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Suzuki type common fixed point theorems for four maps using α-admissible in partial ordered complex partial metric spaces

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Abstract

In this paper we obtain Suzuki type common fixed point theorems for four maps using α-admissible in partial ordered complex partial metric spaces. Also we give examples to illustrate our theorems.

Keywords: Complex partial metric space, α-admissible maps, p_c^*-compatible maps.
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1. Introduction

The existence and uniqueness of fixed and common fixed points of self mappings has been a subject of great interest since the work of Banach [10] in 1922.

The existence of fixed points in ordered metric spaces has been initiated in 2004 by Ran and Reurings [23] and further studied by several authors in this direction, see for example [21, 22].

The concept of a partial metric space was introduced by Matthews [19] in 1994. After that, fixed and common fixed point results in partial metric spaces were studied by many other authors, see for example [6–8, 13, 14, 18, 24]

Azam et al. [9] introduced the notion of a complex valued metric space which is a generalization of the classical metric space and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a rational contractive condition. Later several authors proved fixed and common fixed point theorems in complex valued metric spaces,see for example [1, 3, 11, 16, 17, 20, 25, 29, 33–35, 38].

Recently Dhivya and Marudai [12] introduced the concept of a complex partial metric spaces and studied common fixed point results for two mappings satisfying a rational inequality.

First we give the following known concepts in the literature.

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Let $\mathbb{C}$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \quad \textit{if and only if} \quad Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2).$$

Let $\mathbb{C}^+$ denotes for all $0 \precsim C \in \mathbb{C}$. Through out this paper $\mathbb{N}$ denotes the set of all natural numbers and $\mathbb{R}^+$ denotes the set of all non negative real numbers.

2. Preliminaries

Recently Dhivya and Marudai [12] defined the notion of a complex partial metric space as follows.

Definition 2.1 ([12])**.** A complex partial metric on a non empty set X is a function $p_c : X \times X \to \mathbb{C}^+$ such that for all $x, y, z \in X$:

(p_1) $0 \precsim p_c(x, x) \precsim p_c(x, y)$;
(p_2) $p_c(x, x) = p_c(x, y) = p_c(y, y)$ if and only if $x = y$;
(p_3) $p_c(x, y) = p_c(y, x)$;
(p_4) $p_c(x, y) \precsim p_c(x, z) + p_c(z, y) - p_c(z, z)$.

(X, p_c) is called a complex partial metric space.

Example 2.2. Let $X = [0, \infty)$ and $p_c : X \times X \to \mathbb{C}^+$ be defined by

$$p_c(x, y) = \max\{x, y\} + i \max\{x, y\}$$

for all $x, y \in X$. Then (X, p_c) is a complex partial metric space.

It is clear that $|p_c^*(x, y)| \leq |1 + p_c^*(x, y)|$ for all $x, y \in X$.

Each complex partial metric p_c on X generates a topology τ_{p_c} on X with the base family of open p_c-balls $\{B_{p_c}(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_{p_c}(x, \epsilon) = \{y \in X : p_c(x, y) \prec p_c(x, x) + \epsilon\}$ for all $x \in X$ and $0 < \epsilon \in \mathbb{C}^+$. With this terminology, the complex partial metric space (X, p_c) is a T_0 space.

Definition 2.3. Let (X, p_c) be a complex partial metric space. A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if for every $0 < \epsilon \in \mathbb{C}^+$, there is $N \in \mathbb{N}$ such that $x_n \in B_{p_c}(x, \epsilon)$ for all $n \geq \mathbb{N}$. Here x is said to be a limit of $\{x_n\}$ and we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Lemma 2.4. *Let (X, p_c) be a complex partial metric space. A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if and only if $p_c(x, x) = \lim\limits_{n\to\infty} p_c(x, x_n)$.*

Definition 2.5. Let (X, p_c) be a complex partial metric space. A sequence $\{x_n\}$ in X is said to be Cauchy if there exists $a \in \mathbb{C}^+$ such that for every $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $|p_c(x_n, x_m) - a| < \epsilon$ for all $n, m \geq n_0$.

Definition 2.6. Let (X, p_c) be a complex partial metric space.

(i) X is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_{p_c}, to a point $x \in X$ such that
$$p_c(x, x) = \lim_{n,m\to\infty} p_c(x_n, x_m).$$

(ii) A mapping $T : X \to X$ is said to be continuous to $x_0 \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $T(B_{p_c}(x, \delta)) \subseteq B_p(Tx_0, \epsilon)$.

Lemma 2.7.

(a_1) *Let (X, p_c) be a complex partial metric space. A sequence $\{x_n\}$ is Cauchy in (X, p_c) iff $\{x_n\}$ is Cauchy in (X, d_{p_c}).*

(a$_2$) *(X, p_c) is complete if and only if (X, d_{p_c}) is complete. Moreover,*

$$\lim_{n\to\infty} d_{p_c}(x, x_n) = 0 \quad \Leftrightarrow \quad p_c(x, x) = \lim_{n\to\infty} p_c(x, x_n) = \lim_{n,m\to\infty} p_c(x_n, x_m).$$

Note that if (X, p_c) be a complex partial metric space, then we have

$$\lim_{n\to\infty} p_c(x, x_n) = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} |p_c(x, x_n)| = 0$$

for every $\{x_n\}$, $x \in X$.

One can prove the following.

Lemma 2.8. *Let (X, p_c) be a complex partial metric space. A sequence $\{x_n\}$ in X converges to $x \in X$ such that $p_c(x, x) = 0$. Then $\lim_{n\to\infty} p_c(x_n, y) = p_c(x, y)$ for every $y \in X$.*

Proof. We have $p_c(x_n, y) \precsim p_c(x_n, x) + p_c(x, y) - p_c(x, x) = p_c(x_n, x) + p_c(x, y)$. Thus,

$$\lim_{n\to\infty} p_c(x_n, y) \precsim p_c(x, x) + p_c(x, y) = p_c(x, y). \tag{i}$$

Also, $p_c(x, y) \precsim p_c(x, x_n) + p_c(x_n, y) - p_c(x_n, x_n) \precsim p_c(x, x_n) + p_c(x_n, y)$. So we have

$$p_c(x, y) \precsim p_c(x, x) + \lim_{n\to\infty} p_c(x_n, y) = \lim_{n\to\infty} p_c(x_n, y). \tag{ii}$$

From (i) and (ii), we have $\lim_{n\to\infty} p_c(x_n, y) = p_c(x, y)$. □

Rao et al. [26] modified the definition of partial compatible pair of maps given by Samet et al. [30] as partial* compatible maps in partial metric spaces. In this paper we introduce p_c^*-compatible maps as follows.

Definition 2.9. Let (X, p_c) be a complex partial metric space and $F, g : X \to X$. Then the pair (F, g) is said to be p_c^*-compatible if the following conditions hold:

(i) $p_c(x, x) = 0 \Rightarrow p_c(gx, gx) = 0$ whenever $x \in X$;

(ii) $\lim_{n\to\infty} p_c(Fgx_n, gFx_n) = 0$ whenever there exists a sequence $\{x_n\}$ in X such that $Fx_n \to t$ and $gx_n \to t$ for some $t \in X$ with $p_c(t, t) = 0$.

