved the following result for multivalued nonlinear contractions.

Theorem 1.2 ([7]). *Let (X,d) be a complete metric space and $T : X \to CO(X)$ be a multivalued mapping. If there exists a function $\alpha : (0,\infty) \to [0,1)$ such that*

$$\lim_{t\to s^+} \sup \alpha(t) < 1, \text{ for all } s \in (0,\infty),$$

satisfying

$$H(T(x), T(y)) \leq \alpha(d(x,y))d(x,y),$$

for all $x, y \in X$ with $x \neq y$. Then T has a fixed point.

In 1989, Mizoguchi and Takahashi [4] generalized Nadler's result by establishing the following theorem:

Theorem 1.3 ([4]). *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multivalued mapping. If there exists a function $\alpha : (0,\infty) \to [0,1)$ such that*

$$\lim_{t\to s^+} \sup \alpha(t) < 1, \text{ for all } s \in (0,\infty),$$

satisfying

$$H(T(x), T(y)) \leq \alpha(d(x,y))d(x,y),$$

for all $x, y \in X$ with $x \neq y$. Then T has a fixed point.

We denote by Θ the set of functions $\theta : (0,\infty) \to (1,\infty)$ satisfying conditions (Θ1)-(Θ3) and by Ξ the set of functions $\theta : (0,\infty) \to (1,\infty)$ satisfying conditions (Θ1)-(Θ4),

(Θ1) θ is non-decreasing,

(Θ2) for each sequence $\{t_n\} \subset (0,\infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \text{ if and only if } \lim_{n\to\infty} t_n = 0^+,$$

(Θ3) there exists $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\underset{t\to 0^+}{lim} \frac{\theta(t)-1}{t^r} = \ell$.

(Θ4) $\theta(\inf A) = \inf \theta(A)$ for all $A \subset (0,\infty)$ with $\inf A > 0$.

In 2014 Jleli and Samet [2] introduced attractive generalization of the Banach contraction principle, which throughout this paper, we will call θ-contraction.

Let (X,d) be a metric space and $\theta \in \Theta$. A mapping $T : X \to X$ is said to be a θ-contraction, if there exists a constant $k \in (0,1)$ such that,

$$x, y \in X,\ d(Tx,Ty) \neq 0 \to \theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k.$$

Jleli and Samet [2] established the following fixed point theorem as follows:

Theorem 1.4 (Corollary 2.1, [2]). *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. If T is an θ-contraction, then T has a unique fixed point.*

Example 1.5 ([2])**.** The following functions $\theta : (0, \infty) \to (1, \infty)$ are elements of Θ :

(1) $\theta(t) = e^{\sqrt{t}}$,

(2) $\theta(t) = e^{\sqrt{te^t}}$,

(3) $\theta(t) = 2 - \frac{2}{\pi} \arctan\left(\frac{1}{t^\gamma}\right)$, $0 < \gamma < 1$, $t > 0$.

HanÇer et al. [1] (see also [8]) extended the concept of θ-contraction to multivalued mappings as follows.

Definition 1.6 ([1])**.** Let (X, d) be a metric space, $T : X \to CB(X)$ and $\theta \in \Theta$. Then T is said to be a multivalued θ- contraction if there exists a function $k \in [0, 1)$ such that

$$\theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^k, \tag{1.1}$$

for all $x, y \in X$, with $H(Tx, Ty) > 0$.

Recently, Miknak and Altun [5] introduced the notion of multivalued nonlinear θ-contraction in this way,

Definition 1.7 ([5])**.** Let(X, d) be a metric space, $T : X \to CB(X)$ and $\theta \in \Theta$. Then T is said to be a multivalued nonlinear θ- contraction if there exists a function $k : (0, \infty) \to [0, 1)$ such that

$$\theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^{k(d(x,y))}, \tag{1.2}$$

for all $x, y \in X$, with $H(Tx, Ty) > 0$.

