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Fixed point theorems on a quaternion valued G-metric spaces

Adewale, O. K.[a,*], Olaleru, J. O.[a], Akewe, H.[a]

[a] *Department of Mathematics, University of Lagos, Nigeria.*

Abstract

In this paper, we introduce the concept of a quaternion valued G-metric spaces which generalizes real valued G-metric spaces, complex valued G-metric spaces, real valued metric spaces and complex valued metric spaces known in literature. Analogues of Banach contraction principle, Kannan's and Chatterjea's fixed point theorem are proved. Our results generalize many known results in fixed point theory.

Keywords: G-metric spaces, G^Q-metric spaces, quaternion, fixed point.
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1. Introduction

Quaternion is a number system that extends the complex numbers. It was first defined by Irish mathematician, William Rowan Hamilton in 1843 and applied to mechanics in three-dimensional space. A feature of quaternions is that multiplication of two quaternions is noncommutative.
We write H for the skew field of quaternion and $q \in H$ denotes that q is of the form $q = a + bi + cj + dk$ where $i^2 = j^2 = k^2 = ijk = -1, ij = -ji = k, kj = -jk = -i, ki = -ik = j$ and the modulus of q, $|q| = \sqrt{a^2 + b^2 + c^2 + d^2}$ where a, b, c, and d are real numbers, and i, j, and k are the fundamental quaternion units.
There is possibility of treating quaternion as simply quadruples of real numbers $[a, b, c, d]$, with operation of addition and multiplication suitably defined. The components naturally group into the imaginary part (b, c, d), for which we take this part as a vector and the purely real part, a, which called a scalar. Sometimes, we write a quaternion as $[V, a]$ with $V = (b, c, d)$.

$$[V, a] = [(b, c, d), a] = [a, b, c, d] = a + bi + cj + dk \forall a, b, c, d \in R.$$

Hence, a quaternion may be viewed as a four-dimensional vector (a, b, c, d). For more information about quaternion analysis, see [2] and its references. Motivated by the real life applications of quanternion and

*Corresponding author
Email addresses: adewalekayode2@yahoo.com (Adewale, O. K.), jolaleru@unilag.edu (Olaleru, J. O.), hakewe@unilag.edu (Akewe, H.)

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that of fixed point theorems, the concept of quanternion valued G-metric spaces are recommended in this paper to extend G-metric spaces introduced by [7], quanternion valued metric spaces by [1] and some well known spaces in literature. Now, some basic definitions of concepts which serve as background to this work are stated.

2. Preliminaries

Mustafa and Sims (2006) introduced the definition below as generalization of the usual metric space.

Definition 1.1 [7]. Let X be a non-empty set and $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties:

(i) $G(x, y, z) = 0$ if and only if $x = y = z$

(ii) $G(x, x, y) > 0, \quad \forall x, y \in X$, with $x \neq y$

(iii) $G(x, x, y) \leq G(x, y, z), \quad \forall x, y, z \in X$, with $z \neq y$

(iv) $G(x, y, z) = G(x, z, y) = G(y, x, z) = ...$ (symmetry).

(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z) \quad \forall a, x, y, z \in X$ (rectangle inequality)

The function G is called a G-metric and (X, G), G-metric space.
A G-metric space (X, G) is said to be symmetric if $G(x, y, y) = G(x, x, y)$ for all $x, y \in X$.

Akbar et al. (2011) introduced the definition of the complex valued metric space as follows.

Definition 1.2 [2]. Let X be a non-empty set and $d_C : X \times X \to C$ be a function satisfying the following properties:

(i) $d_C(x, y) = 0$ if and only if $x = y$;

(ii) $d_C(x, y) \geq 0, \quad \forall x, y \in X$;

(iii) $d_C(x, y) = d_C(y, x)$ (symmetry);

(iv) $d_C(x, y) \leq d_C(x, z) + d_C(z, y) \quad \forall x, y, z \in X$ (triangle inequality).

The function d_C is called a complex valued metric on X and (X, d_C), a complex valued metric space.

Ahmed et al. (2014) extended the notion of metric spaces to quanternion valued metric spaces as shown below.

