Available online at http://www.cna-journal.com
Commun. Nonlinear Anal. 1 (2019), 58–72
Research Article

Communications in Nonlinear Analysis
Journal Homepage: www.cna-journal.com

Global Existence of Solutions for A Gierer–Meinhardt System with Two Activators and Two Inhibitors

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Abstract

This paper deals with a Gierer-Meinhardt model with 2 activators and 2 inhibitors described by a reaction–diffusion system with fractional reactions. The purpose of this paper is to prove the existence of a global solution. Our technique is based on a suitable Lyapunov functional.

Keywords: Reaction–diffusion system, Gierer-Meinhardt, Global existence of solutions, Lyapunov functional.
2010 MSC: 35K45, 35K57.

1. Introduction

Due to their interesting behavior and overall dynamics, Gierer–Meinhardt type systems have attracted a lot of attention in last decade. The original system was proposed in 1972 [8] relating to a biological phenomenon discovered by A. Trembley in 1744 [6] and named morphogenesis. Morphogenesis is the study of pattern formation in spatial biological tissue structures, see [15].The system is of the form

$$\begin{cases} u_t - a_1\Delta u = \sigma - \mu u + \frac{u^p}{v^q} \\ v_t - a_2\Delta v = -\nu v + \frac{u^r}{v^s} \end{cases} \qquad \text{for all } x \in \Omega\ , t > 0, \tag{1.1}$$

with the boundary and initial conditions given by

$$\begin{cases} \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, & x \in \partial\Omega, t > 0, \\ u\,(x,0) = \varphi_1(x) > 0, v\,(x,0) = \varphi_2(x) > 0, & x \in \Omega, \end{cases} \tag{1.2}$$

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Received received date

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $a_1, a_2 > 0$, $\mu, \nu, \sigma > 0$, and the indices p, q, r and s are non negative with $p > 1$. A comprehensive summary related to the application of activator–inhibitor type systems in the theory of phyllotaxis, i.e. biological pattern formation in plants, was provided by Meinhardt et al. in [12]. The model (1.1) was studied by Rothe in 1984 [14], who established the global existence of solutions for the specific case where $p = 2,\ q = 1,\ r = 2,\ s = 0$ and $N = 3$. Later, Wu and Li [16] achieved the same result for u, v^{-1} and σ sufficiently small. The boundedness of solutions for (1.1) independent of the initial values was established Mingde *et al.* in 1995 [13] subject to

$$\frac{p-1}{r} < \min\left(\frac{q}{s+1}, 1\right). \tag{1.3}$$

Masuda and Takahashi [11] extended the previous results to the generalized case:

$$\begin{cases} u_t - a_1 \Delta u = \sigma_1(x) - \mu u + \rho_1(x,u) \dfrac{u^p}{v^q}, \\ v_t - a_2 \Delta v = \sigma_2(x) - \nu v + \rho_2(x,u) \dfrac{u^r}{v^s}, \end{cases} \tag{1.4}$$

with $\sigma_1, \sigma_2 \in C^1(\overline{\Omega})$, $\sigma_1 \geq 0,\ \sigma_2 \geq 0$, and $\rho_1, \rho_2 \in C^1\left(\overline{\Omega} \times \mathbb{R}_+^2\right) \cap L^\infty\left(\overline{\Omega} \times \mathbb{R}_+^2\right)$ satisfying $\rho_1 \geq 0,\ \rho_2 > 0$. In 2006, Jiang [10] showed that system (1.4) admits a unique nonnegative global solution (u, v) satisfying (1.2) under the same conditions in (1.3) and with $\varphi_1, \varphi_2 \in W^{2,l}(\Omega)$, $l > \max\{N, 2\}$, $\frac{\partial \varphi_1}{\partial \eta} = \frac{\partial \varphi_2}{\partial \eta} = 0$ on $\partial\Omega$, and $\varphi_1 \geq 0, \varphi_2 > 0$ in $\overline{\Omega}$.

Abdelmalek *et al.* [2] used a Lyapunov functional to show the global existence of solutions for a 3–substance phyllotaxis Gierer Meinhardt system of the form

$$\begin{cases} u_t - a_1 \Delta u = \sigma - b_1 u + \dfrac{u^{p_1}}{v^{q_1}(w^{r_1} + c)}, \\ v_t - a_2 \Delta v = -b_2 v + \dfrac{u^{p_2}}{v^{q_2} w^{r_2}}, \\ w_t - a_3 \Delta w = -b_3 w + \dfrac{u^{p_3}}{v^{q_3} w^{r_3}}, \end{cases} \tag{1.5}$$

for $\sigma > 0$, $c \geq 0$, and

$$0 < p_1 - 1 < \max\left\{p_2 \min\left(\frac{q_1}{q_2+1}, \frac{r_1}{r_2}, 1\right), p_3 \min\left(\frac{r_1}{r_3+1}, \frac{q_1}{q_3}, 1\right)\right\}. \tag{1.6}$$

. This work was extended to m–components in [1]. Another more recent and related study is that of Henine et al. [5] in 2015, in which the boundedness of solutions and large time behavior was established for a more generalized version of (1.5).

Building on the work carried out in [2, 1, 5], the aim of this study is to study the nature of solutions for a 2–activator–2–inhibitor Gierer–Meinhardt type system by means of a suitable Lyapunov functional. We consider the 4–component system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - a_1 \Delta u_1 = f_1 = \sigma_1 - b_1 u_1 + \dfrac{u_1^{p_{11}} u_2^{p_{12}}}{v_1^{p_{13}} v_2^{p_{14}}}, \\ \dfrac{\partial u_2}{\partial t} - a_2 \Delta u_2 = f_2 = \sigma_2 - b_2 u_2 + \dfrac{u_1^{p_{21}} u_2^{p_{22}}}{v_1^{p_{23}} v_2^{p_{24}}} + a(u_1 - v_1 + b), \\ \dfrac{\partial v_1}{\partial t} - a_3 \Delta v_1 = g_1 = -b_3 v_1 + \dfrac{u_1^{p_{31}} u_2^{p_{32}}}{v_1^{p_{33}} v_2^{p_{34}}}, \\ \dfrac{\partial v_2}{\partial t} - a_4 \Delta v_2 = g_2 = -b_4 v_2 + \dfrac{u_1^{p_{41}} u_2^{p_{42}}}{v_1^{p_{43}} v_2^{p_{44}}}, \end{cases} \tag{1.7}$$

for $x \in \Omega, t > 0$ with Neummann boundary conditions

$$\frac{\partial u_1}{\partial \eta} = \frac{\partial u_2}{\partial \eta} = \frac{\partial v_1}{\partial \eta} = \frac{\partial v_2}{\partial \eta} = 0 \qquad \text{on } \partial\Omega \times \{t > 0\}, \tag{1.8}$$

