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Some L_P- Type Inequalities for Polar Derivative of a polynomial

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Abstract

In this paper, we shall prove some L^p inequalities for the polar derivative of a polynomial having zeros in $|z| \leq k \leq 1$ and thereby obtain generalizations and refinements of an integral inequality due to Barchand Charam et al.

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1. Introduction

Let $\mathcal{P}_n$ denote the space of all algebraic polynomials of the form $P(z) = \sum\limits_{\nu=0}^{n} a_\nu z^\nu$ of degree n and let $P'(z)$ be its derivative. The study of inequalities for different norms of derivatives of a univariate complex polynomial in terms of the polynomial norm is a classical topic in analysis. A classical inequality that provides an estimate to the size of the derivative of a given polynomial on the unit disk, relative to size of the polynomial itself on the same disk is the famous Bernstein inequality [6]. It states that: if $P(z)$ is a polynomial of degree n, then on $|z| = 1$,

$$\max_{|z|=1} \left|P'(z)\right| \leq n \max_{|z|=1} \left|P(z)\right| \tag{1.1}$$

Equality holds in (1) iff $P(z)$ has all its zeros at the origin. Concerning the estimation of the lower bound of $Re\left(\frac{zP'(z)}{P(z)}\right)$ on $|z| = 1$, Dubinin[7] proved the following result.

Theorem 1.1: If $P \in \mathcal{P}_n$ having all its zeros in $|z| \leq 1$, then for all z on $|z| = 1$ for which $P(z) \neq 0$

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$$Re\left(\frac{zP'(z)}{P(z)}\right) \geq \frac{n}{2}\left(1+\frac{1}{n}\left(\frac{|a_n|-|a_0|}{|a_n|+|a_0|}\right)\right). \tag{1.2}$$

As a generalization of Theorem 1.1, N. A. Rather etal [11] have proved the following result.

Theorem 1.2: If $P \in \mathcal{P}_n$ having all its zeros in $|z| \leq k$, $k \leq 1$, then for all z on $|z| = 1$ for which $P(z) \neq 0$

$$Re\left(\frac{zP'(z)}{P(z)}\right) \geq \frac{n}{1+k}\left(1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right). \tag{1.3}$$

Also concerning the maximum of $|P'(z)|$ in terms of maximum of $|P(z)|$ on $|z| = 1$, Turán [12] showed that if $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1.4}$$

Equality in inequality (4) holds for those polynomials $P \in \mathcal{P}_n$ which have all their zeros on $|z| = 1$. Dubinin [7] used the boundary Schwarz lemma due to Osserman [10] to obtain an interesting refinement of (4), in fact, proved that if all the zeros of $P \in \mathcal{P}_n$ lie in $|z| \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{2}\left(1+\frac{1}{n}\left(\frac{|a_n|-|a_0|}{|a_n|+|a_0|}\right)\right)\max_{|z|=1}|P(z)|. \tag{1.5}$$

As a generalization of inequality (4) Malik[8] proved the following result.

Theorem 1.3: If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq k \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{1+k}\max_{|z|=1}|P(z)| \tag{1.6}$$

the result is sharp and best possible for $P(z) = (z + k)^n$. It is natural to look for a similar refinement of inequality (6), as the Dubinin[7] refined the inequality (5). In this direction N.A.Rather et al[11] proved the result which is refinement of inequality (6) and generalization of inequality (5). Infact, they proved

Theorem 1.4: If $P(z) = z^s(a_0 + a_1z + \cdots + a_{n-s}z^{n-s}) \in \mathcal{P}_n$, $0 \leq s \leq n$, has all its zeros in $|z| \leq k$, $k \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{1+k}\left(1+\frac{k}{n}\left(s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\right)\max_{|z|=1}|P(z)|. \tag{1.7}$$

The result is sharp and equality holds for $P(z) = z^s(z + k)^{n-s}$, $s < n$. On setting $s = 0$ in Theorem 1.4 they proved the following result,

Corollary 1.5: If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq k \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{1+k}\left(1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right)\max_{|z|=1}|P(z)|. \tag{1.8}$$

The result is sharp and equality holds for $P(z) = (z + k)^n$.
The polar derivative $D_\alpha P(z)$ of $P \in \mathcal{P}_n$ with respect to the point $\alpha \in \mathbb{C}$ is defined by

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha}=P'(z) \tag{1.9}$$

uniformly for $|z|\le R, R>0$.