Samet et al. [31] introduced the notion of α-admissible mappings associated with single map.

Later Karapinar et al. [15], Shahi et al. [32], Abdeljawad [5], and Rao et al. [26] extended α-admissible mappings associated with two and four mappings and proved fixed and common fixed point theorems for mappings on various spaces.

Definition 2.10. Let X be a non empty set and $\alpha : X \times X \to \mathbb{R}^+$. We need the following definitions and notations in the rest of research (see [5, 15, 26, 31, 32] for more detail).

(i) A mapping of $T : X \to X$ is called α-admissible if $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ for all $x, y \in X$.

(ii) A mapping of $T : X \to X$ is called triangular α-admissible if $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$ for all $x, y \in X$ and $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1 \Rightarrow \alpha(x, y) \geq 1$ for all $x, y, z \in X$.

(iii) Let $f, g : X \to X$. Then f is said to be α-admissible with respect to g if $\alpha(gx, gy) \geq 1$ implies $\alpha(fx, fy) \geq 1$ for all $x, y \in X$.

(iv) Let $f, g : X \to X$. Then the pair(f, g) is said to be α-admissible if $\alpha(x, y) \geq 1$ implies $\alpha(fx, gy) \geq 1$ and $\alpha(gx, fy) \geq 1$ for all $x, y \in X$.

(v) Let $f, g, S, T : X \to X$. Then the pair (f, g) is said to be α-admissible w.r.to the pair (S, T) if $\alpha(Sx, Ty) \geq 1$ implies $\alpha(fx, gy) \geq 1$ and $\alpha(Tx, Sy) \geq 1$ implies $\alpha(gx, fy) \geq 1$ for all $x, y \in X$.

Recently Abbas et al. [2, 4] introduced the new concepts in a partially ordered set as follows.

Definition 2.11 ([2, 4])**.** Let $(X, \preceq)$ be a partially ordered set and $f : X \to X$.

(b_1) f is said to be a dominating map if $x \preceq fx$ for all $x \in X$.

(b_2) f is said to be dominated map if $fx \preceq x$ for all $x \in X$.

Suzuki [36, 37] proved and generalized versions of Banach's and Edelsteins basic results. The importance of Suzuki contraction theorem is that the contractive condition required to satisfied not for all points of the domain of mapping involved in it. In this direction several authors have given fixed and common fixed point theorems in various spaces, (see [26–28]).

Recently Rao et al. [27] proved the following.

Theorem 2.12 ([27, Theorem2.1])**.** *Let $(X, d, \preceq)$ be a partially ordered complete complex valued metric space, and $\alpha : X \times X \to \mathcal{R}^+$ be a function. Let $f, g, S, T : X \to X$ be self mappings on X satisfying the following*

(c_1) *f, g are dominating maps and f and g are weak annihilators of T and S, respectively;*

(c_2) *$f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$;*

(c_3) $\frac{1}{2}\min\{|d(fx, Sx)|, |d(gy, Ty)|\} \leq \max\{|d(Sx, Ty)|, |d(fx, gy)|\}$ *implies*

$$\alpha(Sx, Ty)\, d(fx, gy) \precsim a_1 d(Sx, Ty) + a_2 d(Sx, fx) + a_3 d(Ty, gy) + a_4 d(Sx, gy) + a_5 d(Ty, fx) + a_6 \frac{d(fx, Sx)\, d(gy, Ty)}{1 + d(Sx, Ty)} + a_7 \frac{d(Sx, gy)\, d(Ty, fx)}{d(Sx, Ty)}$$

for all comparable elements $x, y \in X$, where $a_i (i = 1, 2, \ldots, 7)$ are non-negative real numbers such that $\sum_{i=1}^{7} a_i < 1$;

(c_4) *the pair (f, g) is α-admissible with respect to the pair (S, T);*

(c_5) *$\alpha(Sx_1, fx_1) \geq 1$ and $\alpha(fx_1, Sx_1) \geq 1$ for some $x_1 \in X$;*

(c_6) (a) *S is continuous, the pair (f, S) is compatible, and the pair (g, T) is weakly compatible and there exists a sequence $\{y_n\}$ in X such that $\alpha(y_n, y_{n+1}) \geq 1$ and $\alpha(y_{n+1}, y_n) \geq 1$ for all $n \in \mathcal{N}$ and $y_n \to z$ for some $z \in X$, then we have $\alpha(Sy_{2n}, y_{2n-1}) \geq 1$ and $\alpha(z, y_{2n-1}) \geq 1$, $\alpha(z, z) \geq 1$, $\alpha(z, Tz) \geq 1$; or*

(c_7) (b) *T is continuous, the pair (g, T) is compatible, and the pair (f, S) is weakly compatible and there exists a sequence $\{y_n\}$ in X such that $\alpha(y_n, y_{n+1}) \geq 1$ and $\alpha(y_{n+1}, y_n) \geq 1$ for all $n \in \mathcal{N}$ and $y_n \to z$ for some $z \in X$, then we have $\alpha(y_{2n}, Ty_{2n-1}) \geq 1$ and $\alpha(y_{2n}, z) \geq 1$, $\alpha(z, z) \geq 1$, $\alpha(Sz, z) \geq 1$;*

(c_8) *if for a non-increasing sequence $\{x_n\}$ in X with $x_n \preceq y_n$ for all $n \in \mathcal{N}$ and $y_n \to u$ for some $u \in X$ implies $x_n \preceq u$ for all $n \in \mathcal{N}$,*

then f, g, S, and T have a common fixed point in X. Further

(c_9) if we assume that $\alpha(u, v) \geq 1$ whenever u and v are common fixed points of f, g, S, and T and the set of common fixed points of f, g, S, and T is well ordered,

then f, g, S, and T have unique common fixed point in X.

The aim of this paper is using alpha-admissible function concept to prove a common fixed point theorem of Suzuki type for two pairs of maps of which only one pair is p_c^*-compatible and one of the maps is continuous in a partial ordered complex partial metric space. We also obtain another common fixed point theorem using closedness of one of the range set of a map instead of p_c^*-compatibility of any pair and continuity of any map. We provide two examples to illustrate our theorems.

Now we give our main results.

3. Main Result

Theorem 3.1. *Let $(X, p_c, \precsim)$ be a partially ordered complete complex partial metric space, $\alpha : X \times X \to \mathbb{R}^+$ be a function, and $f, g, S, T : X \to X$ be mappings satisfying*

(3.1.1) *f, g are dominated and S, T are dominating mappings;*

(3.1.2) *$f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$;*

(3.1.3) $\min\{|p_c(fx, Sx)|, |p_c(gy, Ty)|\} \leq \max\{|p_c(Sx, Ty)|, |p_c(fx, gy)|\}$ *implies*

$$\begin{aligned}\alpha(Sx, Ty)\, p_c(fx, gy) \precsim{}& a_1 p_c(Sx, Ty) + a_2 p_c(Sx, fx) + a_3 p_c(Ty, gy)\\ &+ a_4 p_c(Sx, gy) + a_5 p_c(Ty, fx)\\ &+ a_6 \frac{p_c(Sx, fx)\, p_c(Ty, gy)}{1 + p_c(Sx, Ty) + p_c(fx, gy)} + a_7 \frac{p_c(Sx, gy)\, p_c(Ty, fx)}{1 + p_c(Sx, Ty) + p_c(fx, gy)}\end{aligned}$$

for all comparable elements $x, y \in X$, where $a_i (i = 1, 2, \ldots, 7)$ are non-negative real numbers such that $a_1 + a_2 + a_3 + 2a_4 + 2a_5 + a_6 + a_7 < 1$;