and the initial data

$$\begin{cases} u_1(0,x) = \varphi_1(x) > 0 \\ u_2(0,x) = \varphi_2(x) > 0 \\ v_1(0,x) = \varphi_3(x) > 0 \\ v_2(0,x) = \varphi_4(x) > 0 \end{cases} \text{ on } \Omega, \tag{1.9}$$

and $\varphi_i \in C\left(\overline{\Omega}\right)$ for all $i = 1,2,3,4$. Note that Ω is an open bounded domain of class C^1 in $\mathbb{R}^N$ with boundary $\partial\Omega$, and $\partial/\partial\eta$ denotes the outward normal derivative on $\partial\Omega$. We assume a_i, b_i, and p_{ij} are nonnegative indices for all $i,j = 1,2,3,4$ and $\sigma_1, \sigma_2 > 0$. We also consider the case where

$$\begin{cases} \dfrac{p_{12}}{p_{32}} < \dfrac{(p_{11}-1)}{p_{31}} < \min(\dfrac{p_{13}}{p_{33}+1}, \dfrac{p_{14}}{p_{34}}, 1), \\ \qquad\qquad \text{and} \\ \dfrac{p_{21}}{p_{41}} < \dfrac{(p_{22}-1)}{p_{42}} < \min(\dfrac{p_{24}}{p_{44}+1}, \dfrac{p_{23}}{p_{43}}, 1), \end{cases} \tag{1.10}$$

or

$$\begin{cases} \dfrac{p_{12}}{p_{42}} < \dfrac{(p_{11}-1)}{p_{41}} < \min(\dfrac{p_{14}}{p_{44}+1}, \dfrac{p_{13}}{p_{43}}, 1), \\ \qquad\qquad \text{and} \\ \dfrac{p_{21}}{p_{31}} < \dfrac{(p_{22}-1)}{p_{32}} < \min(\dfrac{p_{23}}{p_{33}+1}, \dfrac{p_{24}}{p_{34}}, 1). \end{cases} \tag{1.11}$$

2. Notations and Important Lemmas

In this section, we will introduce some essential notation to simplify derivations and proofs. For reasons that will become clear at later stages in this paper, we set $A_{ij} = \dfrac{a_i + a_j}{2\sqrt{a_i a_j}}$ for all $i,j = 1,2,3$ and let α, β and γ be positive constants satisfying the three conditions

$$\alpha > 2\max\left\{1, \frac{b_2+b_3}{b_1}\right\}, \ \frac{1}{2\gamma} > A_{13}^2, \tag{2.1}$$

$$B_1 B_2 > B_3^2, \tag{2.2}$$

and

$$\left(B_1B_2 - B_3^2\right)\left(B_1B_4 - B_5^2\right) > \left(B_1B_6 - B_3B_5\right)^2, \tag{2.3}$$

where

$$\begin{cases} B_1 = \frac{1}{2\gamma} - A_{13}^2, \\ B_2 = \frac{\alpha-1}{\alpha}\frac{\beta-1}{\beta} - A_{12}^2, \\ B_3 = \frac{\alpha-1}{\alpha}A_{23} - A_{12}A_{13}, \\ B_4 = \frac{\alpha-1}{\alpha}\frac{\delta+1}{\delta} - A_{14}^2, \\ B_5 = \frac{\alpha-1}{\alpha}A_{34} - A_{13}A_{14}, \\ B_6 = \frac{\alpha-1}{\alpha}A_{24} - A_{14}A_{12}. \end{cases}$$

Also, for completeness, we would like to state the usual norms in spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and $C(\overline{\Omega})$, which are denoted respectively by:

$$\|u\|_p^p = \frac{1}{|\Omega|}\int_\Omega |u(x)|^p\,dx, \ \ \|u\|_\infty = ess\sup_{x\in\Omega}|u(x)|\,,$$

and

$$\|u\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}}|u(x)|\,.$$

The following lemmas are important.

Lemma 2.1. *Assume that p_{1i} and p_{3i} satisfy*

$$\frac{p_{12}}{p_{32}} < \frac{(p_{11}-1)}{p_{31}} < \min(\frac{p_{13}}{p_{33}+1}, \frac{p_{14}}{p_{34}}, 1), \tag{2.4}$$

for $i = 1..4$, then for all $p_{11} > 1$, $p_{31}, p_{32}, p_{34}, \alpha, \beta$, $\gamma > 0$, there exists $C = C\left(p_{11,}p_{31}, p_{32}, \alpha, \beta\right) > 0$ and $\theta_1 = \theta_1\left(\alpha\right) \in (0,1)$, such that

$$\alpha\frac{u_1^{\alpha+p_{11}-1}u_2^{\beta+p_{12}}}{v_3^{\gamma+p_{13}}v_4^{\delta+p_{14}}} \leq \gamma\frac{u_1^{\alpha+p_{31}}u_2^{\beta+p_{32}}}{v_3^{\gamma+p_{33}+1}v_4^{\delta+p_{34}}} + C\left(\frac{u_1^{\alpha}u_2^{\beta}}{v_3^{\gamma}v_4^{\delta}}\right)^{\theta_1}. \tag{2.5}$$

Proof. First, we have for all $u_1 > \mu_1,\ u_2 \geq \mu_2,\ v_1 \geq \mu_3,\ v_2 \geq \mu_4$ the following equality

$$\alpha\frac{u_1^{p_{11}-1}u_2^{p_{12}}}{v_1^{p_{13}}v_2^{p_{14}}} = C\left(\beta\frac{u_1^{p_{31}}}{u_2^{-p_{32}}v_1^{p_{33}+1}v_2^{p_{34}}}\right)^{\dfrac{p_{11}-1}{p_{31}}} u_2^{m_1}v_1^{-m_2}v_2^{-m_3}, \tag{2.6}$$

with

$$m_1 = -p_{32}\frac{p_{11}-1}{p_{31}} + p_{12},$$
$$m_2 = \frac{\left(p_{33}+1\right)\left(p_{11}-1\right)}{p_{31}} - p_{13},$$
$$m_3 = \frac{\left(p_{34}\right)\left(p_{11}-1\right)}{p_{31}} - p_{14},$$

and

$$C = \alpha\beta^{-\dfrac{p_{11}-1}{p_{31}}}.$$

We will apply Young's inequality for (2.6) by taking

$$\begin{aligned}
\alpha\frac{u_1^{p_{11}-1}u_2^{p_{12}}}{v_1^{p_{13}}v_2^{p_{14}}} &= C\beta^{-\varepsilon}\left(\beta\frac{u_1^{p_{31}}}{u_2^{-p_{32}}v_1^{p_{33}+1}v_2^{p_{34}}}\right)^{\dfrac{p_{11}-1}{p_{31}}+\varepsilon}. \\
&\quad u_2^{m_1-p_{32}\varepsilon}v_1^{-m_2+(p_{33}+1)\varepsilon}v_2^{-m_3+p_{34}\varepsilon}u_1^{-p_{31}\varepsilon} \\
&\leq c_1\left(\beta\frac{u_1^{p_{31}}}{u_2^{-p_{32}}v_1^{p_{33}+1}v_2^{p_{34}}}\right)^{\dfrac{p_{11}-1}{p_{31}}+\varepsilon}\left(\frac{u_2^{-\beta}v_1^{\gamma}v_2^{\delta}}{u_1^{\alpha}}\right)1^{\dfrac{p_{31}\varepsilon}{\alpha}},
\end{aligned}$$

