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Existence and Uniqueness Solutions of System Caputo-type Fractional-Order Boundary Value Problems Using Monotone Iterative Method

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Abstract

In this paper, we investigate the existence and uniqueness solutions of nonlinear boundary value problems for system of Caputo type nonlinear fractional differential equations of the form:

$$\begin{cases} {}^cD_{a^+}^{q;\psi} u_i(t) = F_i(t, u_1(t), u_2(t)) \quad t \in J = [a, b], \\ \phi(v_i(a), v_i(b)) = 0. \end{cases}$$

To develop a monotone iterative technique by introducing upper and lower solutions to Caputo type fractional differential equations with nonlinear boundary conditions. The monotone method yield monotone sequences which converges to uniformly and monotonically to extremal solutions.

Keywords: ψ-Caputo fractional derivative,upper and lower solutions, monotone iterative method.
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1. Introduction

Fractional differential equations or fractional differential systems have numerous applications in diverse and widespread field of science and technology[4, 10, 22]. The study of fractional calculus and its applications see more details [12, 13, 20]. The approach to obtain existence and uniqueness of solutions for the nonlinear fractional differential systems in general has been through fixed point theorem method [3, 9, 16, 23, 24, 25,

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26]. In this paper to investigate the existence and uniqueness using the method of lower and upper solutions combined with the monotone iterative technique [5, 6, 24, 28, 29].

The monotone method is useful for nonlinear equations and systems because it reduces the problem to sequences of linear equations. Specifically, if the nonlinear system is unwieldy, too difficult to solve explicitly, then the monotone method may be beneficial. If one can find upper and lower solutions to the original system that are less unwieldy and satisfy the particular requirements, then the monotone method implements a technique for constricting sequences from these upper and lower solutions. These sequences are solutions to linear equations and converge uniformly and monotonically to maximal and minimal solutions [11, 12, 14, 16, 17, 18, 19, 25].

Motivated by the work see [8], we determine the existence criteria of extremal solution for following system ψ-Caputo type fractional differential equations in a Caputo sense with nonlinear boundary conditions

$$\begin{cases} {}^cD^{q;\psi}_{a^+}u_i(t) = F_i(t, u_1(t), u_2(t)) \quad t \in J = [a, b], \\ \phi(v_i(a), v_i(b)) = 0. \end{cases} \tag{1.1}$$

The rest of paper is arranged in the following way.
In section 2, definitions and basic results are discussed that plays vital role in the main results. These results are useful in main results proving that the sequences developed in the generalized monotone method converge to coupled minimal and maximal solutions of the non-linear system of fractional differential equation. Finally under uniqueness assumption, we prove that there exists a unique solution to the non-linear system of ψ-Caputo fractional differential equation.

2. Preliminaries

In this section , we recall some known definitions and known results which are useful to develop our main result.

Definition 2.1[1, 4] The ψ-Riemann-Liouville fractional integral of order q is defined by

$$I^{q;\psi}_{a+}u(t) = \frac{1}{\Gamma(q)}\int_0^t \psi'(\psi(t) - \psi(s))^{q-1}u(s)ds, t > a.$$

Definition 2.2[1] Let $\psi, u \in C^n(J, \mathbb{R})$. The ψ-Riemann-Liouville derivative of order of a function u with $(n-1 < q \leq n)$ can be written as

$$\begin{aligned} D^{q;\psi}_{a+}u(t) &= (\frac{D_t}{\psi'(t)})^n I^{n-q;\psi}_{a+}u(t) \\ &= \frac{1}{\Gamma(n-q)}(\frac{D_t}{\psi'(t)})^n \int_0^t \psi'(\psi(t) - \psi(s))^{n-q-1}u(s)ds, \end{aligned}$$

where $n = [q] + 1 (n \in \mathbb{N})$, and $D_t = \frac{d}{dt}$.

Definition 2.2[1] Let $\psi, u \in C^n(J, \mathbb{R})$. The ψ-Caputo derivative of order of a function u with $(n-1 < q \leq n)$ can be written as

$$D^{q;\psi}_{a+}u(t) = I^{n-q;\psi}_{a+}u^{[n]}_{\psi}(t)$$

where $u^{[n]}_{\psi}(t) = (\frac{D_t}{\psi'(t)})^n u(t)$, $n = [q] + 1$ for $q \notin \mathbb{N}$ and $n = q$ for $q \in \mathbb{N}$.