(3.1.4) *the pair (f, g) is α-admissible with respect to the pair (S, T);*

(3.1.5) *$\alpha(Sx_1, fx_1) \geq 1$ and $\alpha(fx_1, Sx_1) \geq 1$ for some $x_1 \in X$;*

(3.1.6) *if for a non-increasing sequence $\{x_n\}$ in X with $y_n \preceq x_n$ for all $n \in \mathbb{N}$ and $y_n \to u$ for some $u \in X$ implies $u \preceq x_n$ for all $n \in \mathbb{N}$;*

(3.1.7) (a) *the pair (f, S) is p_c^* compatible and f or S is continuous. Further assume that $\alpha(Sy_{2n}, y_{2n-1}) \geq 1$ and $\alpha(p, y_{2n-1}) \geq 1$ for all $n \in \mathbb{N}$ and $\alpha(p, p) \geq 1$ whenever there exists a sequence $\{y_n\}$ in X such that $\alpha(y_n, y_{n+1}) \geq 1$ and $\alpha(y_{n+1}, y_n) \geq 1$ for all $n \in \mathbb{N}$ and $y_n \to p$ for some $p \in X$; or*

(3.1.7) (b) *the pair (g, T) is p_c^* compatible and g or T is continuous. Further assume that $\alpha(y_{2n}, Ty_{2n-1}) \geq 1$ and $\alpha(y_{2n}, p) \geq 1$ for all $n \in \mathbb{N}$ and $\alpha(p, p) \geq 1$ whenever there exists a sequence $\{y_n\}$ in X such that $\alpha(y_n, y_{n+1}) \geq 1$ and $\alpha(y_{n+1}, y_n) \geq 1$ for all $n \in \mathbb{N}$ and $y_n \to p$ for some $p \in X$.*

Then $f, g, S,$ and T have a common fixed point in X.

Proof. From (3.1.5), there exists $x_1 \in X$ such that $\alpha(Sx_1, fx_1) \geq 1$ and $\alpha(fx_1, Sx_1) \geq 1$.

From(3.1.2), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n+1} = fx_{2n+1} = Tx_{2n+2}$ and $y_{2n+2} = gx_{2n+2} = Sx_{2n+3},\ n = 0, 1, 2, \ldots$. Now we have

$$\begin{aligned}\alpha(Sx_1, fx_1) \geq 1 &\Rightarrow \alpha(Sx_1, Tx_2) \geq 1, \text{ from the definition of } \{y_n\}\\ &\Rightarrow \alpha(fx_1, gx_2) \geq 1, \text{ from (2.1.4), i.e., } \alpha(y_1, y_2) \geq 1\\ &\Rightarrow \alpha(Tx_2, Sx_3) \geq 1, \text{ from the definition of } \{y_n\}\\ &\Rightarrow \alpha(gx_2, fx_3) \geq 1, \text{ from (2.1.4), i.e., } \alpha(y_2, y_3) \geq 1\\ &\Rightarrow \alpha(Sx_3, Tx_4) \geq 1, \text{ from the definition of } \{y_n\}\\ &\Rightarrow \alpha(fx_3, gx_4) \geq 1, \text{ from (2.1.4), i.e., } \alpha(y_3, y_4) \geq 1.\end{aligned}$$

Continuing in this way, we have

$$\alpha(y_n, y_{n+1}) \geq 1, \forall n \in \mathbb{N}. \tag{3.1}$$

Similarly by using $\alpha(fx_1, Sx_1) \geq 1$ we can show that

$$\alpha(y_{n+1}, y_n) \geq 1, \forall n \in \mathbb{N}. \tag{3.2}$$

From (3.1.1), we have $x_{2n+1} \preceq Sx_{2n+1} = gx_{2n} \preceq x_{2n} \preceq Tx_{2n} = fx_{2n-1} \preceq x_{2n-1}$. Thus

$$x_{n+1} \preceq x_n, \forall n \in \mathbb{N}.$$

Case (i): Suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. From (3.1), $\alpha(Sx_{2n+1}, Tx_{2n+2}) = \alpha(y_{2n}, y_{2n+1}) \geq 1$. From the definition of $\{y_n\}$ we have

$$\begin{aligned}\min\{|p_c(fx_{2n+1}, Sx_{2n+1})|, |p_c(gx_{2n+2}, Tx_{2n+2})|\} &= \min\{|p_c(Sx_{2n+1}, Tx_{2n+2})|, |p_c(gx_{2n+2}, fx_{2n+1})|\}\\ &\leq \max\{|p_c(Sx_{2n+1}, Tx_{2n+2})|, |p_c(gx_{2n+2}, fx_{2n+1})|\}.\end{aligned}$$

From (3.1.3), we have

$$\begin{aligned}p_c(y_{2n+1}, y_{2n+2}) &= p_c(fx_{2n+1}, gx_{2n+2})\\ &\precsim \alpha(Sx_{2n+1}, Tx_{2n+2})\, p_c(fx_{2n+1}, gx_{2n+2})\\ &\precsim a_1 p_c(y_{2n}, y_{2n+1}) + a_2 p_c(y_{2n}, y_{2n+1}) + a_3 p_c(y_{2n+2}, y_{2n+1})\\ &\quad + a_4 p_c(y_{2n}, y_{2n+2}) + a_5 p_c(y_{2n+1}, y_{2n+1})\\ &\quad + a_6 \frac{p_c(y_{2n}, y_{2n+1})\, p_c(y_{2n+2}, y_{2n+1})}{1 + p_c(y_{2n}, y_{2n+1}) + p_c(y_{2n+1}, y_{2n+2})} + a_7 \frac{p_c(y_{2n}, y_{2n+2})\, p_c(y_{2n+1}, y_{2n+1})}{1 + p_c(y_{2n}, y_{2n+1}) + p_c(y_{2n+1}, y_{2n+2})}.\end{aligned}$$

Using

$$p_c(y_{2n}, y_{2n+2}) \precsim p_c(y_{2n}, y_{2n+1}) + p_c(y_{2n+1}, y_{2n+2}) - p_c(y_{2n+1}, y_{2n+1}) \precsim p_c(y_{2n}, y_{2n+1}) + p_c(y_{2n+1}, y_{2n+2})$$

and $p_c(y_{2n+1}, y_{2n+1}) \precsim p_c(y_{2n+1}, y_{2n})$, we get

$$\begin{aligned}|p_c(y_{2n+1}, y_{2n+2})| &\leq a_1 |p_c(y_{2n}, y_{2n+1})| + a_2 |p_c(y_{2n}, y_{2n+1})| + a_3 |p_c(y_{2n+2}, y_{2n+1})|\\ &\quad + a_4 [|p_c(y_{2n}, y_{2n+1})| + |p_c(y_{2n+1}, y_{2n+2})|] + a_5 |p_c(y_{2n+1}, y_{2n})|\\ &\quad + a_6 |p_c(y_{2n}, y_{2n+1})| + a_7 |p_c(y_{2n+1}, y_{2n})|.\end{aligned}$$