with

$$c_1 = c\beta^{-\varepsilon}.$$

Recall that Young's inequality takes the form

$$fg \leq \varepsilon f^{p_0} + c_\varepsilon g^{q_0}, \tag{2.7}$$

with

$$f;g \geq 0,\ \varepsilon \geq 0;\ c_\varepsilon = \varepsilon^{-\dfrac{1}{p_0-1}},\ p_0, q_0 > 0,\ \frac{1}{p_0} + \frac{1}{q_0} = 1.$$

We let

$$\varepsilon = \frac{1}{c_1},\ f = \left(\beta\frac{u_1^{p_{31}}u_2^{p_{32}}}{v_1{}^{p_{33}+1}v_2^{p_{34}}}\right)^{\dfrac{p_{11}-1}{p_{32}}+\varepsilon},$$

$$g = \left(\frac{u_2^{-\beta} v_1^{\gamma} v_2^{\delta}}{u_1^{\alpha}}\right)^{\frac{p_{31}\varepsilon}{\alpha}} \text{ and } p_0 = \frac{1}{\frac{p_{11}-1}{p_{32}} + \varepsilon}.$$

Since

$$\frac{1}{p_0} + \frac{1}{q_0} = 1,$$

it follows that

$$q_0 = \frac{1}{1 - \frac{p_{11}-1}{p_{32}} + \varepsilon},$$

and

$$c_\varepsilon = \varepsilon^{-\frac{1}{p_0 - 1}} = \left(\frac{1}{c_1}\right)^{\frac{p_{11}-1+p_{32}\varepsilon}{p_{32}-(p_{11}-1)+\varepsilon}}.$$

Substituting in Young's inequality (2.7) yields

$$\alpha \frac{u_1^{p_{11}-1} u_2^{p_{12}}}{v_1^{p_{13}} v_2^{p_{14}}} \leq \left(\beta \frac{u_1^{p_{31}}}{u_2^{-p_{32}} v_1^{p_{33}+1} v_2^{p_{34}}}\right) +$$

$$c_1 \left(\frac{1}{c_1}\right)^{\frac{p_{11}-1+p_{32}\varepsilon}{p_{32}-(p_{11}-1)+\varepsilon}} \left(\frac{u_1^{-\beta} v_1^{\gamma} v_2^{\delta}}{u_2^{\alpha}}\right)^{\frac{p_{32}\varepsilon}{\alpha\left(1 - \frac{p_{11}-1}{p_{32}} + \varepsilon\right)}},$$

which can be simplified to

$$\alpha \frac{u^{p_{11}-1+\alpha} u^{p_{12}+\beta}}{v^{p_{13}+\gamma} v^{p_{14}+\delta}} \leq \beta \frac{u_1^{p_{31}+\alpha} u^{p_{32}+\beta}}{v_1^{p_{31}+1+\gamma} v_2^{p_{34}+\delta}} + c_2 \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{1-\frac{p_{32}\varepsilon}{\alpha\left(1 - \frac{p_{11}-1}{p_{32}} + \varepsilon\right)}}.$$

where

$$c_2 = \left(\frac{1}{c_1}\right)^{1+\frac{p_{11}-1+p_{32}\varepsilon}{p_{31}-(p_{11}-1)+\varepsilon}}.$$

The proof is concluded by taking ε sufficiently small and

$$\theta_1 = \theta_1(\alpha) = 1 - \frac{p_{32}\varepsilon}{\alpha\left(1 - \frac{p_{11}-1}{p_{32}} + \varepsilon\right)}.$$

□

Corollary 2.2. *If $p_{12} = p_{32} = p_{14} = p_{34} = 0$, then condition (2.4) reduces to*

$$\frac{(p_{11}-1)}{p_{31}} < \min\left(\frac{p_{13}}{p_{33}+1}, 1\right). \tag{2.8}$$

Proof. The proof of this corrollary is similar to the previous lemma but with $p_{12} = p_{32} = p_{14} = p_{34} = 0$. □

Lemma 2.3. *Let $\mu, T > 0$ and $f_j = f_j(t)$ be a non-negative integrable function on $[0, T)$ and $0 < \theta_j < 1$ $(j = 1, ..., J)$. Let $W = W(t)$ be a positive function on $[0, T)$ satisfying the differential inequality*

$$\frac{dW(t)}{dt} \leq -\mu W(t) + \sum_{j=1}^{J} f_j(t) W^{\theta_j}(t), \qquad 0 \leq t < T. \tag{2.9}$$

Then, we have

$$W(t) \leq \kappa, \qquad 0 \leq t < T, \tag{2.10}$$

where κ is the maximal root of the algebraic equation:

$$x - \sum_{j=1}^{J} \left(\sup_{0<t<T} \int_0^t e^{-\mu(t-\xi)} f_j(\xi) d\xi \right) x^{\theta_j} = W(0). \tag{2.11}$$

Proof. This Lemma has been proven in [[11], Lemma 2.2] in the more general case. □

Lemma 2.4. *Let $(u_1(t,.), u_2(t,.), v_1(t,.), v_2(t,.))$ be a solution of (1.7)–(1.9). Then, for any (t, x) in $(0, T_{\max}) \times \Omega$,*

$$\begin{cases} u_1(t,x) \geq e^{-b_1 t} \min(\varphi_1(x)) > 0, \\ u_2(t,x) \geq e^{-b_2 t} \min(\varphi_2(x)) > 0, \\ v_1(t,x) \geq e^{-b_3 t} \min(\varphi_3(x)) > 0, \\ v_2(t,x) \geq e^{-b_4 t} \min(\varphi_4(x)) > 0. \end{cases} \tag{2.12}$$

Proof. The proof of this lemma is trivial and can be achieved by means of the maximum principle. □

Lemma 2.5. *Let A be the symmetric matrix defined by*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix},$$

then

$$a_{11}^2 \left(a_{11}a_{22} - a_{12}^2 \right) \det A = \left(PQ - R^2 \right)$$

where

$$\begin{aligned} P &= \left(a_{11}a_{22} - a_{12}^2 \right) \left(a_{11}a_{33} - a_{13}^2 \right) - \left(a_{11}a_{23} - a_{12}a_{13} \right)^2, \\ Q &= \left(a_{11}a_{22} - a_{12}^2 \right) \left(a_{11}a_{44} - a_{14}^2 \right) - \left(a_{11}a_{24} - a_{12}a_{14} \right)^2, \\ R &= \left(a_{11}a_{22} - a_{12}^2 \right) \left(a_{11}a_{34} - a_{13}a_{14} \right) - \left(a_{11}a_{23} - a_{12}a_{13} \right) \left(a_{11}a_{24} - a_{12}a_{14} \right). \end{aligned}$$