One has

$${}^cD^{q;\psi}_{a^+}u(t) = \begin{cases} \int_0^t \psi'(\psi(t) - \psi(s))^{n-q-1}u^{[n]}_{\psi}(s)ds, & if q \notin \mathbb{N}, \\ u^{[n]}_{\psi}(t), & if q \in \mathbb{N} \end{cases}$$

Definition 2.3[2] One and two parameter Mittag-Leffler function is defined as

$$E_q(t) = \sum_{k=0}^{\infty} \frac{(t)^k}{\Gamma(qk+1)} \quad t \in \mathbb{R}, q > 0$$

$$E_{q,\beta}(t) = \sum_{k=0}^{\infty} \frac{(t)^k}{\Gamma(qk+\beta)} \quad q, \beta > 0, t \in \mathbb{R}$$

Lemma 2.1[1] Let $p, q > 0$, and $u \in C(J, \mathbb{R})$, for every $t \in J$

i. ${}^cD_{a^+}^{q;\psi} I_{a^+}^{q;\psi} u(t) = u(t)$,

ii. $I_{a^+}^{q;\psi} {}^cD_{a^+}^{q;\psi} u(t) = u(t) - u(a)$, $0 < q \leq 1$.

iii. $I_{a^+}^{q;\psi} (\psi(t) - \psi(a))^{p-1} = \frac{\Gamma(p)}{\Gamma(p-q)} (\psi(t) - \psi(a))^{p+q-1}$,

iv. ${}^cD_{a^+}^{q;\psi} (\psi(t) - \psi(a))^{p-1} = \frac{\Gamma(p)}{\Gamma(p-q)} (\psi(t) - \psi(a))^{p-q-1}$,

v. ${}^cD_{a^+}^{q;\psi} (\psi(t) - \psi(a))^k = 0$, $\forall k < n \in \mathbb{N}$

Lemma 2.2[27] Let $q \in (0, 1)$ and $x \in \mathbb{R}$, one has i. $E_{q,1}$ and $E_{q,q}$ are non-negative. ii. $E_{q,1}(x) \leq 1, E_{q,q}(x) \leq \frac{1}{\Gamma(q)}$, for any $x < 0$.

Lemma 2.3[8] Let $q \in (0, 1)$, $\lambda \in \mathbb{R}$ and $g \in C(J, \mathbb{R})$, then the linear problem

$$\begin{cases} {}^cD_{a^+}^{q;\psi} u(t) + \lambda u(t) = g(t), & t \in J. \\ u(a) = u_a, \end{cases}$$

has a unique solution as

$$\begin{aligned} u(t) &= u_a E_{q,1}(-\lambda(\psi(t) - \psi(a))^q) \\ &+ \int_0^t \psi'(\psi(t) - \psi(s))^{q-1} E_{q,q}(-\lambda(\psi(t) - \psi(a))^q) g(s) ds, \end{aligned}$$

where $E_{p,q}(.)$ is the two parametric Mittag-Leffler function

Lemma 2.4 [Comprising result][8] Let $q \in (0, 1)$ and $\lambda \in \mathbb{R}$ if $\gamma \in C(J, \mathbb{R})$,

$$\begin{cases} {}^cD_{a^+}^{q;\psi} \gamma(t) \geq -\lambda \gamma(t), & t \in (a, b]. \\ \gamma(a) \geq 0, \end{cases}$$

then $\gamma(t) \geq 0$ for all $t \in J$.

3. Main Result

In this section, we develop a monotone method for the system ψ-Caputo fractional differential equations (3.7) using coupled lower and upper solutions respectively.

Defination 3.1 The functions $f_i \in C(J, \mathbb{R}$ such that ${}^cD_{a^+}^{q;\psi} f_i(t)$ exist and is continuous on J and is known to be a solutions (1.1).Further,f_i gives the statistics of the equation ${}^cD_{a^+}^{q;\psi} u_i(t) = F_i(t, u_1(t), u_2(t))$, for each $t \in J$ and the nonlinear boundary conditions

$$\phi(f_i(a), f_i(b)) = 0$$

Definition 3.2 If the functions $v_i(x,t)$, $w_i(x,t) \in C^{2,q}[Q_T, \mathbb{R}]$ are called the lower and upper solutions of if

$$\begin{cases} {}^cD_{a^+}^{q;\psi} v_i(t) \leq F_i(t, v_1(t), v_2(t)) & t \in [a, b], \\ \phi(v_i(a), v_i(b)) \leq 0 \end{cases}$$

$$\begin{cases} {}^cD^{q;\psi}_{a^+}w_i(t) \geq F_i(t, w_1(t), w_2(t)) & t \in [a,b], \\ \phi(w_i(a), w_i(b)) \geq 0 \end{cases}$$

Theorem 3.1 Let $F : J \times \mathbb{R} \to \mathbb{R}$ be continuous. Assume that
(i) There exist $v_i(t)$ and $w_i(t)$ as lower and upper solutions of problem (1.1) in $C(J,\mathbb{R})$ respectively, with $v_i(t) \leq w_i(t), \quad t \in J$.
(ii) There exist a constant $k_i > 0$ with

$$F_i(t, u_2) - F_i(t, u_1) \geq -k_i(u_2 - u_1) \quad for \quad v_i(t) \leq u_1 \leq u_2 \leq w_i(t), t \in J$$