Thus

$$|p_c(y_{2n+1}, y_{2n+2})| \leq \left(\frac{a_1 + a_2 + a_4 + a_5 + a_6 + a_7}{1 - a_3 - a_4}\right) |p_c(y_{2n}, y_{2n+1})|. \tag{A}$$

From (3.2), $\alpha(Sx_{2n+1}, Tx_{2n}) = \alpha(y_{2n}, y_{2n-1}) \geq 1$. From the definition of $\{y_n\}$ we have

$$\begin{aligned}\min\{|p_c(fx_{2n+1}, Sx_{2n+1})|, |p_c(gx_{2n}, Tx_{2n})|\} &= \min\{|p_c(fx_{2n+1}, gx_{2n})|, |p_c(Sx_{2n+1}, Tx_{2n})|\}\\ &\leq \max\{|p_c(fx_{2n+1}, gx_{2n})|, |p_c(Sx_{2n+1}, Tx_{2n})|\}.\end{aligned}$$

From (3.1.3), we have

$$\begin{aligned}p_c(y_{2n}, y_{2n+1}) &= p_c(fx_{2n+1}, gx_{2n})\\ &\precsim \alpha(Sx_{2n+1}, Tx_{2n})\, p_c(fx_{2n+1}, gx_{2n})\\ &\precsim a_1 p_c(y_{2n}, y_{2n-1}) + a_2 p_c(y_{2n}, y_{2n+1}) + a_3 p_c(y_{2n-1}, y_{2n})\\ &\quad + a_4 p_c(y_{2n}, y_{2n}) + a_5 p_c(y_{2n-1}, y_{2n+1})\\ &\quad + a_6 \frac{p_c(y_{2n}, y_{2n+1})\, p_c(y_{2n-1}, y_{2n})}{1 + p_c(y_{2n}, y_{2n-1}) + p_c(y_{2n}, y_{2n+1})} + a_7 \frac{p_c(y_{2n}, y_{2n})\, p_c(y_{2n-1}, y_{2n+1})}{1 + p_c(y_{2n}, y_{2n-1}) + p_c(y_{2n}, y_{2n+1})},\\ |p_c(y_{2n}, y_{2n+1})| &\leq a_1 |p_c(y_{2n}, y_{2n-1})| + a_2 |p_c(y_{2n}, y_{2n+1})| + a_3 |p_c(y_{2n-1}, y_{2n})|\\ &\quad + a_4 |p_c(y_{2n}, y_{2n-1})| + a_5 [|p_c(y_{2n-1}, y_{2n})| + |p_c(y_{2n}, y_{2n+1})|]\\ &\quad + a_6 |p_c(y_{2n-1}, y_{2n})| + a_7 |p_c(y_{2n}, y_{2n-1})|.\end{aligned}$$

Thus

$$|p_c(y_{2n}, y_{2n+1})| \leq \left(\frac{a_1 + a_3 + a_4 + a_5 + a_6 + a_7}{1 - a_2 - a_5}\right) |p_c(y_{2n-1}, y_{2n})|. \tag{B}$$

Hence

$$|p_c(y_n, y_{n+1})| \leq h\,|p_c(y_{n-1}, y_n)| \text{ for } n = 2, 3, 4, \ldots,$$

where

$$h = \max\left\{\frac{a_1+a_2+a_4+a_5+a_6+a_7}{1-a_3-a_4}, \frac{a_1+a_3+a_4+a_5+a_6+a_7}{1-a_2-a_5}\right\} < 1.$$

Thus

$$|p_c(y_n, y_{n+1})| \leq h^{n-1}\,|p_c(y_1, y_2)| \text{ for } n = 2, 3, 4, \ldots. \tag{3.3}$$

For $m > n$, using (3.3), we have

$$\begin{aligned}|p_c(y_n, y_m)| &\leq |p_c(y_n, y_{n+1})| + |p_c(y_{n+1}, y_{n+2})| + \cdots + |p_c(y_{m-1}, y_m)| \\ &\leq \left(h^{n-1} + h^n + \cdots + h^{m-2}\right)|p_c(y_1, y_2)| \leq \frac{h^{n-1}}{1-h}\,|p_c(y_1, y_2)| \to 0 \text{ as } n \to \infty, m \to \infty,\end{aligned}$$

which implies that

$$\lim_{m,n\to\infty} p_c(y_n, y_m) = 0. \tag{C}$$

Hence $\{y_n\}$ is a Cauchy sequence in X.

Since (X, p_c) is complete, there exists $z \in X$ such that $y_n \to z$ and

$$p_c(z, z) = \lim_{n\to\infty} p_c(z, y_n) = \lim_{m,n\to\infty} p_c(y_n, y_m) = 0, \text{ from (C)}.$$

Hence

$$p_c(z, z) = \lim_{n\to\infty} p_c(fx_{2n+1}, z) = \lim_{n\to\infty} p_c(gx_{2n+2}, z) = \lim_{n\to\infty} p_c(Sx_{2n+1}, z) = \lim_{n\to\infty} p_c(Tx_{2n+2}, z) = 0. \tag{3.4}$$

Suppose (3.1.7) (a) holds. Since tha pair (f, S) is p_c^* compatible, from (3.4), we have

$$p_c(Sz, Sz) = 0 \tag{3.5}$$

and

$$\lim_{n\to\infty} p_c(fSx_{2n+1}, Sfx_{2n+1}) = 0. \tag{3.6}$$

Since S is continuous at z, from (3.5) we have

$$\lim_{n\to\infty} p_c(SSx_{2n+1}, Sz) = p_c(Sz, Sz) = 0 \tag{3.7}$$

and

$$\lim_{n\to\infty} p_c(Sfx_{2n+1}, Sz) = p_c(Sz, Sz) = 0. \tag{3.8}$$

Also

$$|p_c(fSx_{2n+1}, Sz)| \leq |p_c(fSx_{2n+1}, Sfx_{2n+1})| + |p_c(Sfx_{2n+1}, Sz)|.$$

Letting $n \to \infty$, we get from (3.6) and (3.8) that

$$\lim_{n\to\infty} |p_c(fSx_{2n+1}, Sz)| = 0. \tag{3.9}$$

From (3.9) and (3.7) we get

$$|p_c(fSx_{2n+1}, SSx_{2n+1})| \leq |p_c(fSx_{2n+1}, Sz)| + |p_c(Sz, SSx_{2n+1})| \to 0 \text{ as } n \to \infty. \tag{3.10}$$