Proof. The proof is trivial. □

3. Existence of Solutions

This section is concerned with the global existence of solutions for the proposed model (1.7)–(1.9). The existence problem reduces to the derivation of a uniform estimate of $\|f_1\|_p$, $\|f_2\|_p$, $\|g_1\|_p$, and $\|g_2\|_p$ on $[0; T_{\max})$ in $L^p(\Omega)$ for some $p > N/2$ (see [9]). To establish the uniform estimate, we utilize an appropriate Lyapunov functional, thereby producing L^p–bounds on u_1, u_2, v_1, and v_2. Since f_1, f_2, g_1, and g_2 are by design continuously differentiable on $\mathbb{R}_+^4$, it is trivial to confirm their Lipschitz continuity on bounded subsets of the domain of a fractional power of the operator

$$\mathcal{A} = - \begin{pmatrix} a_1\Delta & 0 & 0 & 0 \\ 0 & a_2\Delta & 0 & 0 \\ 0 & 0 & a_3\Delta & 0 \\ 0 & 0 & 0 & a_4\Delta \end{pmatrix}, \tag{3.1}$$

for any initial data in $C(\overline{\Omega})$. Hence, we may directly coin the following local existence result (see, for instance, [9]).

Proposition 3.1. *The system (1.7)–(1.9) admits a local unique classical solution* (u_1, u_2, v_1, v_2) *on* $(0, T_{\max}) \times \Omega$*, i.e. if* $T_{\max} < \infty$*, then*

$$\lim_{t \nearrow T_{\max}} \left(\sum_{i=1}^{2} \|u_i(t,.)\|_\infty + \sum_{i=1}^{2} \|v_i(t,.)\|_\infty \right) = \infty. \tag{3.2}$$

After establishing the local existence of solutions, we move to show the global existence, which is described in the following theorem.

Theorem 3.2. *Suppose that the functions* f_i *and* g_i*,* $i = 1, 2$*, satisfy condition (1.11) and let* $(u_1(t,.), u_2(t,.), v_1(t,.),$ *be a solution of (1.7)–(1.9) and*

$$L(t) = \int_\Omega \frac{u_1^\alpha(t,x)\, u_2^\beta(t,x)}{v_1^\gamma(t,x)\, v_2^\delta(t,x)} dx. \tag{3.3}$$

Then, the functional L *is uniformly bounded on the interval* $[0, T^*], T^* < T_{\max}$*, where*

$$T_{\max}\left(\|\varphi_1(x)\|_\infty, \|\varphi_2(x)\|_\infty, \|\varphi_3(x)\|_\infty, \|\varphi_4(x)\|_\infty\right)$$

denotes the eventual blow–up time.

Proof. We start by differentiating the functional $L(t)$ with respect to t yielding

$$\begin{aligned}
L'(t) &= \int_\Omega \frac{d}{dt}\left(\frac{u_1^\alpha u_2^\beta}{v_1^\gamma v_2^\delta}\right) dx \\
&= \int_\Omega \left[\left(\frac{u_1^\alpha u_2^\beta}{v_1^\gamma v_2^\delta}\right)_{u_1} (u_1)_t + \left(\frac{u_1^\alpha u_2^\beta}{v_1^\gamma v_2^\delta}\right)_{u_2} (u_2)_t \right. \\
&\qquad \left. + \left(\frac{u_1^\alpha u_2^\beta}{v_1^\gamma v_2^\delta}\right)_{v_1} (v_1)_t + \left(\frac{u_1^\alpha u_2^\beta}{v_1^\gamma v_2^\delta}\right)_{v_2} (v_2)_t \right] dx \\
&= \int_\Omega \left[\alpha \frac{u_1^{\alpha-1} u_2^\beta}{v_1^\gamma v_2^\delta} (u_1)_t + \beta \frac{u_1^\alpha u_2^{\beta-1}}{v_1^\gamma v_2^\delta} (u_2)_t - \gamma \frac{u_1^\alpha u_2^\beta}{v_1^{\gamma+1} v_2^\delta} (v_1)_t \right. \\
&\qquad \left. - \delta \frac{u_1^\alpha u_2^\beta}{v_1^\gamma v_2^{\delta+1}} (v_2)_t \right] dx.
\end{aligned}$$

Replacing $\partial_t u_1, \partial_t u_2$, $\partial_t v_1$ and $\partial_t v_2$ by their values in (1.7) leads to

$$\begin{aligned}
L'(t) = \int_\Omega \Bigg(& a_1 \alpha \frac{u_1^{\alpha-1} u_2^\beta}{v_1^\gamma v_2^\delta} \Delta u_1 + a_2 \beta \frac{u_1^\alpha u_2^{\beta-1}}{v_1^\gamma v_2^\delta} \Delta u_2 - a_3 \gamma \frac{u_1^\alpha u_2^\beta}{v_1^{\gamma+1} v_2^\delta} \Delta v_1 \\
& - a_4 \delta \frac{u_1^\alpha u_2^\beta}{v_1^\gamma v_2^{\delta+1}} \Delta v_2 - b_1 \alpha \frac{u_1^\alpha u_2^\beta}{v_1^\gamma v_2^\delta} - b_2 \beta \frac{u_1^\alpha u_2^\beta}{v_1^\gamma v_2^\delta} + b_3 \gamma \frac{u_1^\alpha u_2^\beta}{v_1^\gamma v_2^\delta} \\
& + b_4 \delta \frac{u_1^\alpha u_2^\beta}{v_1^\gamma v_2^\delta} + \alpha \frac{u_1^{p_{11}+\alpha-1} u_2^{p_{12}+\beta}}{v_1^{p_{13}+\gamma} v_2^{p_{14}+\delta}} + \beta \frac{u_1^{p_{21}+\alpha} u_2^{p_{22}+\beta-1}}{v_1^{p_{23}+\gamma} v_2^{p_{24}+\delta}} \\
& - \gamma \frac{u_1^{p_{31}+\alpha} u_2^{p_{32}+\beta}}{v_1^{p_{33}+\gamma+1} v_2^{p_{34}+\delta}} - \delta \frac{u_1^{p_{41}+\alpha} u_2^{p_{42}+\beta}}{v_1^{p_{43}+\gamma} v_2^{p_{44}+\delta+1}} + \sigma_1 \alpha \frac{u_1^{\alpha-1} u_2^\beta}{v_1^\gamma v_2^\delta} \\
& + \sigma_2 \delta \frac{u_1^\alpha u_2^{\beta-1}}{v_1^\gamma v_2^\delta} + \beta a \frac{u_1^{\alpha+1} u_2^{\beta-1}}{v_1^\gamma v_2^\delta} - \beta a \frac{u_1^\alpha u_2^{\beta-1}}{v_1^{\gamma-1} v_2^\delta} + \beta ab \frac{u_1^\alpha u_2^{\beta-1}}{v_1^\gamma v_2^\delta} \Bigg) dx.
\end{aligned} \tag{3.4}$$