Definition 1.3 [1]. Let X be a non-empty set and $d_H : X \times X \to H$ be a function satisfying the following properties:

(i) $d_H(x, y) = 0$ if and only if $x = y$;

(ii) $d_H(x, y) \geq 0, \quad \forall x, y \in X$;

(iii) $d_H(x, y) = d_H(y, x)$ (symmetry);

(iv) $d_H(x, y) \leq d_H(x, z) + d_H(z, y) \quad \forall x, y, z \in X$ (triangle inequality).

The function d_H is called a quaternion valued metric on X and (X, d_H), a quaternion valued metric space.

Definition of interior point, limit point and balls as stated by Ahmed et al. (2014).

Definition 1.4 [1]. Point $x \in X$ is said to be an interior point of set $A \subset X$ whenever there exists $0 \prec r \in H$ such that

$$B(x, r) = \{y \in X : d_H(x, y) \prec r\} \subset A$$

Definition 1.5 [1]. Point $x \in X$ is said to be a limit point of $A \subset X$ whenever for every $0 \prec r \in H$

$$B(x, r) \cap (A - \{x\}) \neq \emptyset$$

Definition 1.6 [1]. Set A is called an open set whenever each element of A is an interior point of A. Subset $B \subset X$ is called a closed set whenever each limit point of B belongs to B. The family

$$F = \{B(x, r) : x \in X, 0 \prec r\}$$

is a subbase for Hausdroff topology τ on X.

Definition 1.7 [1]. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $q \in H$ with $0 \prec q$ there is $n_0 \in N$ such that for all $n > n_0, d_H(x_n, x) \prec q$, then $\{x_n\}$ is said to be convergent if $\{x_n\}$ converges to the limit point x; that is, $x_n \to x$ as $n \to \infty$. If for every $q \in H$ with $0 \prec q$ there is $n_0 \in N$ such that for all $n > n_0, d_H(x_n, x_{n+m}) \prec q$, then $\{x_n\}$ is called Cauchy sequence in (X, d_H). If every Cauchy sequence is convergent in (X, d_H), then (X, d_H) is called a complete quaternion valued metric space.

3. Main Result

Ahmed et al. (2014) introduced the following ordering.
Let H be the set of quaternion and $q_1, q_2 \in H$. Define a partial order $\preceq$ on H as follows:
$q_1 \preceq q_2$ if and only if $Re(q_1) \leq Re(q_2)$ and $Im_s(q_1) \leq Im_s(q_2)$, $q_1, q_2 \in H$, $s = i, j, k$ where $Im_i = b, Im_j = c, Im_k = d$.
It follows that $q_1 \preceq q_2$, if one of the following conditions is satisfied:

(i) $Re(q_1) = R(q_2)$, $Im_{s_1}(q_1) = Im_{s_1}(q_2)$ where $s_1 = j, k$, $Im_i(q_1) < Im_i(q_2)$;
(ii) $Re(q_1) = R(q_2)$, $Im_{s_2}(q_1) = Im_{s_2}(q_2)$ where $s_2 = i, k$, $Im_j(q_1) < Im_j(q_2)$;
(iii) $Re(q_1) = R(q_2)$, $Im_{s_3}(q_1) = Im_{s_3}(q_2)$ where $s_3 = i, j$, $Im_k(q_1) < Im_k(q_2)$;
(iv) $Re(q_1) = R(q_2)$, $Im_{s_1}(q_1) = Im_{s_1}(q_2)$, $Im_i(q_1) = Im_i(q_2)$;
(v) $Re(q_1) = R(q_2)$, $Im_{s_1}(q_1) = Im_{s_1}(q_2)$, $Im_j(q_1) = Im_j(q_2)$;
(vi) $Re(q_1) = R(q_2)$, $Im_{s_1}(q_1) = Im_{s_1}(q_2)$, $Im_k(q_1) = Im_k(q_2)$;
(vii) $Re(q_1) = R(q_2)$, $Im_s(q_1) < Im_s(q_2)$;
(viii) $Re(q_1) < R(q_2)$, $Im_s(q_1) = Im_s(q_2)$;
(ix) $Re(q_1) < R(q_2)$, $Im_{s_1}(q_1) = Im_{s_1}(q_2)$, $Im_i(q_1) < Im_i(q_2)$;
(x) $Re(q_1) < R(q_2)$, $Im_{s_2}(q_1) = Im_{s_2}(q_2)$, $Im_j(q_1) < Im_j(q_2)$;
(xi) $Re(q_1) < R(q_2)$, $Im_{s_3}(q_1) = Im_{s_3}(q_2)$, $Im_k(q_1) < Im_k(q_2)$;
(xii) $Re(q_1) < R(q_2)$, $Im_{s_1}(q_1) < Im_{s_1}(q_2)$, $Im_i(q_1) = Im_i(q_2)$;
(xiii) $Re(q_1) < R(q_2)$, $Im_{s_2}(q_1) < Im_{s_2}(q_2)$, $Im_i(q_1) = Im_i(q_2)$;
(xiv) $Re(q_1) < R(q_2)$, $Im_{s_3}(q_1) < Im_{s_3}(q_2)$, $Im_i(q_1) = Im_i(q_2)$;
(xv) $Re(q_1) < R(q_2)$, $Im_s(q_1) < Im_s(q_2)$;
(xiv) $Re(q_1) = R(q_2)$, $Im_s(q_1) = Im_s(q_2)$.