(iii) There exists nonnegative constants M, N with $v_i(a) \leq x_1 \leq x_2 \leq w_i(a)$, $v_i(b) \leq y_1 \leq y_2 \leq w_i(b)$, such that

$$\phi_i(x_2, y_2) - \phi_i(x_1, y_1 \leq M(x_2 - x_1) - N(y_2 - y_1)$$

Then there exist monotone sequences $\{v_i^n(t)\}$ and $\{w_i^n(t)\}$ such that $v_i^n(t) \to v_i(t)$ and $w_i^n(t) \to w_i(t)$ as $n \to \infty$ uniformly on J, to the extremal solutions of (1.1) in the sector $[v_i, w_i]$ where

$$[v_i, w_i] = \{u_i \in C(J,\mathbb{R}) : v_i(t) \leq u_i(t) \leq w_i(t), t \in J\}$$

Proof.
We construct the sequences $\left\{v_i^{n+1}(t)\right\}$ and $\left\{w_i^{n+1}(t)\right\}$ and $k_i > 0$, we consider the following fractional differential equations

$$\begin{cases} {}^cD^{q;\psi}_{a^+}v_i^{n+1}(t) = F_i(t, v_i^n(t)) - k(v_i^{n+1}(t) - v_i^n(t)) & t \in J, \\ v_i^{n+1}(a) = v_i^n(a) - \frac{1}{c}\phi(v_i^n(a), v_i^n(b)) \end{cases} \tag{3.1}$$

$$\begin{cases} {}^cD^{q;\psi}_{a^+}w_i^{n+1}(t) = F_i(t, w_i^n(t)) - k(w_i^{n+1}(t) - w_i^n(t)) & t \in J, \\ w_i^{n+1}(a) = w_i^n(a)) - \frac{1}{c}\phi(w_i^n(a), w_i^n(b)) \end{cases} \tag{3.2}$$

By Lemma 3 and equation (3.1),(3.2)preserve at most one solution in $C(J,\mathbb{R}$ we have

$$v_i^{n+1}(t) = \left(v_i^n(a)) - \frac{1}{c}\phi(v_i^n(a), v_i^n(b))\right) E_{q,1}(-k_i((\psi)_i(t) - \psi_i(a))^q)$$
$$+ \int_a^t \psi_i'(s)(\psi_i(t) - \psi_i(a))^{q-1}E_{q,q}(-k_i(\psi_i(t) - \psi_i(s))^q)(F_i(s, v_i^n(s)) + k_i(v_i^{n+1}(s)))ds \quad t \in J,$$

$$w_i^{n+1}(t) = \left(w_i^n(a)) - \frac{1}{c}\phi(w_i^n(a), w_i^n(b))\right) E_{q,1}(-k_i((\psi)_i(t) - \psi_i(a))^q)$$
$$+ \int_a^t \psi_i'(s)(\psi_i(t) - \psi_i(a))^{q-1}E_{q,q}(-k_i(\psi_i(t) - \psi_i(s))^q)(F_i(s, w_i^n(s)) + k_i(w_i^{n+1}(s)))ds \quad t \in J.$$

Step 1: The sequences $\left\{v_i^{n+1}(t)\right\}, \left\{w_i^{n+1}(t)\right\}$ $(n \geq 1)$ are lower and upper solutions of () respectively. We prove that $v_i^0(t) \leq v_i^1(t)$. Let $\rho_i(t) = v_i^1(t) - v_i^0(t)$. Then equation (3.1) and Definition 3.2, we have

$$\begin{aligned} {}^cD^{q;\psi}_{a^+}\rho_i(t) &= {}^c D^{q;\psi}_{a^+}v_i^1(t) - {}^c D^{q;\psi}_{a^+}v_i^0(t) \\ &\geq F_i(t, v_i^0(t)) - k_i(v_i^1(t) - v_i^0(t)) - F_i(t, v_i^0(t))) \\ &= -k_i\rho_i(t). \end{aligned}$$

Since $\rho_i(a) = -\frac{1}{c}\phi(v_i^0(a), v_i^0(b)) \geq 0$, $\rho_i(t) \geq 0$, for $t \in J$ by Lemma 4. Thus $v_i^0(t) \leq v_i^1(t)$. Assume that $v_i^{k-1}(t) \leq v_i^k(t)$. Now we show that $v_i^k(t) \leq v_i^{k+1}(t)$. Let $\rho_i(t) = v_i^k(t) - v_i^{k+1}(t)$

$$\begin{aligned}{}^cD_{a^+}^{q;\psi}\rho_i(t) &= {}^c D_{a^+}^{q;\psi} v_i^k(t) - {}^c D_{a^+}^{q;\psi} v_i^{k+1}(t) \\ &\geq F_i(t, v_i^k(t)) - k_i(v_i^{k+1}(t) - v_i^k(t)) - F_i(t, v_i^k(t))) \\ &= -k_i\rho_i(t).\end{aligned}$$

Since $\rho_i(a) = -\frac{1}{c}\phi(v_i^k(a), v_i^k(b)) \geq 0$, $\rho_i(t) \geq 0$, for $t \in J$ by Lemma 4. Thus $v_i^k(t) \leq v_i^{k+1}(t)$. Hence by mathematical induction, we have

$$v_i^0(t) \leq v_i^1(t) \leq ... \leq v_i^k(t) \leq v_i^{k+1}(t) \leq ... \leq v_i^n(t) \tag{3.3}$$