Letting $n \to \infty$ and (3.9) and (3.4) in

$$|p_c(fSx_{2n+1}, gx_{2n}) - p_c(Sz, z)| \leq |p_c(fSx_{2n+1}, Sz)| + |p_c(z, gx_n)|$$

we get
$$\lim_{n\to\infty} p_c(fSx_{2n+1}, gx_{2n}) = p_c(Sz, z). \tag{3.11}$$
Letting $n \to \infty$ and (3.7) and (3.4) in
$$|p_c(SSx_{2n+1}, Tx_{2n}) - p_c(Sz, z)| \le |p_c(SSx_{2n+1}, Sz)| + |p_c(z, Tx_{2n})|$$
we get
$$\lim_{n\to\infty} p_c(SSx_{2n+1}, Tx_{2n}) = p_c(Sz, z). \tag{3.12}$$
Letting $n \to \infty$ and (3.7) and (3.5) in
$$|p_c(SSx_{2n+1}, gx_{2n}) - p_c(Sz, z)| \le |p_c(SSx_{2n+1}, Sz)| + |p_c(z, gx_{2n})|$$
we get
$$\lim_{n\to\infty} p_c(SSx_{2n+1}, gx_{2n}) = p_c(Sz, z). \tag{3.13}$$
Letting $n \to \infty$ and (3.9) and (3.4) in
$$|p_c(fSx_{2n+1}, Tx_{2n}) - p_c(z, Sz)| \le |p_c(fSx_{2n+1}, Sz)| + |p_c(z, Tx_{2n})|$$
we get
$$\lim_{n\to\infty} p_c(fSx_{2n+1}, Tx_{2n}) = p_c(z, Sz). \tag{3.14}$$
Suppose $Sz \ne z$. from (3.1.7)(a) $\alpha(SSx_{2n+1}, Tx_{2n}) = \alpha(Sy_{2n}, y_{2n-1}) \ge 1$.

From (3.1.1), we have $Sx_{2n+1} = gx_{2n} \preceq x_{2n}$. Now using (3.1.3), we get, if
$$\min\{|p_c(fSx_{2n+1}, SSx_{2n+1})|, |p_c(gx_{2n}, Tx_{2n})|\} > \max\{|p_c(SSx_{2n+1}, Tx_{2n})|, |p_c(fSx_{2n+1}, gx_{2n})|\},$$
then letting $n \to \infty$, we get $0 \ge |p_c(Sz, z)|$. It is contradiction. Hence
$$\min\{|p_c(fSx_{2n+1}, SSx_{2n+1})|, |p_c(gx_{2n}, Tx_{2n})|\} \le \max\{|p_c(SSx_{2n+1}, Tx_{2n})|, |p_c(fSx_{2n+1}, gx_{2n})|\},$$

$$\begin{aligned}
|p_c(fSx_{2n+1}, gx_{2n})| &\le \alpha(SSx_{2n+1}, Tx_{2n})\,|p_c(fSx_{2n+1}, gx_{2n})| \\
&\le a_1 |p_c(SSx_{2n+1}, Tx_{2n})| + a_2 |p_c(SSx_{2n+1}, fSx_{2n+1})| \\
&\quad + a_3 |p_c(Tx_{2n}, gx_{2n})| + a_4 |p_c(SSx_{2n+1}, gx_{2n})| + a_5 |p_c(Tx_{2n}, fSx_{2n+1})| \\
&\quad + a_6 \frac{|p_c(SSx_{2n+1}, fSx_{2n+1})|\,|p_c(Tx_{2n}, gx_{2n})|}{|1 + p_c(SSx_{2n+1}, Tx_{2n}) + p_c(fSx_{2n+1}, gx_{2n})|} \\
&\quad + a_7 \frac{|p_c(SSx_{2n+1}, gx_{2n})|\,|p_c(Tx_{2n}, fSx_{2n+1})|}{|1 + p_c(SSx_{2n+1}, Tx_{2n}) + p_c(fSx_{2n+1}, gx_{2n})|},
\end{aligned} \tag{3.15}$$

we have
$$\begin{aligned}
|1 + p_c(Sz, z) + p_c(Sz, z)| &\le \left| \begin{array}{l} 1 + p_c(Sz, SSx_{2n+1}) + p_c(SSx_{2n+1}, Tx_{2n}) + p_c(Tx_{2n}, z) \\ +p_c(Sz, fSx_{2n+1}) + p_c(fSx_{2n+1}, gx_{2n}) + p_c(gx_{2n}, z) \end{array} \right| \\
&\le |1 + p_c(SSx_{2n+1}, Tx_{2n}) + p_c(fSx_{2n+1}, gx_{2n})| + |p_c(Tx_{2n}, z)| \\
&\quad + |p_c(Sz, SSx_{2n+1})| + |p_c(Sz, fSx_{2n+1})| + |p_c(gx_{2n}, z)|.
\end{aligned}$$

Letting $n \to \infty$, we get
$$|1 + p_c(Sz, z) + p_c(Sz, z)| \le \lim_{n\to\infty} |1 + p_c(SSx_{2n+1}, Tx_{2n}) + p_c(fSx_{2n+1}, gx_{2n})|$$

from (3.4), (3.7), and (3.9).

Letting $n \to \infty$ in (3.15) and (3.4), (3.10), (3.11), (3.12), (3.13), and (3.14), we get

$$\begin{aligned}|p_c(Sz,z)| &\le a_1|p_c(Sz,z)| + a_2(0) + a_3(0) + a_4|p_c(Sz,z)| + a_5|p_c(Sz,z)| + a_6(0)\\ &\quad + a_7\frac{|p_c(z,Sz)|\,|p_c(z,Sz)|}{|1+p_c(z,Sz)+p_c(z,Sz)|} < (a_1+a_4+a_5+a_7)|p_c(Sz,z)|,\end{aligned}$$

which implies that $Sz = z$.

Suppose $fz \ne z$. Since $gx_{2n} \preceq x_{2n}$, $gx_{2n} \to z$ by (3.1.6), we have $z \preceq x_{2n}$. Also $\alpha(Sz, Tx_{2n}) = \alpha(z, y_{2n-1}) \ge 1$ from (3.1.7) (a). Since $p_c(z,z) = 0$, by Lemma 2.8, we have $p_c(fz,z) = \lim_{n\to\infty} p_c(fz, gx_{2n})$. If $\min\{|p_c(Sz,fz)|, |p_c(gx_{2n},Tx_{2n})|\} > \max\{|p_c(Sz,Tx_{2n})|, |p_c(fz,gx_{2n})|\}$, then letting $n\to\infty$, we get $0 \ge |p_c(fz,z)|$. It is contradiction. Hence

$$\min\{|p_c(Sz,fz)|, |p_c(gx_{2n},Tx_{2n})|\} \le \max\{|p_c(Sz,Tx_{2n})|, |p_c(fz,gx_{2n})|\}.$$

From (3.1.3), we get

$$\begin{aligned}|p_c(fz,gx_{2n})| &\le a_1|p_c(z,Tx_{2n})| + a_2|p_c(z,fz)| + a_3|p_c(Tx_{2n},gx_{2n})|\\ &\quad + a_4|p_c(z,gx_{2n})| + a_5|p_c(Tx_{2n},fz)|\\ &\quad + a_6\frac{|p_c(z,Sz)|\,|p_c(Tx_{2n},gx_{2n})|}{|1+p_c(z,Tx_{2n})+p_c(fz,gx_{2n})|} + a_7\frac{|p_c(z,gx_{2n})|\,|p_c(Tx_{2n},fz)|}{|1+p_c(z,Tx_{2n})+p_c(fz,gx_{2n})|},\end{aligned} \tag{3.16}$$

$$\begin{aligned}|1+p_c(fz,z)| &\le |1+p_c(fz,gx_{2n})+p_c(gx_{2n},Tx_{2n})+p_c(Tx_{2n},z)|\\ &\le |1+p_c(Tx_{2n},z)+p_c(fz,gx_{2n})| + |p_c(gx_{2n},Tx_{2n})|\\ &\le |1+p_c(Tx_{2n},z)+p_c(fz,gx_{2n})| + |p_c(gx_{2n},z)| + |p_c(z,Tx_{2n})|.\end{aligned}$$