Particularly, we will write $q_1 \precnsim q_2$ if $q_1 \neq q_2$ and one from (i), to (xvi) is satisfied and we will write $q_1 \prec q_2$ if only (xv) is satisfied. It should be noted that

$$q_1 \preceq q_2 \Rightarrow |q_1| \leq |q_2|.$$

Motivated by Ahmed et al.'s work in [1], we introduce the following definitions.
Definition 2.1. Let X be a non-empty set, H, a set of quaternions and $G^Q : X \times X \times X \to H$ be a function satisfying the following properties:

(i) $G^Q(x, y, z) = 0 \qquad$ if and only if $x = y = z$
(ii) $0 \prec G^Q(x, x, y), \qquad \forall x, y \in X$, with $x \neq y$
(iii) $G^Q(x, x, y) \preceq G^Q(x, y, z), \qquad \forall x, y, z \in X$, with $z \neq y$
(iv) $G^Q(x, y, z) = G^Q(y, z, x) = G^Q(x, z, y) = ...$ (symmetry).

(v) There exists a real number $s \geq 1$ such that $G^Q(x,y,z) \preceq s[G^Q(x,a,a) + G^Q(a,y,z)] \quad \forall a,x,y,z \in X$ (rectangle inequality)

Then the function G^Q is called a quaternion G-metric and (X, G^Q) is the quaternion G^Q-metric space. A G^Q - metric space is complete if every Cauchy sequence in it is G^Q - convergent in it.

Remark 1. We obtain quaternion valued metric space in [1] if $y = z$ and we set $d(x,y) = G(x,y,y)$.

Definition 2.2. $\{x_n\}$ is said to converges to the limit point x; that is, $x_n \to x$ as $n \to \infty$ if for every $q \in H$ with $0 \prec q$ there is $n_0 \in N$ such that for all $n > n_0, G^Q(x_n, x, x) \prec q$. If for every $q \in H$ with $0 \prec q$ there is $n_0 \in N$ such that for all $n > n_0, G^Q(x_n, x_{n+m}, x_{n+m+l}) \prec q$, then $\{x_n\}$ is called Cauchy sequence in (X, G^Q). If every Cauchy sequence is convergent in (X, G^Q), then (X, G^Q) is called a complete quaternion valued G-metric space.

Some auxiliary Lemmas with proofs using the concept of quaternion valued G-metric spaces are stated below. These Lemmas will be used to prove some fixed point theorems of contractive mappings in this newly introduced space.

Lemma 2.3. Let (X, G^Q) be a G^Q-metric space and $\{x_n\}$ a sequence in X. $\{x_n\}$ converges to $x \in X$ if and only if $|G^Q(x_n, x, x)| \to 0$ as $n \to \infty$.