To simplify the proof, we write (3.4) in the form

$$L'(t) = I + J,$$

where I contains the Laplacians and J contains the remaining terms, i.e.

$$\begin{aligned} I = {} & a_1\alpha \int_\Omega \frac{u_1^{\alpha-1}u_2^{\beta}}{v_1^{\gamma}v_2^{\delta}}\Delta u_1 dx + a_2\beta \int_\Omega \frac{u_1^{\alpha}u_2^{\beta-1}}{v_1^{\gamma}v_2^{\delta}}\Delta u_2 dx \\ & - a_3\gamma \int_\Omega \frac{u_1^{\alpha}u_2^{\beta}}{v_1^{\gamma+1}v_2^{\delta}}\Delta v_1 dx - a_4\delta \int_\Omega \frac{u_1^{\alpha}u_2^{\beta}}{v_1^{\gamma}v_2^{\delta+1}}\Delta v_2\, dx, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} J = {} & (-b_1\alpha - b_2\beta + b_3\gamma + b_4\delta)\, L(t) + \\ & \alpha \int_\Omega \frac{u_1^{p_{11}+\alpha-1}u_2^{p_{12}+\beta}}{v_1^{p_{13}+\gamma}v_2^{p_{14}+\delta}}\, dx + \beta \int_\Omega \frac{u_1^{p_{21}+\alpha}u_2^{p_{22}+\beta-1}}{v_1^{p_{23}+\gamma}v_2^{p_{24}+\delta}}\, d- \\ & \gamma \int_\Omega \frac{u_1^{p_{31}+\alpha}u_2^{p_{32}+\beta}}{v_1^{p_{33}+\gamma+1}v_2^{p_{34}+\delta}}\, dx - \delta \int_\Omega \frac{u_1^{p_{41}+\alpha}u_2^{p_{42}+\beta}}{v_1^{p_{43}+\gamma}v_2^{p_{44}+\delta+1}}\, dx \\ & + \sigma_1\alpha \int_\Omega \frac{u_1^{\alpha-1}u_2^{\beta}}{v_1^{\gamma}v_2^{\delta}}\, dx + \sigma_2\beta \int_\Omega \frac{u_1^{\alpha}u_2^{\beta-1}}{v_1^{\gamma}v_2^{\delta}}\, dx + \beta a \int_\Omega \frac{u_1^{\alpha+1}u_2^{\beta-1}}{v_1^{\gamma}v_2^{\delta}} dx \\ & - \beta a \int_\Omega \frac{u_1^{\alpha}u_2^{\beta-1}}{v_1^{\gamma-1}v_2^{\delta}} dx + \beta ab \int_\Omega \frac{u_1^{\alpha}u_2^{\beta-1}}{v_1^{\gamma}v_2^{\delta}} dx. \end{aligned} \tag{3.6}$$

Since the aim is to prove the boundedness of the functional $L(t)$, we will examine I and J separately. As for I, we use Green's formula to obtain

$$I = -\frac{u_1^{\alpha-2}u_2^{\beta-2}}{v_1^{\gamma}v_2^{\delta}} \int_\Omega [(QT)\cdot T]\, dx,$$

where Q and T given by

$$Q = \begin{pmatrix} a_1\alpha(\alpha-1) & -\frac{a_1+a_3}{2}\alpha\gamma & \frac{a_1+a_2}{2}\alpha\beta & -\frac{a_1+a_4}{2}\alpha\delta \\ -\frac{a_1+a_3}{2}\alpha\gamma & a_3\gamma(\gamma+1) & -\frac{a_2+a_3}{2}\beta\gamma & \frac{a_3+a_4}{2}\gamma\delta \\ \frac{a_1+a_2}{2}\alpha\beta & -\frac{a_2+a_3}{2}\beta\gamma & a_2\beta(\beta-1) & -\frac{a_2+a_4}{2}\beta\delta \\ -\frac{a_1+a_4}{2}\alpha\delta & \frac{a_3+a_4}{2}\gamma\delta & -\frac{a_2+a_4}{2}\beta\delta & a_4\delta(\delta+1) \end{pmatrix},$$

and

$$T = \begin{pmatrix} \frac{u_2}{v_1v_2}\nabla u_1 \\ \frac{u_1u_2}{v_2}\nabla v_1 \\ \frac{u_1}{v_1v_2}\nabla u_2 \\ \frac{u_1u_2}{v_1}\nabla v_2 \end{pmatrix}.$$

It follows that the matrix Q is positive definite if, and only if, all its principal successive determinants Δ_1, Δ_2, Δ_3, and Δ_4 are positive. The first determinant $\Delta_1 = a_1\alpha(\alpha-1) > 0$ by (2.1). The second determinant

Δ_2 requires more attention. By (2.1), we have

$$\begin{aligned}\Delta_2 &= \begin{vmatrix} a_1\alpha(\alpha-1) & -\dfrac{a_1+a_3}{2}\alpha\gamma \\ -\dfrac{a_1+a_3}{2}\alpha\gamma & a_3\gamma(\gamma+1)\end{vmatrix} \\ &= \alpha^2\gamma^2 a_3 a_1\left[\frac{\alpha-1}{\alpha}\frac{\gamma+1}{\gamma}-A_{13}^2\right] > 0.\end{aligned}$$

Using Theorem 1 of [3], we obtain

$$\begin{aligned}(\alpha-1)\Delta_3 &= a_1a_2a_3\alpha^3\beta^2\gamma^2\left\{\left[\frac{\alpha-1}{\alpha}\frac{\gamma+1}{\gamma}-A_{13}^2\right]\left[\frac{\alpha-1}{\alpha}\frac{\beta-1}{\beta}-A_{12}^2\right]\right. \\ &\qquad \left. -\left[\frac{\alpha-1}{\alpha}A_{23}-A_{12}A_{13}\right]^2\right\}.\end{aligned}$$

Then, using conditions (2.1) and (2.2), it is easy to show that $\Delta_3 > 0$. Similarly, using lemma 2.5 along with conditions (2.1) and (2.3), we obtain the last determinant

$$\Delta_4 = |Q| > 0.$$

This implies that $I \leq 0$ for all $(t,x) \in [0,T^*]\times\Omega$.