A.Aziz [1] and Aziz and Rather[2] obtained several estimates for maximum modulus of $D_\alpha P(z)$ on $|z|=1$ and among other thing , they [3] extended inequality (6) to the polar derivative. Infact, they proved

Theorem 1.6: If $P\in\mathcal{P}_n$ and $P(z)$ has all zeros in $|z|\le k\le 1$, then for $\alpha\in C$ with $|\alpha|\ge k$,

$$\max_{|z|=1}|D_\alpha P(z)|\ge n\frac{|\alpha|-k}{1+k}\max_{|z|=1}|P(z)|. \tag{1.10}$$

The result is sharp and best possible for $P(z)=(z+k)^n$. with $|\alpha|\ge k$. Next N. A. Rather et al [11] establish the following refinement of inequality (10). Infact, they proved

Theorem 1.7: If $P(z)=z^s(a_0+a_1z+\cdots+a_{n-s}z^{n-s})\in\mathcal{P}_n$, $0\le s\le n$, has all its zeros in $|z|\le k$, $k\le 1$, then for $\alpha\in C$ with $|\alpha|\ge k$,

$$\max_{|z|=1}|D_\alpha P(z)|\ge n\frac{|\alpha|-k}{1+k}\left(1+\frac{k}{n}\left(s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\right)\max_{|z|=1}|P(z)|. \tag{1.11}$$

The result is sharp and equality holds for $P(z)=z^s(z+k)^{n-s}$, $s<n$, with $|\alpha|\ge k$. On setting $s=0$ in Theorem 1.7 they proved the following result,

Corollary 1.8: If $P\in\mathcal{P}_n$ and $P(z)$ has all zeros in $|z|\le k\le 1$, then for $\alpha\in C$ with $|\alpha|\ge k$,

$$\max_{|z|=1}|D_\alpha P(z)|\ge n\frac{|\alpha|-k}{1+k}\left(1+\frac{k}{n}\left(\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\right)\max_{|z|=1}|P(z)|. \tag{1.12}$$

The result is sharp and equality holds for $P(z)=(z+k)^n$, with $|\alpha|\ge k$. Recently Barchand Charam et-al [5] extended inequality (5) to its integral analogous for polar derivative of a polynomial. Infact, they proved

Theorem 1.9: If $P\in\mathcal{P}_n$ and $P(z)$ has all zeros in $|z|\le 1$, then for $\alpha\in C$ with $|\alpha|\ge 1$, and $p>0$

$$\begin{aligned}\left\{\int_0^{2\pi}\left|D_\alpha P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}}\ge&\frac{n(|\alpha|-1)}{2}\left(1+\frac{1}{n}\left(\frac{|a_n|-|a_0|}{|a_n|+|a_0|}\right)\right)\\&\times\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}}.\end{aligned} \tag{1.13}$$