Next, we prove that $w_i^1(t) - w_i^0(t)$, $t \in J$. Let $\rho_i(t) = w_i^0(t) - w_i^1(t)$. Then equation (3.1) and Definition 3.2, we have

$$\begin{aligned}{}^cD_{a^+}^{q;\psi}\rho_i(t) &= {}^c D_{a^+}^{q;\psi} w_i^0(t) - {}^c D_{a^+}^{q;\psi} w_i^1(t) \\ &\geq F_i(t, w_i^1(t)) - k_i(w_i^0(t) - w_i^1(t)) - F_i(t, w_i^1(t)) \\ &= -k_i\rho_i(t).\end{aligned}$$

Since $\rho_i(a) = -\frac{1}{c}\phi(w_i^0(a), w_i^0(b)) \geq 0$, $\rho_i(t) \geq 0$, for $t \in J$ by Lemma 4. Thus $w_i^1(t) \leq w_i^0(t)$. Assume that $w_i^k(t) \leq w_i^{k-1}(t)$. Now we show that $w_i^{k+1}(t) \leq w_i^k(t)$. Let $\rho_i(t) = w_i^{k+1}(t) - w_i^k(t)$

$$\begin{aligned}{}^cD_{a^+}^{q;\psi}\rho_i(t) &= {}^c D_{a^+}^{q;\psi} w_i^{k+1}(t) - {}^c D_{a^+}^{q;\psi} w_i^k(t) \\ &\leq F_i(t, w_i^{k+1}(t)) - k_i(w_i^{k+1}(t) - w_i^k(t)) - F_i(t, w_i^{k+1}(t))) \\ &= -k_i\rho_i(t).\end{aligned}$$

Since $\rho_i(a) = -\frac{1}{c}\phi(w_i^{k+1}(a), w_i^{k+1}(b)) \geq 0$, $\rho_i(t) \geq 0$, for $t \in J$ by Lemma 4. Thus $w_i^{k+1}(t) \leq w_i^k(t)$. Hence by mathematical induction, we have

$$w_i^n(t) \leq w_i^{n-1}(t) \leq ... \leq w_i^k(t) \leq w_i^{k-1}(t) \leq ... \leq w_i^1(t) \leq w_i^0(t) \tag{3.4}$$

Now to Prove that $v_i^1(t) \leq w_i^1(t)$. Let $\rho_i(t) = w_i^1(t) - v_i^1(t)$. Using equation (3.1) and (3.2) together with assumption (ii) and (iii) we have

$$\begin{aligned}{}^cD_{a^+}^{q;\psi}\rho_i(t) &= F_i(t, w_i^0(t)) - F_i(t, v_i^0(t)) - k_i(w_i^1(t) - w_i^0(t)) + k_i(v_i^1(t) - v_i^0) \\ &\geq -k_i(w_i^0(t) - v_i^0(t)) - k_i(w_i^1(t) - w_i^0(t)) + k_i(v_i^1(t) - v_i^0(t)) \\ &= -k_i\rho_i(t).\end{aligned}$$

Since

$$\begin{aligned}\rho_i(a) &= (w_i^0(a) - v_i^0(t)) - \frac{1}{c}(\phi(w_i^0(a), w_i^0(b)) - \phi(v_i^0(a), v_i^0(b))) \\ &\geq \frac{d}{c}((w_i^0(b) - v_i^0(b) \\ &\geq 0,\end{aligned}$$

we have $v_i^1(t) \leq w_i^1(t)$, $t \in J$ by Lemma 4. Hence $v_i^0(t) \leq v_i^1(t) \leq w_i^1(t) \leq w_i^0(t)$.

By mathematical inductions and equations (3.3),(3.4) we get

$$v_i^0(t) \leq v_i^1(t) \leq ... \leq v_i^n(t) \leq w_i^n(t) \leq ... \leq w_i^1(t) \leq w_i^0(t) \tag{3.5}$$

We prove that $v_i^0(t), w_i^0(t)$ are extremum solutions of (1.1). Since v_i^0 and w_i^0 are lower and upper solutions of (1.1),assumptions (ii)and (iii), we get

$$\begin{aligned} {}^cD_{a^+}^{q;\psi} v_i^0(t) &= F_i(t, v_i^0(t)) - k_i(v_i^1(t) - v_i^0(t)) \\ &\leq F_i(t, v_i^1(t)) \end{aligned}$$

and

$$\begin{aligned} \phi(v_i^1(a), v_i^1(b)) &\leq \phi(v_i^0(a), v_i^0(b) + c(v_i^1(a) - v_i^0(a)) - d(v_i^1(b) - v_i^0(b)) \\ &= -d(v_i^1(b) - v_i^0(b)) \\ &\leq 0. \end{aligned}$$

$$\begin{aligned} {}^cD_{a^+}^{q;\psi} w_i^0(t) &= F_i(t, w_i^0(t)) - k_i(w_i^1(t) - w_i^0(t)) \\ &\geq F_i(t, w_i^1(t)) \end{aligned}$$

and

$$\begin{aligned} \phi(w_i^1(a), w_i^1(b)) &\geq \phi(w_i^0(a), w_i^0(b) + c(w_i^1(a) - w_i^0(a)) - d(w_i^1(b) - w_i^0(b)) \\ &= -d(w_i^1(b) - w_i^0(b)) \\ &\geq 0. \end{aligned}$$

Therefore, $v_i^1(t)$,$w_i^1(t)$ is lower and upper solution of (1.1)respectively. By induction,Hence $v_i^n(t), w_i^n(t)$ are lower and upper solutions of (1.1) respectively.

