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Some new common fixed point theorems for Geraghty contraction type maps in partial metric spaces

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Abstract

In this paper, we prove some new common fixed point theorems for Geraghtys type contraction mappings on partial metric spaces. Theorems presented are generalizations of fixed point theorems of Altun et al. [Generalized Geraghty type mappings on partial metric spaces and fixed point results, Arab. J. Math. **2**, (2013), no. 3, 247-253]. We also give some examples to illustrate the usability of the obtained results.

Keywords: Fixed point, Geraghty contraction, Partial metric space.
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1. Introduction and Preliminaries

The notion of partial metric spaces (PMS) was initiated by Matthews [18, 19]. Subsequently, many authors established fixed point theorems on partial metric spaces. For more details, see [4, 6, 7, 9, 13, 15, 16, 17, 20, 21, 22, 23] and references contained therein.

Definition 1.1. [18] A mapping $p : X \times X \to [0, \infty)$ is called a partial metric on nonempty set X if the following conditions hold for all $x, y, z \in X$:

(P1) $x = y$ *if and only if* $p(x, x) = p(x, y) = p(y, y)$,

(P2) $p(x, x) \leq p(x, y)$,

(P3) $p(x, y) = p(y, x)$,

(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then, the pair (X, p) is called a partial metric space (PMS).

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For a partial metric p on X, the function $d_p : X \times X \to \mathbf{R}^+$ defined as

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \quad x, y \in X, \tag{1.1}$$

is a metric on X . Each partial metric p on X generates a T_0 topology τ_p, whose base is a family of open p-balls $\{B_p(x, \varepsilon)|x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X|p(x, y) \leq p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 1.2. Let (X, p) be a partial metric space.
(i) A sequence $\{x_n\}$ in the PMS (X, p) converges to $x \in X$ if and only if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$.
(ii) A sequence $\{x_n\}$ in the PMS (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and finite).
(iii) A PMS (X, p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $p(x, x) = lim_{n,m\to\infty} p(x_n, x_m)$.

Lemma 1.3. *[18, 19] Let (X, p) be a partial metric space.*
(i) A sequence $\{x_n\}$ is Cauchy sequence in a PMS (X, p) if and only if $\{x_n\}$ is Cauchy sequence in a metric space (X, d_p).
(ii) A PMS (X, p) is complete if and only if the metric space (X, d_p) is complete. Moreover,

$$\lim_{n\to\infty} d_p(x_n, x) = 0 \text{ if and only if } p(x, x) = \lim_{n,\infty} p(x_n, x) = \lim_{n,m\infty} p(x_n, x_m).$$

Lemma 1.4. *[2] Let (X, p) be a partial metric space and $x_n \to x$ in a PMS (X, p) and $p(x, x) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(x, y)$ for all $y \in X$.*

Lemma 1.5. *[2] Let (X, p) be a partial metric space. Then*
(1) If $p(x, y) = 0$, then $x = y$.
(2) If $x \neq y$, then $p(x, y) > 0$.

Definition 1.6. [1] Let X be a non-empty set and $f, g : X \to X$ are given self-mappings on X. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. Also, self-maps f and g are said to be weakly compatible if they commute at their coincidence point; that is, if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

Lemma 1.7. *[1] Let $f, g : X \to X$ be weakly compatible and f, g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

In 1973, Geraghty [12] introduced a class of functions to generalize the Banach contraction principle. Let $\mathcal{B}$ be the family of all functions $\alpha : [0, \infty) \to [0, 1)$ satisfying the property

$$\lim_{n\to\infty} \alpha(t_n) = 1 \textit{ implies } \lim_{n\to\infty}(t_n) = 0.$$

Theorem 1.8. *[12] Let (X, d) be a complete metric space. Let $T : X \to X$ be given mapping satisfying*

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y), \quad x, y \in X,$$

where $\alpha \in \mathcal{B}$. Then T has a unique fixed point.

In recent years, some authors extended the result of Geraghty in the context of various metric spaces(see [8, 10, 11, 14]). In 2013 Altun et al.[5] proved a version of Geraghtys theorem in partially metric spaces as follows.

Theorem 1.9. *[5] Let (X, p) be a complete partial metric space and T be a self-mapping on X which satisfy,*

$$p(Tx, Ty) \leq \beta(M(x, y)) \max\{p(x, y), p(x, Tx), p(y, Ty)\}, \tag{1.2}$$

for all $x, y \in X$, where $\beta \in \mathcal{B}$ and

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(Tx, y) + p(x, Ty)}{2}\}.$$

Then T has a unique fixed common point in X.