In the same they proved more generalized result,

Theorem 1.10: If $P\in\mathcal{P}_n$ and $P(z)$ has all zeros in $|z|\le 1$, then for $\alpha\in C$ with $|\alpha|\ge 1$, $0\le t<1$ and $p>0$

$$\begin{aligned}\left\{\int_0^{2\pi}\left(\left|D_\alpha P\left(e^{i\theta}\right)\right|-mnt|\alpha|\right)^p d\theta\right\}^{\frac{1}{p}}\ge&\frac{n(|\alpha|-1)}{2}\left(1+\frac{1}{n}\left(\frac{|a_n|-tm-|a_0|}{|a_n|-tm+|a_0|}\right)\right)\\&\times\left\{\int_0^{2\pi}\left(\left|P\left(e^{i\theta}\right)-tm\right)^p d\theta\right\}^{\frac{1}{p}}.\end{aligned} \tag{1.14}$$

where $m=min_{|z|=1}P(z)$

2. Main results

In this paper, we shall prove some L^p inequalities for polynomials with polar derivative. We shall first prove a result that generalizes Theorem 1.9 as well as extends inequality (11) to its integral analogous and there by obtain more results which extended the already proved results to integral analogous. First, we prove the following result, which is the corresponding L^p extension of Theorem 1.7.

Theorem 2.1: If $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s}) \in \mathcal{P}_n$, $0 \le s \le n$, has all its zeros in $|z| \le k$, $k \le 1$, then for $\alpha \in C$ with $|\alpha| \ge k$, and for $p > 0$

$$\left\{\int_0^{2\pi} \left|D_\alpha P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \ge n\frac{|\alpha|-k}{1+k}\left(1+\frac{k}{n}\left(s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\right) \times \left\{\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}}. \tag{2.1}$$

The result is sharp and equality holds for $P(z) = z^s(z+k)^{n-s}$, $s < n$, with $|\alpha| \ge k$. On setting $s = 0$ in Theorem 2.1 we proved the following result which is infact the generalization of Theorem 1.9.

Corollary 2.2: If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \le k \le 1$, then for $\alpha \in C$ with $|\alpha| \ge k$, and for $p > 0$

$$\left\{\int_0^{2\pi} \left|D_\alpha P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \ge n\frac{|\alpha|-k}{1+k}\left(1+\frac{k}{n}\left(\frac{k^{n}|a_{n}|-|a_0|}{k^{n}|a_{n}|+|a_0|}\right)\right) \times \left\{\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}}. \tag{2.2}$$

The result is sharp and equality holds for $P(z) = (z+k)^n$, with $|\alpha| \ge k$.

Remark 1 : For $k = 1$ inequality (2.2) reduces to (1.12) and if we let $p \to \infty$ the inequality (15) and inequality (16) reduces to (10) and (11).

The following interesting results are obtained from Theorem 2.1 and Corollary 2.2 by dividing both sides of (15) and (16) by $|\alpha|$ and letting $|\alpha| \to \infty$

Corollary 2.3: If $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s}) \in \mathcal{P}_n$, $0 \le s \le n$, has all its zeros in $|z| \le k$, $k \le 1$, then for $p > 0$

$$\left\{\int_0^{2\pi} \left|P'(z)\right|^p d\theta\right\}^{\frac{1}{p}} \ge \frac{n}{1+k}\left(1+\frac{k}{n}\left(s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\right) \times \left\{\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}}. \tag{2.3}$$

The result is sharp and equality holds for $P(z) = z^s(z+k)^{n-s}$, $s < n$. On setting $s = 0$ in Corollary 2.3 we prove the following result

Corollary 2.4: If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq k \leq 1$, then for $p > 0$

$$\left\{ \int_0^{2\pi} \left|P'(z)\right|^p d\theta \right\}^{\frac{1}{p}} \geq \frac{n}{1+k}\left(1 + \frac{k}{n}\left(\frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\right) \times \left\{ \int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^p d\theta \right\}^{\frac{1}{p}}. \tag{2.4}$$

The result is sharp and equality holds for $P(z) = (z+k)^n$. Next, we prove the result which is generalization as well as refinement of the Theorem 1.10 and Theorem 2.1.