Theorem 1.8 ([5])**.** *Let (X, d) be a complete metric space, $T : X \to CO(X)$ be a multivalued nonlinear θ-contraction mapping. Then T has a fixed point provided that $\lim_{t \to s^+} \sup k(t) < 1$, for all $s \in [0, \infty)$ holds.*

Lemma 1.9 ([5])**.** *Let (X, d) be a metric space and A be compact subset of X. Then, for $x \in X$, there exists $a \in A$ such that $d(x, a) = d(x, A)$.*

Theorem 1.10 (Theorem 5.1.4, [3])**.** *Let (X, d) be a complete metric space, $T : X \to X$ be a mapping, $r > 0$ and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, 1)$ with*

$$d(T(x), T(y)) \leq kd(x, y), \text{ for all } x, y \in Y = \overline{B(x_0, r)} \tag{1.3}$$

and $d(x_0, T(x_0)) < (1 - k)r$. Then there exists a unique point x^* in $\overline{B(x_0, r)}$ such that $x^* = T(x^*)$.

In this paper, we introduce a new concept of multivalued θ-contraction closed ball in a metric space which is more general than the multivalued nonlinear θ-contraction for multivalued mappings. We establish some fixed point theorems for this type of mappings and give example illustrating our main results. Throughout the article we denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}^+$ the set of all positive real numbers and by $\mathbb{N}$ the set of all positive integers.

2. Main Results

We first introduce a concept of multivalued θ-contraction on closed ball in a metric space.

Definition 2.1. Let (X, d) be a metric space. The mapping $T : X \to CB(X)$ is said to be multivalued θ-contraction on closed ball, if there exists a function $\theta \in \Theta$ such that

$$\theta(H(Tx, Ty)) \leq [\theta(\lambda d(x, y))]^k, \tag{2.1}$$

for all $x, y \in \overline{B(x_0, r)} \subseteq X$, whaere λ, $k \in [0, 1)$.

We now state and prove our main result.

Theorem 2.2. *Let (X,d) be a complete metric space, $T : X \to CO(X)$ be a continuous multivalued θ-contraction on closed ball $\overline{B(x_0,r)}$. Moreover*

$$d(x_0,Tx_0) \leq (1-\lambda)r, \text{ where } \lambda \in [0,1) \text{ and } r>0. \tag{2.2}$$

Then T has a fixed point x^* in $\overline{B(x_0,r)}$.

Proof. Choose a point x_1 in X such that $x_1 \in Tx_0$. continuing in this way, so we get $x_{n+1} \in Tx_n$, for all $n \geq 0$ and this implies that $\{x_n\}$ is a nonincreasing sequence. Now we will prove that $x_n \in \overline{B(x_0,r)}$ for all $n \in \mathbb{N}$, by using mathematical induction. Since from (2.2), we have

$$d(x_0,Tx_0) \leq (1-\lambda)r < r,$$

since Tx_0 is compact, so there exists $x_1 \in Tx_0$ such that $d(x_0,x_1) \leq (1-\lambda)r < r$, thus, $x_1 \in \overline{B(x_0,r)}$. Suppose $x_2....x_j \in \overline{B(x_0,r)}$ for some $j \in \mathbb{N}$. Thus from (2.1), we obtain

$$\begin{aligned}\theta\left(d\left(x_1,Tx_1\right)\right) &\leq \theta\left(H\left(Tx_0,Tx_1\right)\right) \leq \left[\theta\left(\lambda d\left(x_0,x_1\right)\right)\right]^k \\ &< \theta\left(\lambda d\left(x_0,x_1\right)\right).\end{aligned}$$

Which implies,

$$\theta\left(d\left(x_1,Tx_1\right)\right) < \theta\left(\lambda d\left(x_0,x_1\right)\right). \tag{2.3}$$

similar, there exists $x_2 \in Tx_1$ such that

$$\theta\left(d\left(x_1,x_2\right)\right) < \theta\left(\lambda d\left(x_0,x_1\right)\right).$$

From condition $(\Theta 1)$, we get,

$$d\left(x_1,x_2\right) < \lambda d\left(x_0,x_1\right).$$

Repeating these steps for x_2, $x_3, ..., x_j$, we obtain , $x_{j+1} \in Tx_j$,

$$d\left(x_j,x_{j+1}\right) < \lambda d\left(x_{j-1},x_j\right). \tag{2.4}$$

Now, using triangular inequality and (2.4), we have

$$\begin{aligned}d\left(x_0,x_{j+1}\right) &\leq d\left(x_0,x_1\right)+d\left(x_1,x_2\right)+d\left(x_2,x_3\right)+...+d\left(x_j,x_{j+1}\right) \\ &< d\left(x_0,x_1\right)\left[1+\lambda+\lambda^2+...+\lambda^j\right] \\ &< (1-\lambda)\,r\frac{\left(1-\lambda^{j+1}\right)}{1-\lambda} < r.\end{aligned} \tag{2.5}$$