We now move our attention to J as defined in (3.6). According to the maximum principle, there exists C_0 depending on $\|\varphi_1\|_\infty$, $\|\varphi_2\|_\infty$, $\|\varphi_3\|_\infty$ and $\|\varphi_4\|_\infty$such that $u_1,u_2,v_1,v_2 \geq C_0 > 0$ with

$$\begin{aligned}\frac{u_1^{\alpha-1}u_2^\beta}{v_1^\gamma v_2^\delta} &= \left(\frac{u_1^\alpha u_2^\beta}{v_1^\gamma v_2^\delta}\right)^{\frac{\alpha-1}{\alpha}}\left(\frac{1}{v_1}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1}{v_2}\right)^{\frac{\delta}{\alpha}}\left(\frac{1}{u_2}\right)^{\frac{-\beta}{\alpha}} \\ &\leq \left(\frac{u_1^\alpha u_2^\beta}{v_1^\gamma v_2^\delta}\right)^{\frac{\alpha-1}{\alpha}}\left(\frac{1}{C_0}\right)^{\frac{\gamma+\delta-\beta}{\alpha}}.\end{aligned}$$

By defining

$$C_1 = \left(\frac{1}{C_0}\right)^{\frac{\gamma+\delta-\beta}{\alpha}},$$

we can write

$$\frac{u_1^{\alpha-1}u_2^\beta}{v_1^\gamma v_2^\delta} \leq C_1\left(\frac{u_1^\alpha u_2^\beta}{v_1^\gamma v_2^\delta}\right)^{\frac{\alpha-1}{\alpha}}.$$

In a very similar manner, we can obtain

$$\frac{u_1^{\alpha}u_2^{\beta-1}}{v_1^\gamma v_2^\delta} \leq C_2\left(\frac{u_1^\alpha u_2^\beta}{v_1^\gamma v_2^\delta}\right)^{\frac{\beta-1}{\beta}},$$

and

$$\frac{u_1^{\alpha+1}u_2^{\beta-1}}{v_1^\gamma v_2^\delta} \leq C_3\left(\frac{u_1^\alpha u_2^\beta}{v_1^\gamma v_2^\delta}\right)^{\frac{\beta-1}{\beta}},$$

where

$$C_2 = \left(\frac{1}{C_0}\right)^{\frac{\gamma+\delta-\alpha}{\beta}} \quad \text{and} \quad C_3 = \left(\frac{1}{C_0}\right)^{\frac{\gamma+\delta-\alpha-1}{\beta}}.$$

Using lemma 2.1 for all $(t,x) \in [0,T^*] \times \Omega$, we get

$$\begin{cases} \alpha \dfrac{u_1^{\alpha+p_{11}-1} u_2^{\beta+p_{12}}}{v_1^{\gamma+p_{13}} v_2^{\delta+p_{14}}} \le \gamma \dfrac{u_1^{\alpha+p_{31}} u_2^{\beta+p_{32}}}{v_1^{\gamma+p_{33}+1} v_2^{\delta+p_{34}}} + C_1' \left(\dfrac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{\theta_1}, \\ \beta \dfrac{u_1^{\alpha+p_{21}} u_2^{\beta+p_{22}-1}}{v_1^{\gamma+p_{23}} v_2^{\delta+p_{24}}} \le \delta \dfrac{u_1^{\alpha+p_{41}} u_2^{\beta+p_{42}}}{v_1^{\gamma+p_{43}} v_2^{\delta+p_{44}+1}} + C_2' \left(\dfrac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{\theta_2}, \end{cases}$$

or

$$\begin{cases} \alpha \dfrac{u_1^{\alpha+p_{11}-1} u_2^{\beta+p_{12}}}{v_1^{\gamma+p_{13}} v_2^{\delta+p_{14}}} \le \delta \dfrac{u_1^{\alpha+p_{41}} u_2^{\beta+p_{42}}}{v_1^{\gamma+p_{43}} v_2^{\delta+p_{44}+1}} + C_3' \left(\dfrac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{\theta_3}, \\ \beta \dfrac{u_1^{\alpha+p_{21}} u_2^{\beta+p_{22}-1}}{v_1^{\gamma+p_{23}} v_2^{\delta+p_{24}}} \le \gamma \dfrac{u_1^{\alpha+p_{31}} u_2^{\beta+p_{32}}}{v_1^{\gamma+p_{33}+1} v_2^{\delta+p_{34}}} + C_4' \left(\dfrac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{\theta_4}. \end{cases}$$

Employing these inequalities along with the definition of J in (3.6) leads to

$$\begin{aligned} J \quad &\le \quad (-b_1\alpha - b_2\beta + b_3\gamma + b_4\delta) L(t) + \int_\Omega C_1' \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{\theta_1} dx \\ &\quad + \int_\Omega C_2' \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{\theta_2} dx + \sigma_1\alpha \int_\Omega C_1 \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{\frac{\alpha-1}{\alpha}} dx \\ &\quad + \beta(\sigma_2 + ab) \int_\Omega C_2 \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{\frac{\beta-1}{\beta}} dx + \beta a \int_\Omega C_3 \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{\frac{\beta-1}{\beta}} dx. \\ &\le \quad (-b_1\alpha - b_2\beta + b_3\gamma + b_4\delta) L(t) + \int_\Omega C_1' \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{\theta_1} dx \\ &\quad + \int_\Omega C_2' \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{\theta_2} dx + \sigma_1\alpha \int_\Omega C_1 \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{\frac{\alpha-1}{\alpha}} dx \\ &\quad + \beta(\sigma_2 + ab + a) \int_\Omega C_4 \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{\frac{\beta-1}{\beta}} dx. \end{aligned} \tag{3.7}$$

By applying Hölder's inequality for all $t \in [0,T^*]$, we obtain

$$\int_\Omega C \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{\theta} dx \le C \left(\int_\Omega \frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}} dx\right)^{\theta} \left(\int_\Omega C^{\frac{1}{1-\theta}} dx\right)^{1-\theta},$$

or more compactly

$$\int_\Omega C \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{\theta} dx \le C_5 L^{\theta}(t), \tag{3.8}$$

where

$$C_5 = |\Omega|^{1-\theta}.$$

Similarly, it can be shown that

$$\int_\Omega C_1' \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}}\right)^{\theta_1} dx \le C_6 L^{\theta_1}(t), \tag{3.9}$$

and

$$\int_{\Omega} C_2' \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}} \right)^{\theta_2} dx \leq C_7 L^{\theta_2}(t), \tag{3.10}$$

with

$$C_6 = |\Omega|^{1-\theta_1} \quad \text{and} \quad C_7 = |\Omega|^{1-\theta_2}.$$