Theorem 2.5: If $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s}) \in \mathcal{P}_n$, $0 \leq s \leq n$, has all its zeros in $|z| \leq k$, $k \leq 1$, then for $\alpha \in C$ with $|\alpha| \geq k$, and for $p > 0$

$$\left\{ \int_0^{2\pi} \left(|D_\alpha P\left(e^{i\theta}\right)| - mnt|\alpha|\right)^p d\theta \right\}^{\frac{1}{p}} \geq n\frac{|\alpha| - k}{1+k}\left(1 + \frac{k}{n}\left(s + \frac{k^{n-s}|a_{n-s}| - k^{n-s}mt - |a_0|}{k^{n-s}|a_{n-s}| - k^{n-s}mt + |a_0|}\right)\right) \times \left\{ \int_0^{2\pi} \left(|P\left(e^{i\theta}\right)| - tm\right)^p d\theta \right\}^{\frac{1}{p}}. \tag{2.5}$$

Where $m = min_{|z|=k}P(z)$

If we let $p \to \infty$ we obtain a result which is refinement of Theorem 1.7. Infact, we prove

Corollary 2.6: If $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s}) \in \mathcal{P}_n$, $0 \leq s \leq n$, has all its zeros in $|z| \leq k$, $k \leq 1$, then for $\alpha \in C$ with $|\alpha| \geq k$,

$$\max_{|z|=1}\left(|D_\alpha P(z)| - mnt|\alpha|\right) \geq n\frac{|\alpha| - k}{1+k}\left(1 + \frac{k}{n}\left(s + \frac{k^{n-s}|a_{n-s}| - k^{n-s}mt - |a_0|}{k^{n-s}|a_{n-s}| - k^{n-s}mt + |a_0|}\right)\right) \times \max_{|z|=1}\left(|P(z)| - tm\right). \tag{2.6}$$

Where $m = min_{|z|=k}P(z)$.

Remark 2 : For $t = 0$ Theorem 2.5 and Corollary 2.7 reduces to Theorem 2.1 and Theorem 1.7.

3. Lemmas

For the proof of the following Theorems we need the following lemmas. The first lemma is a special case of a result due to Aziz and Rather [4].

Lemma 1: If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq k \leq 1$, then for $|z| = 1$

$$\left|Q'(z)\right| \leq k\left|P'(z)\right| \tag{3.1}$$

Where $Q(z) = z^n\overline{P\left(1/\overline{z}\right)}$.

The Second lemma is due N.A. Rather et al [11]

Lemma 2: If $P(z) = z^s(a_0 + a_1 z + \cdots + a_{n-s}z^{n-s}) \in \mathcal{P}_n$, $0 \leq s \leq n$, has all its zeros in $|z| \leq k$, $k \leq 1$, then for all z on $|z| = 1$ for which $P(z) \neq 0$

$$Re\left(\frac{zP'(z)}{P(z)}\right) \geq \frac{n}{1+k}\left(1 + \frac{k}{n}\left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right) \tag{3.2}$$

4. Proof of Main Results

Proof of Theorem 2.1 : Let $Q(z) = z^n\overline{P(1/\overline{z})}$, then $P(z) = z^n\overline{Q(1/\overline{z})}$ and it can be easily verified that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)| \tag{4.1}$$

and

$$|P'(z)| = |nQ(z) - zQ'(z)| \tag{4.2}$$

Since $P(z)$ has all its zeros in $|z| \leq k, \quad k \leq 1$, it follows by Lemma 1 , for $|z| = 1$

$$\begin{aligned} k|P'(z)| &\geq Q'(z) \\ &= |nP(z) - zP'(z)| \end{aligned} \tag{4.3}$$

Now for every $\alpha \in C$ with $|\alpha| \geq k$,

$$\begin{aligned} |D_\alpha P(z)| &\geq |\alpha||P'(z)| - |Q'(z)| \\ &\geq |\alpha||P'(z)| - |nP(z) - zP'(z)| \end{aligned}$$

Using (25), we get for $|z| = 1$

$$|D_\alpha P(z)| \geq (|\alpha| - k)|P'(z)|. \tag{4.4}$$

For any $p > 0$ and $0 \leq \theta < 2\pi$, we have from (26)

$$|D_\alpha P\left(e^{i\theta}\right)|^p \geq (|\alpha| - k)^p|P'\left(e^{i\theta}\right)|^p.$$