Step 2:$v_i^n \to v_i$ and $w_i^n \to w_i$

First, we prove that $\{v_i^n\}$ is uniformly bounded. By considering supposition Hypothesis 2, we have

$$F_i(t, v_i^0(t)) + k_i v_i^0(t) \leq F_i(t, v_i^n(t)) + k_i v_i^n(t) \leq F_i(t, w_i^0(t)) + k_i w_i^0(t), \quad t \in J$$

That is

$$\begin{aligned} 0 &\leq F_i(t, v_i^n(t)) - F_i(t, v_i^0(t)) + k_i(v_i^n(t) - v_i^0(t)) \\ &\leq F_i(t, w_i^0(t)) - F_i(t, v_i^0(t)) + k_i(w_i^0(t) - v_i^0(t)) \end{aligned}$$

Hence, we have

$$\begin{aligned} |F_i(t, v_i^n(t)) - F_i(t, v_i^0(t)) + k_i(v_i^n(t) - v_i^0(t))| \leq |F_i(t, w_i^0(t)) - F_i(t, v_i^0(t)) \\ + k_i(w_i^0(t) - v_i^0(t))|. \end{aligned}$$

Thus

$$\begin{aligned} |F_i(t, v_i^n(t)) + k_i(v_i^n(t)| &\leq |F_i(t, v_i^n(t)) - F_i(t, v_i^0(t)) + k_i(v_i^n(t) - v_i^0(t))| \\ &+ |F_i(t, v_i^0(t)) + k_i(v_i^0(t)| \\ &\leq |F_i(t, w_i^0(t)) - F_i(t, v_i^0(t)) + k_i(w_i^0(t) - v_i^0(t))| \\ &+ |F_i(t, v_i^0(t)) + k_i(v_i^0(t)| \\ &+ \leq 2|F_i(t, v_i^0(t)) + k_i(v_i^0(t)| + |F_i(t, v_i^0(t)) + k_i(v_i^0(t)|. \end{aligned}$$

Since v_i^0, F_i are continuous on J, we can see a constant C independent of n with

$$|F_i(t, v_i^n(t)) + k_i(v_i^n(t)| \leq C \tag{3.6}$$

Furthermore, from Hypothesis 3, we have

$$v_i^0(a) - \frac{1}{c}\phi(v_i^0(a), v_i^0(b)) \leq v_i^n(a) - \frac{1}{c}\phi(w_i^0(a), w_i^0(b)) \leq w_i^0(a) - \frac{1}{c}\phi(v_i^0(a), v_i^0(b))$$

That is

$$\begin{aligned} 0 &\leq v_i^n(a) - v_i^0(a) - \frac{1}{c}\phi(v_i^n(a), v_i^n(b)) - \phi(v_i^0(a), v_i^0(b)) \\ &\leq w_i^0(a) - v_i^0(a) - \frac{1}{c}\phi(w_i^n(0), w_i^0(b)) - \phi(v_i^0(a), v_i^0(b)). \end{aligned}$$

Hence, we have

$$\begin{aligned} &|v_i^n(a) - v_i^0(a) - \frac{1}{c}\phi(v_i^n(a), v_i^n(b)) - \phi(v_i^0(a), v_i^0(b))| \\ &\leq |v_i^n(a) - v_i^0(a) - \frac{1}{c}\phi(v_i^n(a), v_i^n(b)) - \phi(v_i^0(a), v_i^0(b))| \\ &\leq |v_i^n(a) - v_i^0(a) - \frac{1}{c}\phi(v_i^n(a), v_i^n(b)) - \phi(v_i^0(a), v_i^0(b))|. \end{aligned}$$

Thus

$$\begin{aligned} |v_i^n(a) - \frac{1}{c}\phi(v_i^n(a), v_i^n(b))| &\leq |v_i^n(a) - v_i^0(a) - \frac{1}{c}\phi(v_i^n(a), v_i^n(b)) - \phi(v_i^0(a), v_i^0(b))| \\ &+ |v_i^0(a) - \frac{1}{c}\phi(v_i^0(a), v_i^0(b))| \\ &\leq 2|v_i^0(a) - \frac{1}{c}\phi(v_i^0(a), v_i^0(b))| + |w_i^0(a) - \frac{1}{c}\phi(w_i^0(a), w_i^0(b))|.- \end{aligned}$$