Letting $n\to\infty$, we get

$$|1+p_c(fz,z)| \le \lim_{n\to\infty}|1+p_c(Tx_{2n},z)+p_c(fz,gx_{2n})|, \text{ from (3.4)}.$$

Letting $n\to\infty$ in (3.16), we get

$$|p_c(fz,z)| \le a_1(0) + a_2|p_c(fz,z)| + a_3(0) + a_4(0) + a_5|p_c(z,fz)| + a_6(0) + a_7(0) < (a_2+a_5)|p_c(fz,z)|,$$

which implies that $fz = z$. Thus

$$Sz = z = fz. \tag{3.17}$$

Since $f(X) \subseteq T(X)$, there exists a point $w \in X$ such that $fz = Tw$. From (3.1.1), we have $w \preceq Tw = fz = z$. Suppose $z \ne gw$. From (3.1.7) (a), we have $\alpha(Sz,Tw) = \alpha(z,z) \ge 1$, and

$$\begin{aligned}\min\{|p_c(Sz,fz)|, |p_c(gw,Tw)|\} &= \min\{|p_c(z,z)|, |p_c(gw,z)|\}\\ &= 0 \text{ from (3.5)}\\ &< \max\{|p_c(Sz,Tw)|, |p_c(fz,gw)|\}.\end{aligned}$$

From (3.1.3), we have

$$\begin{aligned}|p_c(z,gw)| &= |p_c(fz,gw)|\\ &\le a_1|p_c(z,z)| + a_2|p_c(z,z)| + a_3|p_c(z,gw)| + a_4|p_c(z,gw)|\\ &\quad + a_5|p_c(z,z)| + a_6\frac{|p_c(z,z)|\,|p_c(z,gw)|}{|1+p_c(z,z)+p_c(z,gw)|} + a_7\frac{|p_c(z,gw)|\,|p_c(z,z)|}{|1+p_c(z,z)+p_c(z,gw)|}\\ &\le a_1(0) + a_2(0) + a_3|p_c(z,gw)| + a_4|p_c(z,gw)| + a_5(0) + a_6(0) + a_7(0)\\ &< (a_3+a_4)|p_c(z,gw)|,\end{aligned}$$

which implies that $gw = z = Tw$. Since T is dominating, and g is dominated, we have $w \preceq Tw = z$ and

$z = gw \preceq w$. Hence $z = w$. Thus

$$gz = z = Tz. \tag{3.18}$$

From (3.17) and (3.18), it follows that z is a common fixed point of f, g, S, and T.

Similarly, we can prove theorem when (3.1.7) (b) holds.

Case (ii): Suppose $y_n = y_{n-1}$ for some n. Without loss of generality assume that $n = 2m$. Then $y_{2m} = y_{2m-1}$. From (A) in Case (i), we have $y_{2m} = y_{2m+1}$. Then From (B) in Case (i), we have $y_{2m+1} = y_{2m+2}$. Continuing in this way, we get $y_{2m-1} = y_{2m} = y_{2m+1} = y_{2m+2} = \cdots$. Thus $\{y_n\}$ is a constant Cauchy sequence in X.

The rest of the proof follows as in Case (i). □

Now we give an example to illustrate our Theorem 3.1.

Example 3.2. Let $X = [0, 1]$ and $p_c : X \times X \to \mathbb{C}^+$ be defined by

$$p_c(x, y) = \max\{x, y\} + i \max\{x, y\} \text{ for all } x, y \in X.$$

Let $\preceq$ be the ordinary $\leq$. Let f, g, S, and T be defined by

$$fx = \frac{x}{8},\ \forall\, x \in [0, 1], \qquad gx = \begin{cases} 0, & \text{if } x \in \left[0, \frac{1}{2}\right), \\ \frac{1}{16}, & \text{if } x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

$$Sx = \begin{cases} \frac{3x}{2}, & \text{if } x \in \left[0, \frac{1}{2}\right), \\ x, & \text{if } x \in \left[\frac{1}{2}, 1\right], \end{cases} \qquad Tx = \begin{cases} 2x, & \text{if } x \in \left[0, \frac{1}{2}\right), \\ x, & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Let $\alpha : X \times X \to \mathbb{R}^+$ be defined by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \in X, y = 1, \\ 2, & \text{otherwise.} \end{cases}$$

Then $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$. From the following table it is clear that f, g are dominated and S, T are dominating mappings.

$x \in [0, \frac{1}{2})$	$fx = \frac{x}{8} \leq x$	$gx = 0 \leq x$	$x \leq \frac{3}{2}x = Sx$	$x \leq 2x = Tx$
$x \in [\frac{1}{2}, 1]$	$fx = \frac{x}{8} \leq x$	$gx = \frac{1}{16} < x$	$x = x = Sx$	$x \leq 1 = Tx$

Now we will verify the condition (3.1.3) as follows.

(i) Let $x, y \in [0, \frac{1}{2})$. Then $p_c(Sx, Ty) = p_c(\frac{3x}{2}, 2y) = \max\{(\frac{3x}{2}, 2y)\} + i \max\{(\frac{3x}{2}, 2y)\}$, $\alpha(Sx, Ty) = \alpha(\frac{3x}{2}, 2y) = 1$,

$$\begin{aligned} \alpha(Sx, Ty)p_c(fx, gy) = (1)p_c(\frac{x}{8}, 0) = \left(\frac{x}{8} + i\frac{x}{8}\right) = \frac{1}{12}(\frac{3x}{2} + i\frac{3x}{2}) \\ \precsim \frac{1}{12}[\max\{\frac{3x}{2}, 2y\} + i \max\{\frac{3x}{2}, 2y\}] = \frac{1}{12}p_c(Sx, Ty) \precsim \frac{1}{4}p_c(Sx, Ty). \end{aligned}$$

(ii) Let $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1]$. Then $p_c(Sx, Ty) = p_c(\frac{3x}{2}, 1) = 1 + i$, $\alpha(Sx, Ty) = \alpha(\frac{3x}{2}, 1) = 1$,

$$\alpha(Sx, Ty)p_c(fx, gy) = p_c(\frac{x}{8}, \frac{1}{16}) = \frac{1}{16} + i\frac{1}{16} \precsim \frac{1}{16}(1 + i) = \frac{1}{16}p_c(Sx, Ty) \precsim \frac{1}{4}p_c(Sx, Ty).$$

(iii) Let $x \in [\frac{1}{2}, 1]$ and $y \in [0, \frac{1}{2})$. Then $p_c(Sx, Ty) = p_c(x, 2y) = \max\{x, 2y\} + i \max\{x, 2y\}$, $\alpha(Sx, Ty) = \alpha(x, 2y) = 2$,

$$\alpha(Sx, Ty)p_c(fx, gy) = 2p_c(\frac{x}{8}, 0) = 2(\frac{x}{8} + i\frac{x}{8} = \frac{1}{4}(x + ix) \precsim \frac{1}{4}[\max\{x, 2y\} + i \max\{x, 2y\}] = \frac{1}{4}p_c(Sx, Ty).$$

(iv) Let $x, y \in [\frac{1}{2}, 1]$. Then $p_c(Sx, Ty) = p_c(x, 1) = 1 + i$, $\alpha(Sx, Ty) = \alpha(x, 1) = 1$,

$$\alpha(Sx, Ty)p_c(fx, gy) = p_c(\frac{x}{8}, \frac{1}{16}) = \frac{x}{8} + i\frac{x}{8} \precsim \frac{1}{8}(1 + i) = \frac{1}{8}p_c(Sx, Ty) \precsim \frac{1}{4}p_c(Sx, Ty).$$

Thus (3.1.3) is satisfied with $a_1 = \frac{1}{4}, a_2 = \cdots = a_7 = 0$. Hence f is continuous and g, S, T are discontinuous.