Also, since

$$\int_{\Omega} C_1 \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}} \right)^{\frac{\alpha-1}{\alpha}} dx \leq \left(\int_{\Omega} \frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}} dx \right)^{\frac{\alpha-1}{\alpha}} \left(\int_{\Omega} (C_1)^{\alpha} dx \right)^{\frac{1}{\alpha}},$$

we can state that

$$\int_{\Omega} C_1 \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}} \right)^{\frac{\alpha-1}{\alpha}} dx \leq C_8 L^{\frac{\alpha-1}{\alpha}}(t), \tag{3.11}$$

and

$$\int_{\Omega} C_4 \left(\frac{u_1^{\alpha} u_2^{\beta}}{v_1^{\gamma} v_2^{\delta}} \right)^{\frac{\beta-1}{\beta}} dx \leq C_9 L^{\frac{\beta-1}{\beta}}(t), \tag{3.12}$$

where

$$C_8 = C_1 |\Omega|^{\frac{1}{\alpha}} \quad \text{and} \quad C_9 = C_4 |\Omega|^{\frac{1}{\beta}}.$$

Using inequalities (3.8)–(3.12) along with (3.7) yields the reduced inequality

$$\begin{aligned} J \leq\ & (-b_1\alpha - b_2\beta + b_3\gamma + b_4\delta) L(t) + C_6 L^{\theta_1}(t) \\ & + C_7 L^{\theta_2}(t) + \sigma_1 \alpha C_8 L^{\frac{\alpha+1}{\alpha}}(t) + \beta (\sigma_2 + ab + a) C_9 L^{\frac{\beta-1}{\beta}}(t). \end{aligned} \tag{3.13}$$

Since $-b_1\alpha - b_2\beta + b_3\gamma + b_4\delta < 0$, inequality (3.13) leads to

$$L'(t) \leq C_6 L^{\theta_1}(t) + C_7 L^{\theta_2}(t) + \sigma_1 \alpha C_8 L^{\frac{\alpha+1}{\alpha}}(t) + \beta (\sigma_2 + ab + a) C_9 L^{\frac{\beta-1}{\beta}}(t). \tag{3.14}$$

Finally, using Lemma 2.3 in $[0, T^*]$, we deduce that $L(t)$ is bounded on $(0, T_{\max})$, i.e. $L(t) \leq \gamma_1$ where γ_1 depends on the L^{∞}–norms of φ_1, φ_2 , φ_3 and φ_4. This concludes the proof of Theorem 3.2. □

Corollary 3.3. *Under the assumptions of Theorem 3.2, all solutions of problem (1.7)–(1.9) with positive initial data in $C(\overline{\Omega})$ are global. In addition, if $b_1, b_2, b_3, b_4, \sigma_1, \sigma_2 > 0$, then u_1, u_2, v_1, and v_2 are uniformly bounded in $\overline{\Omega} \times [0, \infty)$.*

Proof. Since $L(t)$ is bounded on $(0, T_{\max})$ and the functions $\frac{u^{p_{11}} u^{p_{12}}}{v_1^{p_{13}} v_2^{p_{14}}}$, $\frac{u^{p_{21}} u^{p_{22}}}{v_1^{p_{23}} v_2^{p_{24}}}$, $\frac{u^{p_{31}} u^{p_{32}}}{v_1^{p_{33}} v_2^{p_{34}}}$, and $\frac{u^{p_{41}} u^{p_{42}}}{v_1^{p_{43}} v_2^{p_{44}}}$ are in $L^{\infty}((0, T_{\max}), L^m(\Omega))$ for all $m > \frac{N}{2}$, then as a consequence of the arguments in [9] or [7], we conclude that the solution of the system (1.7)–(1.9) is global and uniformly bounded on $\Omega \times (0, +\infty)$. □

4. Numerical Example

Consider the four–component Gierer–Meinhardt system with a Conway–Cooper modification arranged in a circle of 18 units with each unit possessing four reaction–diffusion concentrations, see [4]. The system takes the form

$$\begin{cases} \frac{\partial a}{\partial t} - D_a \Delta a = p_0 p + c_0 p \frac{a^2}{b} - ua, \\ \frac{\partial b}{\partial t} - D_b \Delta b = c' p' a^2 - vb, \\ \frac{\partial c}{\partial t} - D_c \Delta c = p_0 p + c_0 p \frac{c^2}{d} - uc + k y_3 (a - b + floor), \\ \frac{\partial d}{\partial t} - D_d \Delta d = c' p' c^2 - vd, \end{cases} \tag{4.1}$$

Table 1: First set of simulation parameters for the 4–component activator–inhibitor Gierer–Meinhardt type system in (4.1).

Parameters	Values	Parameters	Values
p	0.4	$floor$	0.5
p_0	0.21	$a(x,y,0)$	1
p'	0.14	$b(x,y,0)$	0.7
c_0	0.1	$c(x,y,0)$	0.9
c'	0.1	$d(x,y,0)$	0.3
u	0.07	D_a	0.15
v	0.07	D_b	0.12
y_3	0.009	D_c	0.05
k	1	D_d	0.06

Table 2: Second set of simulation parameters for the 4–component activator–inhibitor Gierer–Meinhardt type system in (4.1).

Parameters	Values	Parameters	Values
p	0.5	$floor$	0.5
p_0	0.15	$a(x,y,0)$	10
p'	0.04	$b(x,y,0)$	7
c_0	0.1	$c(x,y,0)$	9
c'	0.1	$d(x,y,0)$	7.3
u	0.07	D_a	0.15
v	0.04	D_b	0.12
y_3	0.009	D_c	0.05
k	1	D_d	0.06

with a and c representing activator concentrations and b and d denoting inhibitor concentrations in a certain scenario. It is easy to see that this system is a special case of our proposed model (1.7) by taking

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

The resulting system satisfies conditions (1.10) using Corollary 2.2, which according to Theorem 3.2 guarantees the global existence of solutions for the system.

A Matlab computer simulation was carried out to obtain the numerical solutions of (4.1) given two sets of parameters as illustrated in Tables 1 and 2. Figures 1–3 show snapshots of the solutions for the first set of parameters given a two–dimensional spatial diffusion taken at times $t = 1s$, $10s$, and $500s$, respectively. Note that a zero mean Gaussian spatial noise with variance $\sigma^2 = 0.04$ was added to the initial values shown in Table 1 in order to introduce non–uniformity to the concentrations. As can be observed, the solutions reach a constant steady state in a short duration.

Figures 4 and 5 show the solutions of (4.1) in the 2–dimensional case given the parameters shown in 2. The solutions are depicted at times $t = 10s$ and $t = 500s$. Although the solutions perturbate over time never reaching a constant steady state, solutions still exist globally in time. The numerical simulations for both examples were run for a duration as long as $5000s$ to confirm the global existence of solutions.

Acknowledgement

The authors would like to thank Pr. A. Youkana of Batna University, Algeria, for his continued help and assistance throughout this work.