Since v_i^0, w_i^0 and ϕ are continuous functions, we can see a constant D independent of n with

$$|v_i^n(a) - \frac{1}{c}\phi(v_i^n(a), v_i^n(b))| \leq D \tag{3.7}$$

Moreover, by (3.1) and (3.2) we have

$$\begin{aligned} |v_i^{n+1}(t)| &= |v_i^n(a) - \frac{1}{c}\phi(v_i^n(a), v_i^n(b))| E_{q,1}(-k_i(\psi(t) - \psi(a))^q) \\ &+ \int_a^t \psi'(s)(\psi(t) - \psi(s))^{v-1} E_{q,q}(-k_i(\psi(t) - \psi(s)))^q |F(s, v_i^n(s) + k_i v_i^n(t)| ds, \end{aligned}$$

Using Lemma 2 along with (3.6) and (3.7), we have

$$\begin{aligned} |v_i^{n+1}(t)| &= D + \frac{C}{\Gamma(q)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{v-1} ds, \\ &\leq D + \frac{C(\psi(t) - \psi(s))^q}{\Gamma(q+1)}. \end{aligned}$$

Hence, v_i^n is uniformly bounded in $C(J, \mathbb{R})$. Similarly w_i^n is uniformly bounded $C(J, \mathbb{R})$. Next to prove that the sequence v_i^n and w_i^n are equi-continuous on J. Choosing $t_1, t_2 \in J$, with $t_1 \leq t_2$. By (3.6),(3.7) and

Lemma 2, we have

$$\begin{aligned}
|v_i^{n+1}(t_2) - v_i^{n+1}(t_1)| &\le |v_i^n(a) - \frac{1}{c}\phi(v_i^n(a), v_i^n(b))||E_{q,1}(-k_i(\psi(t_2) - \psi(a))^q) \\
&- E_{q,1}(-k_i(\psi(t_1) - \psi(a))^q)| \\
&\le \int_a^{t_1} \frac{\psi^{'}(s)[(\psi(t_1) - \psi(s))^{v-1} - (\psi(t_2) - \psi(s))^{v-1}]}{\gamma(q)} |F(s, v_i^n(s) + k_i v_i^n(s)|ds \\
&+ \int_{t_1}^{t_2} \frac{\psi^{'}(s)[(\psi(t_2) - \psi(s))^{v-1}]}{\Gamma(q)} |F(s, v_i^n(s) + k_i v_i^n(s)|ds \\
&\le D|E_{q,1}(-k_i(\psi(t_2) - \psi(a))^q) - E_{q,1}(-k_i(\psi(t_1) - \psi(a))^q)| \\
&+ \frac{2C(\psi(t_2) - \psi(t_1))^q}{\Gamma(q+1)}.
\end{aligned}$$

By the continuity of $E_{q,1}(-k_i(\psi(t_1) - \psi(a))^q)$ on J, the right-hand-side of the preceding inequality approaches zero, when $t_1 \to t_2$. This implies that $\{v_i^{n+1}(t)\}$ is equi-continuous on J. Similarly $\{w_i^{n+1}(t)\}$ is equi-continuous on J. Hence, by using Ascoli-Arzelas theorem, the subsequences converges to $v_i^*(t)$ and $w_i^*(t)$. Hence the monotonic sequences combined with $v_i^n(t)$ and $w_i^n(t)$ yields $\lim_{n\to\infty} v_i^n(t) = v_i^*(t)$ and $\lim_{n\to\infty} w_i^n(t) = w_i^*(t)$, uniformly on $t \in J$ and limit functions v_i^*, w_i^* satisfy (1.1)

Step 3: v_i^* and w_i^* are maximal solutions of (1.1)in $[v_i^0, w_i^0]$ Let $u_i \in [v_i^0, w_i^0]$ be any solution of (1.1). Suppose that

$$v_i^n(t) \le u_i(t) \le w_i^n(t), \quad t \in J \tag{3.8}$$

for some $n \in \mathbb{N}$. To prove that $u_i(t) \le v_i^n(t)$ Let $\rho_i(t) = u_i(t) - v_i^n(t)$. Then from we have

$$\begin{aligned}
{}^cD_{a^+}^{q;\psi} \rho_i(t) &= F_i(t, u_i(t)) - F_i(t, v_i^0(t)) - k_i(v_i^{n+1}(t) - v_i^n(t)) \\
&\ge -k_i(_i(t) - v_i^n(t)) + k_i(v_i^{n+1}(t) - v_i^n(t)) \\
&= -k_i\rho_i(t).
\end{aligned}$$