This implies that $x_{j+1} \in \overline{B(x_0,r)}$. Hence $x_n \in \overline{B(x_0,r)}$ for all $n \in \mathbb{N}$ and

$$\theta\left(d\left(x_n,x_{n+1}\right)\right) \leq \theta\left(H\left(Tx_{n-1},Tx_n\right)\right).$$

From the above inequality, we get,

$$\theta\left(d\left(x_n,x_{n+1}\right)\right) \leq \theta\left(H\left(Tx_{n-1},Tx_n\right)\right) \leq \left[\theta\left(\lambda d\left(x_{n-1},x_n\right)\right)\right]^k < \theta\left(\lambda d\left(x_{n-1},x_n\right)\right), \text{ for all } n \in \mathbb{N}.$$

Thus, by taking into account $(\theta 1)$, the sequence $\{d\left(x_n,x_{n+1}\right)\}$ is decreasing and hence convergent, we get

$$\begin{aligned}1 &< \theta\left(d\left(x_n,x_{n+1}\right)\right) \\ &\leq \left[\theta\left(\lambda d\left(x_{n-1},x_n\right)\right)\right]^k \leq \left[\theta\left(d\left(x_{n-1},x_n\right)\right)\right]^k \\ &\leq \left[\theta\left(\lambda d\left(x_{n-2},x_{n-1}\right)\right)\right]^{k^2} \leq \left[\theta\left(d\left(x_{n-2},x_{n-1}\right)\right)\right]^{k^2} \\ &\;\;. \\ &\;\;. \\ &\leq \left[\theta\left(d\left(x_0,x_1\right)\right)\right]^{k^n}.\end{aligned}$$

Thus, we obtain,

$$1 < \theta\left(d\left(x_n, x_{n+1}\right)\right) \leq \left[\theta\left(d\left(x_0, x_1\right)\right)\right]^{k^n}, \text{ for all } n \in \mathbb{N}. \tag{2.6}$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} \theta\left(d\left(x_n, x_{n+1}\right)\right) = 1, \tag{2.7}$$

that together with (Θ2) gives as

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0.$$

From condition (Θ3), there exist $r \in (0,1)$ and $\ell \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta\left(d\left(x_n, x_{n+1}\right)\right) - 1}{\left[\theta\left(d\left(x_n, x_{n+1}\right)\right)\right]^r} = \ell.$$

Suppose that $\ell < \infty$. In this case, let $B = \frac{\ell}{2} > 0$. From the definition of the limit, there exists $n_0 \geq 1$ such that

$$\left|\frac{\theta\left(d\left(x_n, x_{n+1}\right)\right) - 1}{\left[d\left(x_n, x_{n+1}\right)\right]^r} - \ell\right| \leq B \text{ for all } n \geq n_0.$$

This implies

$$\frac{\theta\left(d\left(x_n, x_{n+1}\right)\right) - 1}{\left[d\left(x_n, x_{n+1}\right)\right]^r} \geq \ell - B = B \text{ for all } n \geq n_0.$$

Then

$$k\left[d\left(x_n, x_{n+1}\right)\right]^r \leq Ak\left[\theta\left(d\left(x_n, x_{n+1}\right)\right) - 1\right] \text{ for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Suppose now that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta\left(d\left(x_n, x_{n+1}\right)\right) - 1}{\left[d\left(xn, x_{n+1}\right)\right]^r} \geq B \text{ for all } n \geq n_0,$$

which implies

$$k\left[d\left(x_n, x_{n+1}\right)\right]^r \leq Ak\left[\theta\left(d\left(x_n, x_{n+1}\right)\right) - 1\right] \text{ for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$k\left[d\left(x_n, x_{n+1}\right)\right]^r \leq Ak\left[\theta\left(d\left(x_n, x_{n+1}\right)\right) - 1\right] \text{ for all } n \geq n_0.$$

By using (2.6), we get

$$k\left[d\left(x_n, x_{n+1}\right)\right]^r \leq Ak\left(\left[\theta\left(d(x_0, x_1)\right)\right]^{k^n} - 1\right) \text{ for all } n \geq n_0. \tag{2.8}$$