Proof: Suppose that $\{x_n\}$ converges to x. For any real number $\epsilon > 0$, let $q = \frac{\epsilon}{2} + \frac{\epsilon}{2}i + \frac{\epsilon}{2}j + \frac{\epsilon}{2}k$. Then, $0 \prec q \in H$ and there is a natural number N such that $G^Q(x_n, x, x) \prec q$ for all $n \in N$. Therefore, $|G^Q(x_n, x, x)| < |q| = \sqrt{4(\frac{\epsilon}{2})^2} = \epsilon$ for all $n \in N$. Hence, $|G^Q(x_n, x, x)| \to 0$ as $n \to \infty$.
Conversely, suppose that $|G^Q(x_n, x, x)| \to 0$ as $n \to \infty$. Then, given $q \in H$ with $0 \prec q$, there exists a real number $\delta > 0$, such that, for $h \in H$, $|h| < \delta \Rightarrow h \prec q$. For this δ, there is a natural number N such that $|G^Q(x_n, x, x)| < \delta$ for all $n > N$ which implies that $G^Q(x_n, x, x) \prec q$ for all $n > N$, hence $\{x_n\}$ converges to $x \in X$.

Lemma 2.4. Let (X, G^Q) be a G^Q-metric space and $\{x_n\}$ a sequence in X. $\{x_n\}$ is Cauchy sequence if and only if $|G^Q(x_n, x_m, x_l)| \to 0$ as $n, m, l \to \infty$.

Proof: Suppose that $\{x_n\}$ is a Cauchy sequence. For any real number $\epsilon > 0$, let $q = \frac{\epsilon}{2} + \frac{\epsilon}{2}i + \frac{\epsilon}{2}j + \frac{\epsilon}{2}k$. Then, $0 \prec q \in H$ and there is a natural number N such that $G^Q(x_n, x_{n+m}, x_{n+m+l}) \prec q$ for all $n \in N$. Therefore, $|G^Q(x_n, x_{n+m}, x_{n+m+l})| < |q| = \sqrt{4(\frac{\epsilon}{2})^2} = \epsilon$ for all $n \in N$. Hence, $|G^Q(x_n, x_{n+m}, x_{n+m+l})| \to 0$ as $n \to \infty$.
Conversely, suppose that $|G^Q(x_n, x_{n+m}, x_{n+m+l})| \to 0$ as $n \to \infty$. Then, given $q \in H$ with $0 \prec q$, there exists a real number $\delta > 0$, such that, for $h \in H$, $|h| < \delta \Rightarrow h \prec q$. For this δ, there is a natural number N such that $|G^Q(x_n, x_{n+m}, x_{n+m+l})| < \delta$ for all $n > N$ which implies that $G^Q(x_n, x_{n+m}, x_{n+m+l}) \prec q$ for all $n > N$, hence $\{x_n\}$ is a Cauchy sequence.

The following example shows that G_Q-metric space is not necessarily a G-metric space.

Example 1 Let $X = S \cup T$ where $S = N$, $T = \{\frac{1}{2n}, n \in N\}$ and $G^Q : X \times X \times X \to H$ be defined as follows:

$$G^Q(a,b,c) = G^Q(b,c,a) = G^Q(a,c,b) = ...$$

for $a, b.c \in X$ and

$$G^Q(a,b,c) = \begin{cases} 0, & if \quad a = b = c; \\ 1 + 2v, & if \quad a,b,c \in S \; and \; a = b, \; b = c \; or \; a = c; \\ 1 + 7v, & Otherwise. \end{cases}$$

where $v \in \{ai + bj + ck : i^2 = j^2 = k^2 = ijk = -1, a, b, c \in R\}$. Then G^Q is a G^Q - metric on X but not G - metric on X.

Example 2 Let $X = \{\frac{1}{n}, n \in N\}$ with

$$G^Q(a,b,c) = G^Q(b,c,a) = G^Q(a,c,b) = ...$$

for $a, b.c \in X$. $G^Q : X^3 \to H$ defined as follows:

$$G^Q(q_1,q_2,q_3) = 1 + |x_2 - x_1| + |y_3 - y_2| + |z_1 - z_3|$$

where
$q_1 = 1 + x_1 i + y_1 j + z_1 k$,
$q_2 = 1 + x_2 i + y_2 j + z_2 k$
$q_3 = 1 + x_3 i + y_3 j + z_3 k$
with $x_1 + x_2 + x_3 > 1$, $y_1 + y_2 + y_3 > 1$ and $z_1 + z_2 + z_3 = 1$
G^Q is a quanternion valued G-metric on X but not G - metric on X.