2. Main Result

Now, we announce our first new result in this context.

Theorem 2.1. *Let (X, p) be a partial metric space and let the mappings $f, g : X \to X$ satisfy the condition*

$$p(gx, gy) \leq \beta(M(x, y))m(x, y), \qquad \text{for all } x, y \in X, \tag{2.1}$$

where $\beta \in \mathcal{B}$ and

$$m(x, y) = \max\{p(fx, fy), p(fx, gx), p(fy, gy)\},$$

$$M(x, y) = \max\{p(fx, fy), p(fx, gx), p(fy, gy), \frac{p(fx, gy) + p(fy, gx}{2}\}. \tag{2.2}$$

Suppose also that $g(X) \subset f(X)$ and $f(X)$ is a complete subspace of X. Then f, g have a unique point of coincident in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Since $g(X) \subset f(X)$, inductively, we can define a sequence $\{x_n\}$ in X such that

$$fx_{n+1} = gx_n, \quad n = 0, 1, 2, \ldots. \tag{2.3}$$

Using (2.1) and (2.3) with $x = x_n$ and $y = x_{n+1}$, we get

$$p(fx_{n+1}, fx_{n+2}) = p(gx_n, gx_{n+1}) \leq \beta(M(x_n, x_{n+1}))m(x_n, x_{n+1}), \tag{2.4}$$

for all $n = 0, 1, 2, \ldots$. Thus, by (2.2) and using (P4), we have

$$\begin{aligned} m(x_n, x_{n+1}) &= \max\{p(fx_n, fx_{n+1}), p(fx_n, gx_n), p(fx_{n+1}, gx_{n+1})\} \\ &= \max\{p(fx_n, fx_{n+1}), p(fx_{n+1}fx_{n+2})\} \\ &= \max\{p(fx_nfx_{n+1}), p(fx_{n+1}fx_{n+2})\}, \end{aligned}$$

for all $n = 0, 1, 2, \ldots$. If $m(x_n, x_{n+1}) = p(fx_{n+1}, fx_{n+2})$, then from the inequality (2.4), we have

$$p(fx_{n+1}, fx_{n+2}) \leq \beta(M(x_n, x_{n+1}))p(fx_{n+1}, fx_{n+2}) < p(fx_{n+1}, fx_{n+2}),$$

which is a contradiction. Thus $m(x_n, x_{n+1}) = p(fx_n, fx_{n+1})$, so we have $p(fx_{n+1}, fx_{n+2}) \leq p(fx_n, fx_{n+1})$. Then $p(fx_n, fx_{n+1})$ is a nonincreasing sequence and hence it is convergent. Consequently, there exists $r \geq 0$ such that $\lim_{n\to\infty} p(fx_n, fx_{n+1}) = r$. We prove $r = 0$. Suppose, on the contrary, that $r > 0$. Then, we have

$$\frac{p(fx_{n+1}, fx_{n+2})}{p(fx_n, fx_{n+1})} \leq \beta(M(x_n, x_{n+1})) < 1.$$

Then $1 \leq \lim_{n\to\infty} \beta(M(x_n, x_{n+1})) \leq 1$, which implies $\lim_{n\to\infty} \beta(M(x_n, x_{n+1})) = 1$. Since $\beta \in \mathcal{B}$, we have $\lim_{n\to\infty} M(x_n, x_{n+1}) = 0$ and hence,

$$\lim_{n\to\infty} p(fx_{n+1}, fx_n) = 0. \tag{2.5}$$

From (1.1) and (2.5), we get $\lim_{n\to\infty} d_p(fx_{n+1}, fx_n) = 0$. We claim that fx_n is a Cauchy sequence in (X, d_p). Suppose fx_n is not Cauchy sequence. Then there exists some $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that $d_p(fx_{m(k)}, fx_{n(k)}) \geq \varepsilon$. From (1.1), we have $\varepsilon \leq d_p(fx_{m(k)}, fx_{n(k)}) \leq 2p(fx_{m(k)}, fx_{n(k)})$. Without loss of generality, we can suppose that also

$$n(k) > m(k) > 0, \qquad p(fx_{m(k)}, fx_{n(k)}) \geq \frac{\varepsilon}{2}, \qquad p(fx_{m(k)}, fx_{n(k)-1}) < \frac{\varepsilon}{2}. \tag{2.6}$$