Letting $n \to \infty$ in the inequality (2.8), we obtain

$$\lim_{n\to\infty} k\left[d\left(x_n, x_{n+1}\right)\right]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$d\left(x_n, x_{n+1}\right) \leq \frac{1}{n^{\frac{1}{r}}} \text{ for all } n \geq n_1. \tag{2.9}$$

Now, we will prove that $\{x_n\}$ is a Cauchy sequence, $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (2.9), we get

$$\begin{aligned} d\left(x_n, x_m\right) &\leq d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right) + \ldots + d\left(x_{m-1}, x_m\right) \\ &= \sum_{i=n}^{m-1} d\left(x_i, x_{i+1}\right) \leq \sum_{i=n}^{\infty} d\left(x_i, x_{i+1}\right) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}. \end{aligned}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}$ is convergent (since $\frac{1}{r} > 1$), we deduce that $\{x_n\}$ is a Cauchy sequence. Hence $\{x_n\}$ is a Cauchy sequence in $(\overline{B(x_0, r)}, d)$. Since $(\overline{B(x_0, r)}, d)$ is a complete metric space, so there exists $x_* \in \overline{B(x_0, r)}$ such that $x_n \to x_*$ as $n \to \infty$. Since T is a continuous, then $x_{n+1} \in Tx_n \to Tx_*$ as $n \to \infty$. That is, $x_* \in Tx_*$. Hence x_* is a fixed point of T in $\overline{B(x_0, r)}$. □

Definition 2.3. Let K be a nonempty subset of metric space X and let $x \in X$. An element $y_0 \in K$ is called a best approximation in K if

$$d(x, K) = d(x, y_0), \text{ where } d(x, K) = \inf_{y \in K} d(x, y).$$

If each $x \in X$ has at least one best approximation in K, then K is called a proximinal set. We denote $P(X)$ be the set of all proximinal subsets of X. We cannot take $P(X)$ instead of $CO(X)$ in Theorem 2.2. However, by adding the condition (Θ4) on Θ, we can introduce the following Theorem:

Theorem 2.4. *Let (X, d) be a complete metric space, $T : X \to P(X)$ be a continuous multivalued θ-contraction on closed ball $\overline{B(x_0, r)}$. Moreover, $\theta \in \Xi$ and*

$$d(x_0, Tx_0) \leq (1 - \lambda)r, \text{ where } \lambda \in [0, 1) \text{ and } r > 0. \tag{2.10}$$

Then T has a fixed point x^* in $\overline{B(x_0, r)}$.

Proof. Choose a point x_1 in X such that $x_1 \in Tx_0$. continuing in this way, so we get $x_{n+1} \in Tx_n$, for all $n \geq 0$ and this implies that $\{x_n\}$ is a nonincreasing sequence. Now we will prove that $x_n \in \overline{B(x_0, r)}$ for all $n \in \mathbb{N}$, by using mathematical induction. Since from (2.10), we have

$$d(x_0, Tx_0) \leq (1 - \lambda)r < r.$$

There exists $x_1 \in Tx_0$ such that $d(x_0, x_1) \leq (1 - \lambda)r < r$, thus, $x_1 \in \overline{B(x_0, r)}$. Suppose $x_2 .... x_j \in \overline{B(x_0, r)}$ for some $j \in \mathbb{N}$. Thus from (2.1), we obtain

$$\begin{aligned} \theta(d(x_1, Tx_1)) \leq \theta(H(Tx_0, Tx_1)) &\leq [\theta(\lambda d(x_0, x_1))]^k \\ &< \theta(\lambda d(x_0, x_1)). \end{aligned} \tag{2.11}$$

Which implies,

$$\theta(d(x_1, Tx_1)) < \theta(\lambda d(x_0, x_1)).$$

From condition (Θ4), we can write,

$$\theta(d(x_1, Tx_1)) = \inf_{y \in Tx_1} \theta(d(x_1, y))$$

Hence from (2.11) we get,

$$\begin{aligned} \inf_{y \in Tx_1} \theta(d(x_1, y)) &\leq [\theta(\lambda d(x_0, x_1))]^k \\ &< [\theta(\lambda d(x_0, x_1))]^{\sqrt{k}}. \end{aligned} \tag{2.12}$$