Which is equivalent to

$$\left\{\int_0^{2\pi}\left|D_\alpha P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \geq (|\alpha| - k)\left\{\int_0^{2\pi}\left|P'\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}}.$$

By Lemma 2, for all z on $|z| = 1$ for which $P(z) \neq 0$

$$Re\left(\frac{zP'(z)}{P(z)}\right) \geq \frac{n}{1+k}\left(1 + \frac{k}{n}\left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right)$$

Which implies for all z on $|z| = 1$ for which $P(z) \neq 0$

$$\left|\frac{zP'(z)}{P(z)}\right| \geq Re\left(\frac{zP'(z)}{P(z)}\right) \geq \frac{n}{1+k}\left(1 + \frac{k}{n}\left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right)$$

which gives, for all z on $|z| = 1$

$$\left|P'(z)\right| \geq Re\left(\frac{zP'(z)}{P(z)}\right) \geq \frac{n}{1+k}\left(1 + \frac{k}{n}\left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right)\left|P(z)\right| \tag{4.5}$$

Further, it is evident that inequality (27) follows trivially for those z on $|z| = 1$ at which $P(z) = 0$ as well.

Also from (27), for any $p > 0$ and $0 \leq \theta < 2\pi$, we have

$$\begin{aligned} \left\{\int_0^{2\pi}\left|P'(z)\right|^p d\theta\right\}^{\frac{1}{p}} \geq & \frac{n}{1+k}\left(1 + \frac{k}{n}\left(s + \frac{k^{n-s}|a_{n-s}| - |a_0|}{k^{n-s}|a_{n-s}| + |a_0|}\right)\right) \\ & \times\left\{\int_0^{2\pi}\left|P(z))\right|^p d\theta\right\}^{\frac{1}{p}} \end{aligned} \tag{4.6}$$

Combining (26) and (28), we get

$$\left\{\int_0^{2\pi}\left|D_\alpha P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \geq n\frac{|\alpha|-k}{1+k}\left(1+\frac{k}{n}\left(s+\frac{k^{n-s}|a_{n-s}|-|a_0|}{k^{n-s}|a_{n-s}|+|a_0|}\right)\right) \times\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}}.$$

That completes the proof of Theorem 2.1. □

Proof of Theorem 2.5 : Let $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $|z| \leq k \leq 1$. If $P(z)$ has a zero on $|z| = k$, then $m = min_{|z|=k}P(z) = 0$ and the result follows from Theorem 2.1 in this case. Henceforth, we suppose that all the zeros of $P(z)$ lie in $|z| < k$ so that $m > 0$. Since $m \leq |P(z)|$ for $|z| = 1$, therefore if β is any complex number with $|\beta| \leq 1$, then for $|z| = 1$ we have

$$|m\beta z^n| < |P(z)| \tag{4.7}$$

Since all the zeros of $P(z)$ are $|z| \leq k \leq 1$, it follows by Rouche's Theorem all the zeros of $P(z) - m\beta z^n$ are $|z| \leq k \leq 1$. Hence, by Theorem 2.1, we have for $|\alpha| \geq k$ and for any $p > 0$,

$$\left\{\int_0^{2\pi}\left|D_\alpha P\left(e^{i\theta}\right)-\beta mn\alpha\left(e^{i(n-1)\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \geq n\frac{|\alpha|-k}{1+k}\left(1+\frac{k}{n}\left(s+\frac{k^{n-s}|a_{n-s}-\beta m|-|a_0|}{k^{n-s}|a_{n-s}-\beta m|+|a_0|}\right)\right) \times\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)-\beta m\left(e^{in\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}}. \tag{4.8}$$