Now, the main theorems are stated with proofs:
Theorem 2.5. Let X be a complete G^Q-metric space and $T : X \to X$ a map for which there exist a real number k satisfying $0 \le k < 1$, such that for each pair $x, y, z \in X$:
$G^Q(Tx, Ty, Tz) \preceq kG^Q(x, y, z)$.
Then T has a unique fixed point.
Proof: Let $x_0 \in X$ be an arbitrary point and define the sequence $\{x_n\}$ by $x_n = T^n x_0 \forall n \in N$, then we have

$$G^Q(x_n, x_{n+1}, x_{n+1}) \preceq kG^Q(x_{n-1}, x_n, x_n). \tag{3.1}$$

We deduce that

$$\begin{aligned} G^Q(x_n, x_{n+1}, x_{n+1}) &\preceq kG^Q(x_{n-1}, x_n, x_n) \\ &\preceq k^2 G^Q(x_{n-2}, x_{n-1}, x_{n-1}) \\ G^Q(x_n, x_{n+1}, x_{n+1}) &\preceq k^3 G^Q(x_{n-3}, x_{n-2}, x_{n-2}) \\ G^Q(x_n, x_{n+1}, x_{n+1}) &\preceq k^n G^Q(x_0, x_1, x_1). \end{aligned}$$

By repeated use of rectangle inequality with $m > n$, we have

$$\begin{aligned} G^Q(x_n, x_m, x_m) &\preceq G^Q(x_n, x_{n+1}, x_{n+1}) + G^Q(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G^Q(x_{n+2}, x_{n+3}, x_{n+3}) + ... + G^Q(x_{m-1}, x_m, x_m). \end{aligned}$$

From (3.2) and (3.3), we have

$$\begin{aligned} G^Q(x_n, x_m, x_m) &\preceq G^Q(x_n, x_{n+1}, x_{n+1}) + G^Q(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G^Q(x_{n+2}, x_{n+3}, x_{n+3}) + ... + G^Q(x_{m-1}, x_m, x_m) \\ &\preceq [k^n + k^{n+1} + k^{n+2} + ... + k^{m-1}] G^Q(x_0, x_1, x_1) \\ &\preceq k^n [1 + k + k^2 + ... + k^{m-n-1}] G^Q(x_0, x_1, x_1) \\ &\preceq [\frac{k^n}{1-k}] G^Q(x_0, x_1, x_1). \end{aligned}$$

Taking the limit of $G^Q(x_n, x_m, x_m)$ as $n, m \to \infty$, we have

$$\lim_{n,m\to\infty} |G^Q(x_n, x_m, x_m)| = \lim_{n,m\to\infty} [\frac{k^n}{1-k}] |G^Q(x_0, x_1, x_1)| = 0. \tag{3.2}$$

For $n, m, l \in N$

$$G^Q(x_n, x_m, x_l) \preceq G^Q(x_n, x_m, x_m) + G^Q(x_m, x_m, x_l). \tag{3.3}$$

Taking the limit of $G^Q(x_n, x_m, x_l)$ as $n, m, l \to \infty$,
we have

$$G^Q(x_n, x_m, x_l) \preceq \lim_{n,m,l\to\infty} |G^Q(x_n, x_m, x_m) + G^Q(x_m, x_m, x_l)| = 0. \tag{3.4}$$