Using (2.6) and using (P4), we have

$$\begin{aligned}\frac{\varepsilon}{2} &\leq p(fx_{m(k)}, fx_{n(k)})\\ &\leq p(fx_{m(k)}, fx_{m(k)-1}) + p(fx_{m(k)-1}, fx_{n(k)-1}) + p(fx_{n(k)-1}, fx_{n(k)})\\ &\leq p(fx_{m(k)}, fx_{m(k)-1}) + p(fx_{m(k)-1}, fx_{m(k)}) + p(fx_{m(k)}, fx_{n(k)-1}) + p(fx_{n(k)-1}, fx_{n(k)}),\end{aligned}$$

for all $k > 1$. Letting $k \to \infty$ in above inequality and using (2.5), we get

$$lim_{k\to\infty} p(fx_{m(k)-1}, fx_{n(k)-1}) = \frac{\varepsilon}{2}. \tag{2.7}$$

Also, by (P4), we have

$$p(fx_{m(k)-1}, fx_{n(k)}) \leq p(fx_{m(k)-1}, fx_{n(k)-1}) + p(fx_{n(k)-1}, fx_{n(k)}).$$

Letting $k \to \infty$ in the above inequality and using (2.5) and (2.7), we obtain

$$\lim_{n\to\infty} p(fx_{m(k)-1}, fx_{n(k)}) \leq \frac{\varepsilon}{2}. \tag{2.8}$$

Using (2.1), with $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$, we get

$$\begin{aligned}p(fx_{m(k)}, fx_{n(k)}) &= p(gx_{m(k)-1}, gx_{n(k)-1})\\ &\leq \beta(M(x_{m(k)-1}, x_{n(k)-1}))m(x_{m(k)-1}, x_{n(k)-1}),\end{aligned} \tag{2.9}$$

where, we have

$$m(x_{m(k)-1}, x_{n(k)-1}) = \max\{p(fx_{m(k)-1}, fx_{n(k)} - 1), p(fx_{m(k)-1}, gx_{m(k)-1}), p(fx_{n(k)-1}, gx_{n(k)-1})\}.$$

Letting $k \to \infty$ in relationship above and using (2.5-2.8), we obtain

$$\lim_{k\to\infty} m(x_{m(k)-1}, x_{n(k)-1}) \leq \frac{\varepsilon}{2}. \tag{2.10}$$

By using (2.9) and (2.10), we get

$$\frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} \lim_{k\to\infty} \beta(M(x_{m(k)-1}, x_{n(k)-1})).$$

Then, we derive that, $1 \leq \lim_{k\to\infty} \beta(M(x_{m(k)-1}, x_{n(k)-1})) \leq 1$, and so $\lim_{k\to\infty} \beta(M(x_{m(k)-1}, x_{n(k)-1})) = 1$. Since $\beta \in \mathcal{B}$, we conclude that $\lim_{k\to\infty} p(fx_{m(k)-1}, fx_{n(k)-1}) = 0$, which is a contradiction with (2.7). Hence, we concluded that $\{fx_n\}$ is a Cauchy sequence in (X, d_p). Since $(f(X), p)$ is complete, by Lemma 1.3, $(f(X), d_p)$ is complete. Then there exists $z \in f(X)$ such that

$$\lim_{n\to\infty} d_p(fx_n, z) = 0. \tag{2.11}$$

From the property (ii) in Lemma 1.3, we have $p(z, z) = \lim_{n\to\infty} p(fx_n, z) = \lim_{n,m\to\infty} p(fx_n, fx_m)$. From (P2) and using (2.5), we have

$$\lim_{n\to\infty} p(fx_n, fx_n) = 0. \tag{2.12}$$

Applying (1.1), we have

$$d_p(fx_n, fx_m) = 2p(fx_n, fx_m) - p(fx_n, fx_n) - p(fx_m, fx_m).$$

Letting $n, m \to \infty$ in the above inequality, using (2.11) and (2.12), we have $\lim_{n\to\infty} p(fx_n, fx_m) = 0$. Then, we have

$$p(z,z) = \lim_{n\to\infty} p(fx_n, z) = \lim_{n\to\infty} p(fx_n, fx_m) = 0.$$

Since $z \in f(X)$, we can find $t \in X$ such that $ft = z$. We prove that $gt = z$. From (2.1), we have

$$p(fx_n, gt) = p(gx_{n-1}, gt) \leq \beta(M(x_{n-1}, t))m(x_{n-1}, t), \quad n = 1, 2, 3, ..., \tag{2.13}$$

where

$$m(x_{n-1}, t) = \max\{p(fx_{n-1}, ft), p(fx_{n-1}, gx_{n-1}), p(ft, gt)\}. \tag{2.14}$$