Since the function $\frac{x-|a_0|}{x+|a_0|}$ is non decreasing function of x, we have

$$\frac{k^{n-s}|a_{n-s}-\beta m|-|a_0|}{k^{n-s}|a_{n-s}-\beta m|+|a_0|} \geq \frac{k^{n-s}|a_{n-s}|-k^{n-s}|\beta|m-|a_0|}{k^{n-s}|a_{n-s}|-k^{n-s}|\beta|m+|a_0|} \tag{4.9}$$

Also by triangles inequality , for $|z| = 1$, we have

$$\left|P(z)-\beta mz^n\right| \geq \left|P(z)\right|-|\beta|m \tag{4.10}$$

Applying the argument of (31) and (32) to (30) respectively , we have

$$\left\{\int_0^{2\pi}\left|D_\alpha P\left(e^{i\theta}\right)-\beta mn\alpha\left(e^{i(n-1)\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}} \geq n\left(\frac{|\alpha|-k}{1+k}\right) \times\left(1+\frac{k}{n}\left(s+\frac{k^{n-s}|a_{n-s}|-k^{n-s}|\beta|m-|a_0|}{k^{n-s}|a_{n-s}|-k^{n-s}|\beta|m+|a_0|}\right)\right)\left\{\int_0^{2\pi}\left(\left|P\left(e^{i\theta}\right)\right|-|\beta|m\right)^p d\theta\right\}^{\frac{1}{p}}. \tag{4.11}$$

It is a simple consequence of Laguerre Theorem [[9], p.52] on the polar derivative of polynomial that for every α with $|\alpha| \geq 1$, the polynomial

$$D_\alpha\left(P(z)-\beta mz^n\right) = D_\alpha P(z)-\beta mn\alpha z^{n-1} \tag{4.12}$$

has all zeros in $|z| \leq k \leq 1$. This implies that, for $|z| \geq 1$

$$|D_\alpha P(z)| \geq \beta mn|\alpha||z|^{n-1} \tag{4.13}$$

Now choosing the argument of β suitably on the left hand side of (33) such that for $|z| = 1$

$$\left|D_\alpha P(z) - \beta mn\alpha z^{n-1}\right| = \left|D_\alpha P(z)\right| - |\beta| mn|\alpha| \tag{4.14}$$

which is possible by (35), we get

$$\left\{\int_0^{2\pi}\left(\left|D_\alpha P\left(e^{i\theta}\right)\right| - |\beta| mn|\alpha|\right)^p d\theta\right\}^{\frac{1}{p}} \geq n\left(\frac{|\alpha| - k}{1+k}\right)$$
$$\times\left(1 + \frac{k}{n}\left(s + \frac{k^{n-s}|a_{n-s}| - k^{n-s}|\beta| m - |a_0|}{k^{n-s}|a_{n-s}| - k^{n-s}|\beta| m + |a_0|}\right)\right)\left\{\int_0^{2\pi}\left(\left|P\left(e^{i\theta}\right)\right| - |\beta| m\right)^p d\theta\right\}^{\frac{1}{p}}. \tag{4.15}$$

Put $|\beta| = t$ in (36), we get

$$\left\{\int_0^{2\pi}\left(|D_\alpha P\left(e^{i\theta}\right) - mnt|\alpha|\right)^p d\theta\right\}^{\frac{1}{p}} \geq n\frac{|\alpha| - k}{1+k}\left(1 + \frac{k}{n}\left(s + \frac{k^{n-s}|a_{n-s}| - k^{n-s}mt - |a_0|}{k^{n-s}|a_{n-s}| - k^{n-s}mt + |a_0|}\right)\right)$$
$$\times\left\{\int_0^{2\pi}\left(|P\left(e^{i\theta}\right)| - tm\right)^p d\theta\right\}^{\frac{1}{p}}. \tag{4.16}$$

Where $0 \leq t < 1$ and this completes proof of Theorem 2.5. □

Conflict of Interest

The authors have no conflict of interest regarding the publication of this article.

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