Furthermore

$$\begin{aligned}
v_i^{n+1}(a) &= (v_i^n(a) - \frac{1}{c}(\phi(v_i^n(a), v_i^n(b)) \\
&= (v_i^n(a) - \frac{1}{c}(\phi(u_i(a), u_i(b)) - \frac{1}{c}(\phi(v_i^n(a), v_i^n(b)) \\
&\le u_i(a) - \frac{d}{c}((u_i(b) - v_i^n(b)) \\
&\le u_i(a)
\end{aligned}$$

that is $\rho_i \ge 0$. By Lemma 4, we have $\rho_i \ge 0$, $t \in J$ which implies that

$$v_i^n(t) \le u_i(t), \quad t \in J$$

Next we prove that $w_i^{n+1}(t) \le u_i(t)$ Let $\rho_i(t) = w_i^{n+1}(t) \le u_i(t)$. Then from we have

$$\begin{aligned}
{}^cD_{a^+}^{q;\psi} \rho_i(t) &= F_i(t, w_i^{n+1}(t)) - F_i(t, u_i(t)) - k_i(u_i(t) - u_i(t)) \\
&\ge -k_i(w_i 6n(t) - u_i(t)) + k_i(u_i(t) - u_i(t)) \\
&= -k_i\rho_i(t).
\end{aligned}$$

Furthermore

$$\begin{aligned}u_i(a) &= (u_i(a) - \frac{1}{c}(\phi(u_i(a), u_i(b))\\ &= (u_i(a) - \frac{1}{c}(\phi(w_i^n(a), w_i^n(b)) - \frac{1}{c}(\phi(u_i(a), u_i(b))\\ &\leq w_i^n(a) - \frac{d}{c}((w_i^n(b) - u_i(b))\\ &\leq w_i^n(a)\end{aligned}$$

that is $\rho_i \geq 0$. By Lemma 4, we have $\rho_i \geq 0$, $t \in J$ which implies that

$$u_i(t) \leq w_i^n(t), \quad t \in J$$

Hence,

$$v_i^n(t) \leq u_i(t) \leq w_i^n(t), \quad t \in J$$

By (4.8) is satisfied on J for all $n \in \mathbb{N}$.For $n \to \infty$ on (3.8) we have

$$v_i^* \leq u_i \leq w_i^*.$$

Hence v_i^*, w_i^* are the extremal solutions of (1.1)in $[v_i^0, w_i^0]$

Theorem 3.2 Let all the assumptions of the Theorem 3.1 hold.Further,there exists nonnegative constants M, N such that the function f_i satisfies the condition

$$f_i(x, u_1, u_2) - f_i(x, v_1, v_2) \leq M(u_1 - v_1) + N((u_2 - v_2),$$

for $v_i^0(t) \leq u_i \leq w_i^0(t)$. Then the problem $u_i(t)$ of (1.1) has a unique solution.

Proof. We know $v_i^0(t) \leq w_i^0(t)$ on J. It is sufficient to prove that $v_i(t)^0 \geq w_i^0(t)$ on J. Consider $\rho_i(t) = w_i^0(t) - v_i^0(t)$.Then we have

$$\begin{aligned}{}^cD_{a^+}^{q;\psi}\rho_i(t) &= F_i(t, w_i^0(t)) - F_i(t, v_i^0(t)) - k_i(w_i^0(t) - w_i^0(t)) + k_i(v_i^0(t) - v_i^0)\\ &\geq -k_i(w_i^0(t) - v_i^0(t)) - k_i(w_i^0(t) - w_i^0(t)) + k_i(v_i^0(t) - v_i^0(t))\\ &= -k_i\rho_i(t).\end{aligned}$$

Since

$$\begin{aligned}\rho(a) &= (w_i^0(a) - v_i^0(t)) - \frac{1}{c}(\phi(w_i^0(a), w_i^0(b)) - \phi(v_i^0(a), v_i^0(b)))\\ &\geq \frac{d}{c}((w_i^0(b) - v_i^0(b)\\ &\geq 0,\end{aligned}$$

we have $w_i^0(t) \geq v_i^0(t)$, $t \in J$. By Lemma 4, we know $p_i \geq 0$, implying that $w_i^0(t) \geq v_i^0(t)$ on J. Hence $v_i(t) = u_i(t) = w_i(t)$

4. Conclusion

In this work, initially we have investigate by using monotone iterative method together with upper and lower solutions for boundary value problems involving a generalized system of Caputo derivative of fractional order. The monotone method yields monotone sequences which converges uniformly and monotonically to extremal(maximal and minimal) solutions of (1.1). We have prove that the uniqueness solution of $u_i(t)$ of the system.

Conflict of Interest

The authors have no conflict of interest regarding the publication of this article.

References

[1] Almeida,R.A. Caputo fractional derivative of a function with respect to another functions. Commun.Nonlinear Sci. Numerical Simul. 2017,44,460-481. 2

[2] A.A.Kilbas, F.Mainardi and S.V.Rogosin, Mittaf-Leffer Functions, Related Topic and Applications. Springer Monographs in Mathematics. 2014. 2

[3] A.A.Kilbas and S. A. Marzan, Cauchy problem for differential equation with Caputo derivative. Fract. Calc. Appl. Anal.,7(3),(2004),297-321. 1

[4] A.A.Kilbas, H.M.Srivastava and J.J.Trujillo, Theory and Applications of Fractional Differential Equations, Amsterdam, Elsevier, 2006. 1, 2

[5] A.S.Vatsala and Donna Stutson, Generalized Monotone Method For Fractional Reaction-Diffusion Equation. Communications in Applied Analysis,16(2)(2012)165-174 1

[6] A.S.Vatsala and Donna Stutson , A representations solution is obtained for the one dimentional Caputo fractional reaction-diffusion equation. Dynamic system and applictions 6(2012). 1

[7] Benchohra, M. Cabada, A. Initial value problem for nonlinear fractional differential equations with ψ-Caputo derivative via monotone iterative technique. Axioms 2020, 9, 57.