Letting $n \to \infty$ in (2.14) and using (2.5), we get $\lim_{n\to\infty} m(x_{n-1}, z) = p(z, gt)$. From (2.13) and using Lemma 1.4, we obtain $p(z, gt) \leq \lim_{n\to\infty} \beta(M(x_{n-1}, t))p(z, gt)$, which implies that

$$1 \leq \lim_{n\to\infty} \beta(M(x_{n-1}, t)) \leq 1.$$

By the property β, we have $\lim_{n\to\infty} M(x_{n-1}, t) = 0$, which yields $p(z, gt) = 0$, that is $ft = gt = z$. Hence t is a coincidence point and z is a point of coincidence of f and g. Now we show that f and g have a unique point of coincidence. For this, assume that there exists another point q in X such that $z_1 = fq = gq$. Suppose, to the contrary, $p(z, z_1) > 0$. Using (2.1), we have

$$p(z, z_1) = p(gt, gq) \leq \beta(M(t, q))m(t, q),$$

where $m(t, q) = \max\{p(ft, fq), p(ft, gt), p(fq, gq)\} = p(z, z_1)$. Then, we get

$$p(z, z_1) \leq \beta(M(t, q))p(z, z_1) < p(z, z_1),$$

which is a contradiction and hence $p(z, z_1) = 0$ and we get that $z = z_1$. Therefore, z is the unique point of coincidence of f and g. Now, if f and g are weakly compatible then by Lemma 1.7, z is the unique common fixed point of f and g. □

If we take $fx = x$ for all $x \in X$, in Theorem 2.1, we obtain the following result.

Corollary 2.2. *[5] Let (X, p) be a complete partial metric space and let $g : X \to X$ be a mapping such that*

$$p(gx, gy) \leq \beta(M(x, y)) \max\{p(x, y), p(x, gx), p(y, gy)\},$$

for all $x, y \in X$, where $\beta \in \mathcal{B}$ and

$$M(x, y) = \max\{p(x, y), p(x, gx), p(y, gy), \frac{p(x, gy) + p(y, gx)}{2}\}.$$

Then g has a unique fixed point.

Corollary 2.3. *Let (X, p) be a complete partial metric space and let $g : X \to X$ be a mapping such that*

$$p(gx, gy) \leq \beta(M(x, y))p(x, y),$$

for all $x, y \in X$, where $\beta \in \mathcal{B}$ and

$$M(x, y) = \max\{p(x, y), p(x, gx), p(y, gy), \frac{p(x, gy) + p(y, gx)}{2}\}.$$

Then g has a unique fixed point.

Example 2.4. Let $X = [0, \infty)$ and $p : X \times X \to [0, \infty)$ be given as $p(x, y) = \max\{x, y\}$. Let $f, g : X \to X$ be defined respectively as $fx = e^x - 1, gx = \frac{e^x-1}{x+3}$ and $\beta(t) = \frac{1}{2}$. Suppose $0 \leq x < y$, then we have

$$m(x,y) = \max\{\max\{e^x - 1, e^y - 1\}, \max\{e^x - 1, \frac{e^x-1}{x+3}\}, \max\{e^y - 1, \frac{e^y-1}{y+3}\}\} = e^y - 1,$$

and

$$\begin{aligned} p(gx, gy) &= \max\{\frac{e^x-1}{x+3}, \frac{e^y-1}{y+3}\} \leq \frac{e^y-1}{2} \\ &= \beta(M(x,y))m(x,y). \end{aligned}$$

Then the conditions of Theorem 2.1 are satisfied and f, g have a unique common fixed point.

Theorem 2.5. *Let (X, p) be a complete partial metric space and T, S be self-mappings on X which satisfy,*

$$p(Tx, Sy) \leq \beta(M(x,y))m(x,y), \quad for\ all\ x, y \in X, \tag{2.15}$$

where $\beta \in \mathcal{B}$ and

$$m(x,y) = \max\{p(x,y), p(Tx,x), p(y,Sy)\},$$
$$M(x,y) = \max\{p(x,y), p(Tx,x), p(y,Sy), \frac{p(Tx,y) + p(x,Sy)}{2}\}.$$

Then T and S have a unique common fixed point in X.