So, x_n is a G^Q-Cauchy Sequence.
By completeness of (X, G^Q), there exist $u \in X$ such that $\{x_n\}$ is G^Q-convergent to u.
Suppose $Tu \neq u$

$$G^Q(x_n, Tu, Tu) \preceq kG^Q(x_{n-1}, u, u). \tag{3.5}$$

Taking the limit as n $\to \infty$, we get

$$G^Q(u, Tu, Tu) \preceq kG^Q(u, u, u). \tag{3.6}$$

Hence,

$$G^Q(u, Tu, Tu) \preceq 0. \tag{3.7}$$

This is a contradiction. So, $Tu = u$.
To show the uniqueness, suppose $v \neq u$ is such that $Tv = v$, then

$$G^Q(Tu, Tv, Tv) \preceq kG^Q(u, v, v). \tag{3.8}$$

Since $Tu = u$ and $Tv = v$, we have

$$G^Q(u, v, v) \preceq 0. \tag{3.9}$$

which implies that $v = u$

Remark 2. If $\forall q = a + bi + cj + dk \in H$, $b = c = d = 0$ in Theorem 2.5, Banach contraction principle in a real valued G-metric space is obtained. Setting $d(x, y) = G(x, y, y)$, the theorem reduces to Banach contraction principle in [3].

Theorem 2.6. Let X be a complete G^Q - metric space and $T : X \to X$ a map for which there exist a real number k satisfying $0 \leq k < \frac{1}{2}$, such that for each pair $x, y, z \in X$:
$G^Q(Tx, Ty, Tz) \preceq k[G^Q(x, Tx, Tx) + G^Q(y, Ty, Ty) + G^Q(z, Tz, Tz)]$.
Then T has a unique fixed point.
Proof: Let $x_0 \in X$ be an arbitrary point and define the sequence $\{x_n\}$ by $x_n = T^n x_0 \forall n \in N$, then we have

$$\begin{aligned} G^Q(x_n, x_{n+1}, x_{n+1}) &\preceq k[G^Q(x_{n-1}, x_n, x_n) + 2G^Q(x_n, x_{n+1}, x_{n+1})] \\ &\preceq \frac{k}{1-2k} G^Q(x_{n-1}, x_n, x_n). \end{aligned}$$

Let $q = \frac{k}{1-2k}$
We deduce that

$$\begin{aligned} G^Q(x_n, x_{n+1}, x_{n+1}) &\preceq qG^Q(x_{n-1}, x_n, x_n) \\ &\preceq q^2 G^Q(x_{n-2}, x_{n-1}, x_{n-1}) \\ G^Q(x_n, x_{n+1}, x_{n+1}) &\preceq q^3 G^Q(x_{n-3}, x_{n-2}, x_{n-2}) \\ G^Q(x_n, x_{n+1}, x_{n+1}) &\preceq q^n G^Q(x_0, x_1, x_1). \end{aligned}$$

By repeated use of rectangle inequality with $m > n$, we have

$$\begin{aligned} G^Q(x_n, x_m, x_m) \preceq\ & G^Q(x_n, x_{n+1}, x_{n+1}) + G^Q(x_{n+1}, x_{n+2}, x_{n+2}) \\ & + G^Q(x_{n+2}, x_{n+3}, x_{n+3}) + ... + G^Q(x_{m-1}, x_m, x_m). \end{aligned}$$

From (3.14) and (3.15), we have

$$\begin{aligned}
G^Q(x_n,x_m,x_m) &\preceq G^Q(x_n,x_{n+1},x_{n+1})+G^Q(x_{n+1},x_{n+2},x_{n+2})\\
&\quad +G^Q(x_{n+2},x_{n+3},x_{n+3})+...+G^Q(x_{m-1},x_m,x_m)\\
&\preceq [q^n+q^{n+1}+q^{n+2}+...+q^{m-1}]G^Q(x_0,x_1,x_1)\\
&\preceq q^n[1+q+q^2+...+q^{m-n-1}]G^Q(x_0,x_1,x_1)\\
&\preceq [\frac{q^n}{1-q}]G^Q(x_0,x_1,x_1).
\end{aligned}$$

Taking the limit of $G^Q(x_n,x_m,x_m)$ as $n,m\to\infty$, we have

$$\lim_{n,m\to\infty}|G^Q(x_n,x_m,x_m)| = \lim_{n,m\to\infty}[\frac{q^n}{1-q}]|G^Q(x_0,x_1,x_1)| = 0. \tag{3.10}$$