Suppose $p_c(x, x) = 0$. Then $x = 0$ and hence $p_c(Sx, Sx) = p_c(0, 0) = 0$. Thus $p_c(x, x) = 0 \Rightarrow p_c(Sx, Sx) = 0$. Suppose there exists a sequence $\{x_n\}$ in X such that $fx_n \to t$ and $Sx_n \to t$ for some $t \in X$ with $p_c(t, t) = 0$. Then $p_c(t, t) = 0 \Rightarrow t = 0$,

$$fx_n \to t \Rightarrow \lim_{n\to\infty} p_c(fx_n, t) = p_c(t, t) = 0 \Rightarrow \lim_{n\to\infty} p_c(fx_n + ifx_n) = 0 \Rightarrow \lim_{n\to\infty} fx_n = 0 \Rightarrow \lim_{n\to\infty} x_n = 0.$$

Similarly $Sx_n \to t \Rightarrow \lim_{n\to\infty} x_n = 0$. Consider

$$\lim_{n\to\infty} p_c(fSx_n, Sfx_n) = \lim_{n\to\infty} [\max\{fSx_n, Sfx_n\} + i \max\{fSx_n, Sfx_n\}] = 0 \text{ by definitions of } f \text{ and } S.$$

Thus the pair (f, S) is p_c^*-compatible. One can easily verify the remaining conditions. Clearly 0 is a common fixed point of f, g, S, and T.

Remark 3.3. Our main Theorem 3.1 is an improvement of Theorem 2.1 of [27] which is in complex valued metric space.

Now replacing continuity and p_c^* compatibility assumptions with one of $f(X)$, $g(X)$, $S(X)$, and $T(X)$ being a closed subspace of X, we prove the following theorem.

Theorem 3.4. *Assume the conditions* (3.1.1), (3.1.2), (3.1.3), (3.1.4), (3.1.5), *and* (3.1.6) *hold. Further assume the following.*

(3.4.1) (a) *Suppose $S(X)$ is a closed subset of X. Further assume that $\alpha(p, y_{2n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\alpha(p, p) \geq 1$ whenever there exists a sequence $\{y_n\}$ in X such that $\alpha(y_n, y_{n+1}) \geq 1$ and $\alpha(y_{n+1}, y_n) \geq 1$ for all $n \in \mathbb{N}$ and $y_n \to p$ for some $p \in X$; or*

(3.4.1) (b) *suppose $T(X)$ is a closed subset of X. Further assume that $\alpha(y_{2n}, p) \geq 1$ for all $n \in \mathbb{N}$ and $\alpha(p, p) \geq 1$ whenever there exists a sequence $\{y_n\}$ in X such that $\alpha(y_n, y_{n+1}) \geq 1$ and $\alpha(y_{n+1}, y_n) \geq 1$ for all $n \in \mathbb{N}$ and $y_n \to p$ for some $p \in X$.*

Then $f, g, S,$ and T have a common fixed point in X.

Proof. As in Theorem 3.1, there exists a Cauchy sequence $\{y_n\}$ in X such that $y_{2n+1} = fx_{2n+1} = Tx_{2n+2}$ and $y_{2n+2} = gx_{2n+2} = Sx_{2n+3}$, $n = 0, 1, 2, \ldots$, and $y_n \to z \in X$ such that

$$p_c(z, z) = \lim_{n\to\infty} p_c(fx_{2n+1}, z) = \lim_{n\to\infty} p_c(Sx_{2n+1}, z) = \lim_{n\to\infty} p_c(gx_{2n+2}, z) = \lim_{n\to\infty} p_c(Tx_{2n+2}, z) = 0. \quad (3.19)$$

Suppose (3.4.1) (a) holds. Since $S(X)$ is a closed subset of X. Then there exists $u \in X$ such that $z = Su$. Since S is dominating we have $u \preceq Su = z$. Since g is dominated we have $gx_{2n+2} \preceq x_{2n+2}$ and $gx_{2n+2} \to z$ by (3.1.6), $z \preceq x_{2n+2}$. Thus $u \preceq x_{2n+2}$.

Suppose $fu \neq z$, $\alpha(Su, Tx_{2n+2}) = \alpha(z, y_{2n+1}) \geq 1$. If

$$\min\{|p_c(fu, Su)|, |p_c(gx_{2n+2}, Tx_{2n+2})|\} > \max\{|p_c(Su, Tx_{2n+2})|, |p_c(fu, gx_{2n+2})|\},$$

then letting $n \to \infty$, we get $0 \geq |p_c(fu, z)|$. It is contradiction. Hence

$$\min\{|p_c(fu, Su)|, |p_c(gx_{2n+2}, Tx_{2n+2})|\} > \max\{|p_c(Su, Tx_{2n+2})|, |p_c(fu, gx_{2n+2})|\}.$$

Now From (3.1.3), we have

$$\begin{aligned}|p_c(fu, gx_{2n+2})| &\le \alpha(Su, Tx_{2n+2})\, |p_c(fu, gx_{2n+2})| \\ &\le a_1 |p_c(z, Tx_{2n+2})| + a_2 |p_c(z, fu)| + a_3 |p_c(Tx_{2n+2}, gx_{2n+2})| \\ &\quad + a_4 |p_c(z, gx_{2n+2})| + a_5 |p_c(Tx_{2n+2}, fu)| \\ &\quad + a_6 \frac{|p_c(z, fu)|\, |p_c(Tx_{2n+2}, gx_{2n+2})|}{|1 + p_c(z, Tx_{2n+2}) + p_c(fu, gx_{2n+2})|} \\ &\quad + a_7 \frac{|p_c(z, gx_{2n+2})|\, |p_c(Tx_{2n+2}, fu)|}{|1 + p_c(z, Tx_{2n+2}) + p_c(fu, gx_{2n+2})|},\end{aligned} \tag{3.20}$$

$$1 + p_c(fu, z) \precsim 1 + p_c(fu, gx_{2n+2}) + p_c(gx_{2n+2}, Tx_{2n+2}) + p_c(Tx_{2n+2}, z),$$
$$|1 + p_c(fu, z)| \le |1 + p_c(fu, gx_{2n+2}) + p_c(Tx_{2n+2}, z)| + |p_c(gx_{2n+2}, Tx_{2n+2})|.$$