Proof. Let $x_0 \in X$ and consider the sequence $\{x_n\}$ in which $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for all $n = 0, 1, 2, \ldots$. If n is odd, for all $n = 0, 1, 2, \ldots$, we have

$$p(x_{n+1}, x_{n+2}) = p(Tx_n, Sx_{n+1}) \leq \beta(M(x_n, x_{n+1}))m(x_n, x_{n+1}), \tag{2.16}$$

where

$$\begin{aligned} m(x_n, x_{n+1}) &= \max\{p(x_n, x_{n+1}), p(x_n, Tx_n), p(x_{n+1}, x_{Sn+1})\} \\ &= \max\{p(x_n, x_{n+1}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} \\ &= \max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\}. \end{aligned}$$

Suppose $m(x_n, x_{n+1}) = p(x_{n+1}, x_{n+2})$. Since $\beta \in \mathcal{B}$, from (2.16), we have $p(x_{n+1}, x_{n+2}) < p(x_{n+1}, x_{n+2})$, which is a contradiction. Thus $m(x_n, x_{n+1}) = p(x_n, x_{n+1})$, so we have $p(x_{n+1}, x_{n+2}) \leq p(x_n, x_{n+1})$. Similarly, we can also prove that $p(x_{n+1}, x_{n+2}) \leq p(x_n, x_{n+1})$ holds for case that n is even. Thus $p(x_n, x_{n+1})$ is a nonincreasing sequence and hence it is convergent. Hence, there exists $r \geq 0$ such that $p(x_n, x_{n+1}) \to r$. Next, we claim that $r = 0$. Assume on the contrary that $r > 0$. From (2.16), we have

$$\frac{p(x_{n+1}, x_{n+2})}{p(x_n, x_{n+1})} \leq \beta(M(x_n, x_{n+1})) < 1.$$

Which shows $\lim_{n\to\infty} \beta(M(x_n, x_{n+1})) = 1$. Since $\beta \in \mathcal{B}$, we have

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{2.17}$$

Using (1.1) and (2.17), we get $\lim_{n\to\infty} d_p(x_{n+1}, x_n) = 0$. Now, we claim that x_n is a Cauchy sequence in (X, d_p). It suffices to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then there exists some $\varepsilon > 0$ for which we can find subsequences $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ of $\{x_{2n}\}$ such that $n(k)$ is the smallest index for $m(k) > n(k) > k$ such that $d_p(x_{2n(k)}, x_{2m(k)}) \geq \varepsilon$. From (1.1), we obtain $\varepsilon \leq d_p(x_{2n(k)}, x_{2m(k)}) \leq 2p(x_{n(k)}, x_{m(k)})$. Without loss of generality, we can suppose that also

$$n(k) > m(k) > 0, \qquad p(x_{2n(k)}, x_{2m(k)}) \geq \frac{\varepsilon}{2}, \qquad p(x_{2n(k)}, x_{2m(k)-2}) < \frac{\varepsilon}{2}. \tag{2.18}$$

Using (2.18) and (P4), we have

$$\frac{\varepsilon}{2} \leq p(x_{2n(k)}, x_{2m(k)}) \leq p(x_{2n(k)}, x_{2n(k)+1}) + p(x_{2n(k)+1}, x_{2m(k)}) \\ \leq p(x_{2n(k)}, x_{2n(k)+1}) + \beta(M(x_{2m(k)-1}, x_{2n(k)}))m(x_{2m(k)-1}, x_{2n(k)}), \tag{2.19}$$

where

$$m(x_{2m(k)-1}, x_{2n(k)}) = \max\{p(x_{2m(k)-1}, x_{2n(k)}), p(x_{2m(k)-1}, Tx_{2m(k)-1}), p(x_{2n(k)}, Sx_{2n(k)})\} \\ \leq \max\{p(x_{2m(k)-1}, x_{2m(k)-2}) + p(x_{2m(k)-2}, x_{2n(k)}) \\ , p(x_{2m(k)-1}, x_{2m(k)}), p(x_{2n(k)}, x_{2n(k)+1})\}.$$