[8] Choukri Derbazi, Zidane Baitiche, Mohammed S. Abbo, Kamal Shah, Bahaaeldian Abdalla Extrmal Solution of Generalized Caputo-Type Fractional-Order Boundary Value Problem Using Monotone Iterative Method Fractal Fract.2022,6,146. 1, 2

[9] Dhaigude D.B. Nanware J.A., Monotone Technique for Weakly Coupled System of Caputo Fractional Differential Equations with Periodic Boundary Conditions, Dyn.Contin.Discrete Impuls. Syst.Ser.A Math.Anal. 19(2012),575 - 584. 1

[10] I.Podlubny, Fractional Differential Equations. Academics Press. 198,San Diego,1999. 1

[11] J.D.Ramirez and A.S.Vatsala, Generalized Monotone Iterative Technique for Caputo Fractional Differential Equations with Periodic Boundary Conditions via Initial Value Problem. International Journal of Differential Equations. 17(2012),842-843 1

[12] Jagdish A Nanware and Pandurang D Kundgar, Generalized Monotone Method for System of Riemann-Liouville Fractional Reaction-Diffusion Equations, Communication in Nonlinear Analysis.(2021),1-20

[13] Jagdish A Nanware and Pandurang D Kundgar, Existence and Uniqueness of Solutions for System of Fractional Reaction Diffusion Equations [Communicated] 1

[14] J.A.Nanware and D.B. Dhaigude, Monotone Technique for Finite Weakly Coupled System of Caputo Fractional Differential Equations with Periodic Boundary Conditions, Dyn.Contin.Discrete Impuls.Syst. Ser.A Math.Anal. 22(1)(2015),13-23. 1

[15] J.A.Nanware and D.B. Dhaigude, Existence of Uniqueness of Solutions of Riemann- Liouville Fractional Differential Equations with Integral Bounadry Conditions.International Journal of Nonlinear Science. 14(4)(2012),410 - 415. 1

[16] J.A.Nanware and D.B. Dhaigude, System of initial value problems involving Riemann-Liouville sequential fractional derivative, Communications in Applied Analysis. 22(3)(2018),353-368.

[17] J.A.Nanware,N.B.Jadhav and D.B. Dhaigude, Monotone iterative technique for finite system of Riemann-Liouville fractional differential equations with integral boundary conditions, Proc.Internal.Conf.on Mathematical Sciences,India,2014,pp. 235- 238. 1

[18] J.A.Nanware,N.B.Jadhav and D.B. Dhaigude, Initial value problems for fractional differential equations involving Riemann-Liouville derivative, Malaya J. Mat. 5(2)(2017) 337-345. 1

[19] J.A.Nanware,N.B. Jadhav and D.B.Dhaigude, Existence and uniqueness of solutions of nonlinear implicit fractional differential equations, Dyn, Contin.Discrete Impuls.Syst.Ser.A: Math.Anal. 27(4)(2020) 275-282. 1

[20] K.B. Oldham, J.spanier, The Fractional Calculus, Dover Publications,INC,New York,2002. 1

[21] Ladde G.S., Lakshmikntham, V. and Vatsala, A.S. Monotone Iterative Techniques for Nonliner Differential Equations, Pitman, New York,1985. 1

[22] M.Sowmya and A.S.Vatsala, Generalized Monotone Method for Caputo Fractional Differential Equations via Coupled Lower and Upper Solutions with Superlinear Convergence. Nonlinear Dynamics and System theory. 15(2)(2015),198-208.

[23] R.Hilfer. Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000. 1

[24] V.Lakshmikantham S.Leela and D.J.Vasundhara Devi, Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers,2009 1

[25] V.Lakshmikantham and A.S.Vatsala, General Monotone Method For fractional Reaction-Diffusion Equations. Communications in Applied Analysis. 16(2012) 165-174. 1

[26] V.Lakshmikantham Theory of fractional functional differential equations, Nonlinear Anal.69(2008),3337 - 3343. 1

[27] Wei, Z.; Li, Q.; Che, J. Initial value problems for fractional differential equations involving Riemann-Liouville sequential fractional derivative. J. Math. Anal. Appl. 2010, 367, 260272. 1

[28] Z.Denton and A.S.Vatsala, Monotone Iterative Technique For Finite System of Non- linear Riemann- Liouville Fractional Differential Equations. Opuscula Mathematica. 31(3)(2011),327-339. 2

[29] Z.Denton P.W.Ngand A.S.Vatsala, Quasilinearization Method Via Lower and Upper Solutions for Riemann-Liouville Fractional Differential Equations. Nonlinear Dynamics and System Theory. 11(3)(2011),239-251. 1