Letting $n \to \infty$ and using (3.21), we get

$$|1 + p_c(fu, z)| \le \lim_{n\to\infty} |1 + p_c(z, Tx_{2n+2}) + p_c(fu, gx_{2n+2})|.$$

Letting $n \to \infty$ in (3.20) and using (3.19), we get

$$|p_c(fu, z)| \le a_1(0) + a_2 |p_c(z, fu)| + a_3(0) + a_4(0) + a_5 |p_c(z, fu)| + a_6(0) + a_7(0) < (a_2 + a_5)\, |p_c(z, fu)|,$$

which in turn yields that $fu = z$. Thus $fu = z = Su$. Since f is dominated and S is dominating maps, we have $z = fu \preceq u$ and $u \preceq Su = z$. Thus $u = z$. Hence

$$fz = z = Sz. \tag{3.21}$$

Since $f(X) \subseteq T(X)$, there exists $v \in X$ such that $z = fz = Tv$. Since T is dominating $v \preceq Tv = z$. Suppose $z \ne gv$. Now $\alpha(Sz, Tv) = \alpha(z, z) \ge 1$,

$$\min\{|p_c(fz, Sz)|, |p_c(gv, Tv)|\} = 0 < \max\{|p_c(Sz, Tv)|, |p_c(fz, gv)|\}, \text{ from (3.4)}.$$

From (3.1.3), we have

$$\begin{aligned}|p_c(z, gv)| = |p_c(fz, gv)| &\le \alpha(Sz, Tv)\, |p_c(fz, gv)| \\ &\le a_1 |p_c(z, z)| + a_2 |p_c(z, z)| + a_3 |p_c(z, gv)| + a_4 |p_c(z, gv)| + a_5 |p_c(z, z)| \\ &\quad + a_6 \frac{|p_c(z, z)|\, |p_c(z, gv)|}{|1 + p_c(z, z) + p_c(z, gv)|} + a_7 \frac{|p_c(z, gv)|\, |p_c(z, z)|}{|1 + p_c(z, z) + p_c(z, gv)|} \\ &< (a_3 + a_4)\, |p_c(z, gv)|,\end{aligned}$$

which in turn yields that $z = gv$. Thus $gv = z = Tv$. Since g is dominated and T is dominating maps, we have $z = gv \preceq v$ and $v \preceq Tv = z$. Thus $v = z$. Hence

$$gz = z = Tz. \tag{3.22}$$

From (3.21) and (3.22), it follows that z is a common fixed point of f, g, S, and T. Similarly, we can prove this theorem when (3.4.1) (b) holds. □

The following example illustrates our Theorem 3.4.

Example 3.5. Let $X = [0,3]$ and $p_c : X \times X \to \mathbb{C}^+$ be defined by $p_c(x,y) = \max\{x,y\} + i\max\{x,y\}$ for all $x, y \in X$. Let $\preceq$ be the ordinary $\leq$. Let f, g, S, and T be defined by

$$fx = \begin{cases} \frac{x}{6}, & \text{if } x \in [0,1)\,, \\ \frac{1}{8}, & \text{if } x \in [1,3]\,, \end{cases} \qquad gx = \begin{cases} 0, & \text{if } x \in [0,1)\,, \\ \frac{1}{8}, & \text{if } x \in [1,3]\,, \end{cases}$$

$$Sx = \begin{cases} 2\sqrt{x}, & \text{if } x \in [0,1)\,, \\ 3, & \text{if } x \in [1,3]\,, \end{cases} \qquad Tx = \begin{cases} 2\sqrt{x}, & \text{if } x \in [0,1)\,, \\ x, & \text{if } x \in [1,3]\,. \end{cases}$$

Let $\alpha : X \times X \to \mathbb{R}^+$ be defined by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } y \in X, x = 3, \\ 2, & \text{otherwise.} \end{cases}$$

Then $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ and $T(X)$ is a closed subset of X. From the following table it is clear that f, g are dominated and S, T are dominating mappings.

$x \in [0,1)$	$fx = \frac{x}{6} \leq x$	$gx = 0 \leq x$	$x \leq 2\sqrt{x} = Sx$	$x \leq 2\sqrt{x} = Tx$
$x \in [1,3]$	$fx = \frac{1}{8} \leq x$	$gx = \frac{1}{8} < x$	$x \leq 3 = Sx$	$x = x = Tx$

Now we will verify the condition (3.1.3) as follows.

(i) Let $x, y \in [0,1]$. Then $p_c(Sx,Ty) = p_c(2\sqrt{x}, 2\sqrt{y}) = \max\{2\sqrt{x}, 2\sqrt{y}\} + i\max\{2\sqrt{x}, 2\sqrt{y}\}$, $\alpha(Sx,Ty) = \alpha(2\sqrt{x}, 2\sqrt{y}) = 2$,

$$\begin{aligned} \alpha(Sx,Ty)p_c(fx,gy) = 2p_c(\frac{x}{6},0) = 2(\frac{x}{6} + i\frac{x}{6}) = \frac{1}{3}\,(x+ix) &\leq \frac{1}{3}(2\sqrt{x} + i2\sqrt{x}) \\ &\precsim \frac{1}{3}[\max\{2\sqrt{x},2\sqrt{y}\} + i\max\{2\sqrt{x},2\sqrt{y}\}] = \frac{1}{3}p_c(Sx,Ty). \end{aligned}$$

(ii) Let $x \in [0,1)$ and $y \in [1,3]$. Then $p_c(Sx,Ty) = p_c(2\sqrt{x}, y) = \max\{2\sqrt{x}, y\} + i\ \max\{2\sqrt{x}, y\}$, $\alpha(Sx,Ty) = \alpha(2\sqrt{x}, y) = 1$,

$$\alpha(Sx,Ty)p_c(fx,gy) = p_c(\frac{x}{6},\frac{1}{8}) = \frac{x}{6} + i\frac{x}{6} \precsim \frac{1}{6}\left[2\sqrt{x} + i2\sqrt{x}\right] \precsim \frac{1}{6}p_c(Sx,Ty) \precsim \frac{1}{3}p_c(Sx,Ty).$$

(iii) Let $x \in [1,3]$ and $y \in [0,1)$. Then $p_c(Sx,Ty) = p_c(3, 2\sqrt{y}) = 3 + 3i$, $\alpha(Sx,Ty) = \alpha(3, 2\sqrt{y}) = 2$,

$$\alpha(Sx,Ty)p_c(fx,gy) = 2p_c(\frac{1}{8},0) = \frac{1}{4} + i\frac{1}{4} = \frac{1}{12}(3+3i) \precsim \frac{1}{3}p_c(Sx,Ty).$$

(iv) Let $x, y \in [1,3]$. Then $p_c(Sx,Ty) = p_c(3,y) = 3 + 3i$, $\alpha(Sx,Ty) = \alpha(3,y) = 1$,

$$\alpha(Sx,Ty)p_c(fx,gy) = p_c(\frac{1}{8},\frac{1}{8}) = \frac{1}{8} + i\frac{1}{8} = \frac{1}{24}(3+3i) \precsim \frac{1}{3}p_c(Sx,Ty).$$

Thus (3.1.3) is satisfied with $a_1 = \frac{1}{3}, a_2 = \cdots = a_7 = 0$. One can easily verify the remaining conditions. Clearly 0 is a common fixed point of f, g, S, and T.

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