Using (2.17) and (2.18), we get, $lim_{k\to\infty} m(x_{2m(k)-1}, x_{2n(k)}) \leq \frac{\varepsilon}{2}$. Then, letting $n \to \infty$, from (2.19) we have $\frac{\varepsilon}{2} \leq \lim_{k\to\infty} \beta(M(x_{2m(k)-2}, x_{2n(k)}))\frac{\varepsilon}{2}$, which implies that $1 \leq \lim_{k\to\infty} \beta(M(x_{2m(k)-2}, x_{2n(k)})) < 1$. Since $\beta \in \mathcal{B}$, we have $\lim_{k\to\infty} M(x_{2m(k)-2}, x_{2n(k)}) = 0$, which yields $\lim_{k\to\infty} p(x_{2m(k)-1}, x_{2n(k)}) = 0$. Using (P4), we have

$$p(x_{2n(k)}, x_{2m(k)}) \leq p(x_{2n(k)}, x_{2m(k)-1}) + p(x_{2m(k)-1}, x_{2m(k)}).$$

Letting $k \to \infty$, and using (2.17), we have $\lim_{k\to\infty} p(x_{2n(k)}, x_{2m(k)}) = 0$ which is a contradiction to (2.18). Hence $\{x_n\}$ is a Cauchy sequence in (X, d_p). Since (X, p) is complete, by Lemma 1.3, (X, d_p) is complete. Then there exists $z \in X$ such that

$$\lim_{n\to\infty} d_p(x_n, z) = 0. \tag{2.20}$$

From the property (ii) in Lemma 1.3, we have $p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m)$. By the property (P2) and using (2.17), we have $\lim_{n\to\infty} p(x_n, x_n) = 0$. Using (1.1) and (2.20), we obtain $\lim_{n\to\infty} p(x_n, x_m) = 0$. Then, we have $p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n\to\infty} p(x_n, x_m) = 0$. We assert that $Tz = z$. Assume on the contrary that $p(Tz, z) = \varepsilon$ and $\varepsilon > 0$. From (2.15), we have

$$p(Tz, Sx_{2n}) \leq \beta(M(z, x_{2n}))m(z, x_{2n}), \tag{2.21}$$

where $m(z, x_{2n}) = \max\{p(z, x_{2n}), p(Tz, z), p(x_{2n}, Sx_{2n})\}$. Again, taking limit as $n \to \infty$ in (2.21) and using Lemma 1.4, we obtain $\varepsilon \leq \varepsilon \lim_{n\to\infty} \beta(M(z, x_{2n})) < 1$. Since $\beta \in \mathcal{B}$, we get $\lim_{n\to\infty} M(z, x_{2n}) = 0$, consequently $p(z, Tz) = 0$ and so $Tz = z$. Similarly, we can prove that $Sz = z$. For the uniqueness, suppose that $Tw = Sw = w$ and $z \neq w$. From (2.15), we have $p(z, w) = p(Tz, Sw) \leq \beta(M(z, w))m(z, w)$, where

$$m(z, w) = \max\{p(z, w), p(z, Tz), p(w, Sw)\} = p(z, w).$$

Then $1 \leq \beta(M(z, w)) \leq 1$. Now, as $\beta \in \mathcal{B}$ we conclude that $M(z, w) = 0$, consequently, $P(z, w) = 0$ and we get that $z = w$. □

In theorem 2.5, if $T = S$ is on X, then, we obtain the following result. Following corollary is Theorem 1.9 due to Altun and Sadarangani [5].

Corollary 2.6. *[5] Let (X, p) be a complete partial metric space and T be a self-mapping on X which satisfy,*

$$p(Tx, Ty) \leq \beta(M(x, y)) \max\{p(x, y), p(x, Tx), p(y, Ty)\},$$

for all $x, y \in X$, where $\beta \in \mathcal{B}$ and

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(Tx, y) + p(x, Ty)}{2}\}.$$

Then T has a unique fixed point in X.

Example 2.7. Let $X = [0,\infty)$ and $p : X \times X \to [0,\infty)$ be defined as follows: $p(x,y) = \max\{x,y\}$. Then (X,p) is a complete partial metric space. Define the mappings $T,S : X \to X$ by $Tx = \frac{x}{5}, Sx = \frac{x}{3}$ and set $\beta(t) = \frac{1}{2}$. Suppose $0 \leq x < y$, Then, we have

$$p(Tx,Sy) = \max\{\frac{x}{5},\frac{y}{3}\} \leq \frac{y}{2} = \beta(M(x,y))m(x,y),$$

and

$$p(Ty,Sx) = \max\{\frac{y}{5},\frac{x}{3}\} \leq \frac{y}{2} = \beta(M(x,y))m(x,y),$$

where $m(x,y) = y$. Then the conditions of Theorem 2.5 are satisfied.

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