Then, from (2.12) there exists $x_2 \in Tx_1$ such that

$$\theta(d(x_1, x_2)) \leq [\theta(\lambda d(x_0, x_1))]^{\sqrt{k}} < \theta(\lambda d(x_0, x_1)).$$

From condition (Θ1), we get

$$d(x_1, x_2) < \lambda d(x_0, x_1).$$

Repeating these steps for $x_2, x_3, ..., x_j$, we obtain, $x_{j+1} \in Tx_j$,

$$d(x_j, x_{j+1}) < \lambda d(x_{j-1}, x_j). \tag{2.13}$$

Now, using triangular inequality and (2.13), we have

$$\begin{aligned} d(x_0, x_{j+1}) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + ... + d(x_j, x_{j+1}) \\ &< d(x_0, x_1)\left[1 + \lambda + \lambda^2 + ... + \lambda^j\right] \\ &< (1-\lambda) r \frac{(1-\lambda^{j+1})}{1-\lambda} < r. \end{aligned} \tag{2.14}$$

This implies that $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in \mathbb{N}$,

$$\theta(d(x_n, x_{n+1})) \leq [\theta(\lambda d(x_{n-1}, x_n))]^{\sqrt{k}} < \theta(\lambda d(x_{n-1}, x_n)),$$

for all $n \in \mathbb{N}$. The rest of the proof can be completed as in the proof of Theorem 2.2. □

In case of single valued mapping $T : X \to X$, we have the following result:

Corollary 2.5. *Let (X, d) be a complete metric space, $T : X \to X$ be a continuous θ-contraction on closed ball $\overline{B(x_0, r)}$. That is, if there exists a function $\theta \in \Theta$ such that*

$$\theta(d(Tx, Ty)) \leq [\theta(\lambda d(x, y))]^k, \tag{2.15}$$

for all $x, y \in \overline{B(x_0, r)} \subseteq X$, where λ, $k \in [0, 1)$. Moreover,

$$d(x_0, Tx_0) \leq (1-\lambda)r < r, \text{ where } r > 0. \tag{2.16}$$

Then T has a unique fixed point x^* in $\overline{B(x_0, r)}$.

Example 2.6. Let $X = [0, \infty)$. Define $T : X \to P(X)$, and $\theta \in \Xi$ by

$$Tx = \begin{cases} \left[0, \frac{x}{100}\right], & \text{if } x \in [0, 1], \\ \{2x\} & \text{otherwise,} \end{cases}$$

and $\theta(t) = e^{\sqrt{t}}$, with $t > 0$. Also, $x_0 = \frac{1}{4}$, $r = 1$, $\overline{B(x_0, r)} = [0, 1]$, then

$$d\left(\frac{1}{4}, T\left(\frac{1}{4}\right)\right) = \left|\frac{1}{4} - \frac{1}{400}\right| = \frac{99}{400} \leq (1-\lambda) r = \frac{1}{3} < 1 = r.$$

If $x, y \in \overline{B(x_0, r)}$, then

$$\begin{aligned} \theta(H(Tx, Ty)) &= \theta\left(\left|\frac{x}{100} - \frac{y}{100}\right|\right) \\ &\leq \left[\theta\left(\frac{2}{3}|x-y|\right)\right]^{\frac{2}{3}} \\ &= [\theta(\lambda d(x, y))]^k, \text{ where, } k = \lambda = \frac{2}{3}, \end{aligned}$$

which implies that

$$\theta(H(Tx, Ty)) \leq [\theta(\lambda d(x, y))]^k, \text{ for all } x, y \in \overline{B(x_0, r)}.$$

Hence, the hypotheses of Theorem 2.4 hold on closed ball and $x = 0$ is a fixed point of T in $\overline{B(x_0, r)}$. If $x \notin \overline{B(x_0, r)}$ or $y \notin \overline{B(x_0, r)}$, then

$$\begin{aligned} \theta(2|x-y|) &> [\theta(|x-y|)]^{\frac{2}{3}}, \\ \theta(|2x-2y|) &> [\theta(|x-y|)]^{\frac{2}{3}}, \\ \theta(H(Tx, Ty)) &> [\theta(d(x, y))]^k. \end{aligned}$$

Hence the multivalued θ- contraction condition (1.1) does not hold on X

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