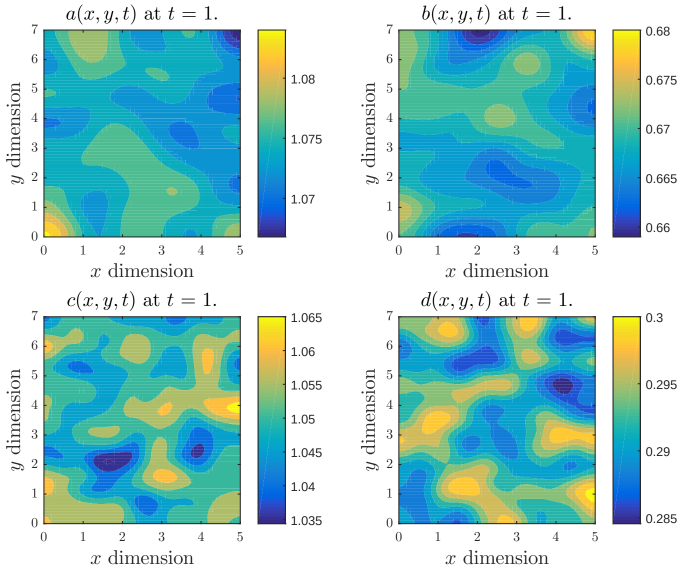

Figure 1: Solutions of system (4.1) taken at time $t = 1s$ with the parameters shown in Table (1).

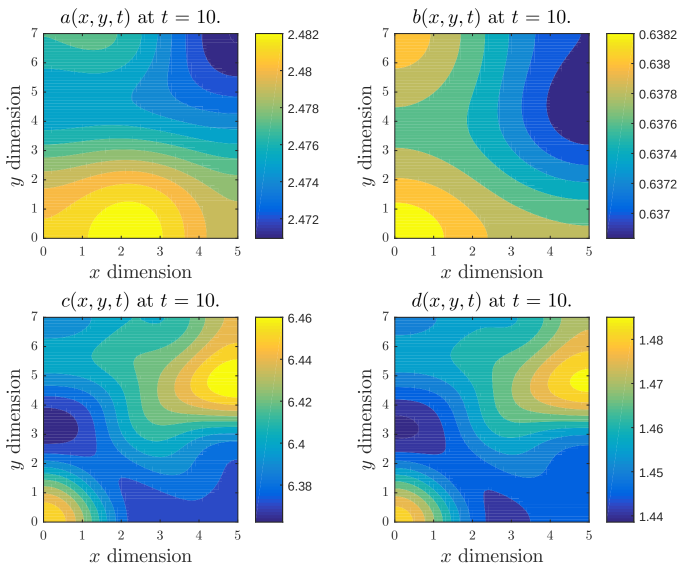

Figure 2: Solutions of system (4.1) taken at time $t = 10s$ with the parameters shown in Table (1).

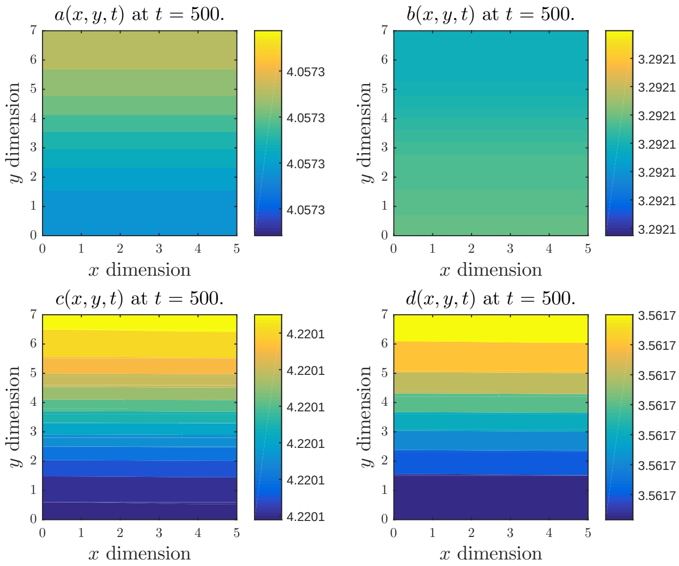

Figure 3: Solutions of system (4.1) taken at time $t = 500s$ with the parameters shown in Table (1).

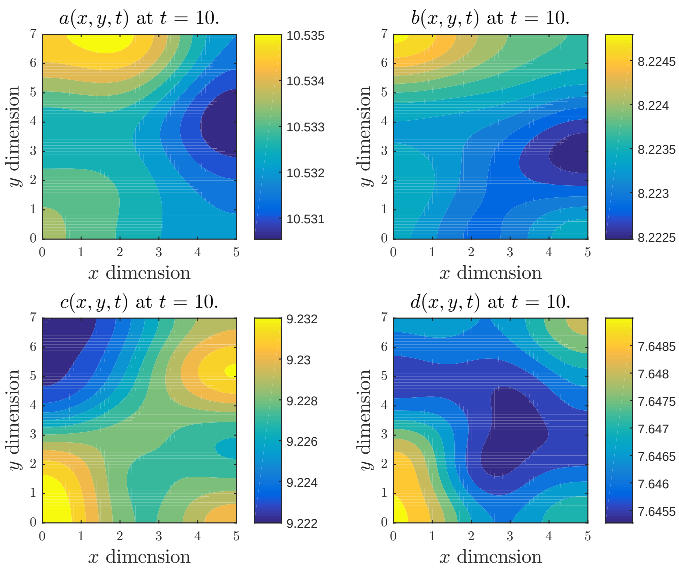

Figure 4: Solutions of system (4.1) taken at time $t = 10s$ with the parameters shown in Table (2).

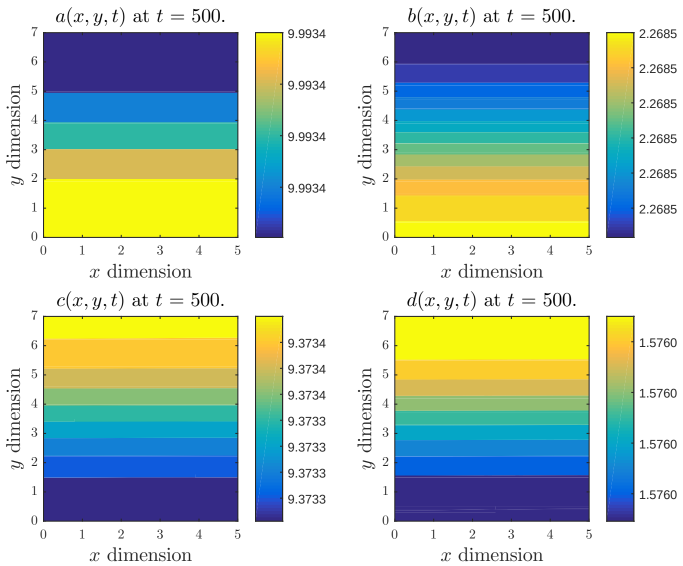

Figure 5: Solutions of system (4.1) taken at time $t = 500s$ with the parameters shown in Table (2).

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