For $n,m,l\in N$

$$G^Q(x_n,x_m,x_l)\preceq G^Q(x_n,x_m,x_m)+G^Q(x_m,x_m,x_l). \tag{3.11}$$

Taking the limit of $G^Q(x_n,x_m,x_l)$ as $n,m,l\to\infty$,
we have

$$G^Q(x_n,x_m,x_l)\preceq \lim_{n,m,l\to\infty}|G^Q(x_n,x_m,x_m)+G^Q(x_m,x_m,x_l)| = 0. \tag{3.12}$$

So, x_n is a G^Q-Cauchy Sequence.
By completeness of (X,G^Q), there exist $u\in X$ such that $\{x_n\}$ is G^Q-convergent to u.
Suppose $Tu\neq u$

$$G^Q(x_n,Tu,Tu)\preceq k[G^Q(x_{n-1},x_n,x_n)+2G^Q(u,Tu,Tu)]. \tag{3.13}$$

Taking the limit as $n\to\infty$, we get

$$G^Q(u,Tu,Tu)\preceq 2kG^Q(u,Tu,Tu). \tag{3.14}$$

Hence,

$$G^Q(u,Tu,Tu)\preceq 0. \tag{3.15}$$

This is a contradiction. So, $Tu=u$.
To show the uniqueness, suppose $v\neq u$ is such that $Tv=v$, then

$$G^Q(Tu,Tv,Tv)\preceq k[G^Q(u,Tu,Tu)+2G^Q(v,Tv,Tv)]. \tag{3.16}$$

Since $Tu=u$ and $Tv=v$, we have

$$G^Q(u,v,v)\preceq 0. \tag{3.17}$$

which implies that $v=u$

Remark 3. If $\forall q=a+bi+cj+dk\in H$, $b=c=d=0$ in Theorem 2.6, Kannan's fixed point theorem in a real valued G-metric space is obtained. Setting $d(x,y)=G(x,y,y)$, the theorem reduces to Kannan's fixed point theorem in [6].

Theorem 2.7. Let X be a complete G^Q - metric space and $T:X\to X$ a map for which there exist a real number k satisfying $0\le k<\frac{1}{2}$, such that for each pair $x,y,z\in X$:
$G^Q(Tx,Ty,Tz) \preceq k[G^Q(x,Ty,Ty)+G^Q(y,Tx,Tx)+G^Q(z,Tx,Tx)]$.
Then T has a unique fixed point.
Proof: Let $x_0\in X$ be an arbitrary point and define the sequence $\{x_n\}$ by $x_n=T^nx_0\forall n\in N$, then we have

$$\begin{aligned}
G^Q(x_n,x_{n+1},x_{n+1}) &\preceq kG^Q(x_{n-1},x_{n+1},x_{n+1})\\
&\preceq k[G^Q(x_{n-1},x_n,x_n)+G^Q(x_n,x_{n+1},x_{n+1})]\\
&\preceq \frac{k}{1-k}G^Q(x_{n-1},x_n,x_n).
\end{aligned}$$

Let $q = \frac{k}{1-k}$
We deduce that

$$\begin{aligned} G^Q(x_n, x_{n+1}, x_{n+1}) &\preceq qG^Q(x_{n-1}, x_n, x_n) \\ &\preceq q^2G^Q(x_{n-2}, x_{n-1}, x_{n-1}) \\ G^Q(x_n, x_{n+1}, x_{n+1}) &\preceq q^3G^Q(x_{n-3}, x_{n-2}, x_{n-2}) \\ G^Q(x_n, x_{n+1}, x_{n+1}) &\preceq q^nG^Q(x_0, x_1, x_1). \end{aligned}$$

By repeated use of rectangle inequality, we have

$$\begin{aligned} G^Q(x_n, x_m, x_m) \preceq\ & G^Q(x_n, x_{n+1}, x_{n+1}) + G^Q(x_{n+1}, x_{n+2}, x_{n+2}) \\ & + G^Q(x_{n+2}, x_{n+3}, x_{n+3}) + ... + G^Q(x_{m-1}, x_m, x_m). \end{aligned}$$

From (3.26) and (3.27), we have

$$\begin{aligned} G^Q(x_n, x_m, x_m) &\preceq G^Q(x_n, x_{n+1}, x_{n+1}) + G^Q(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G^Q(x_{n+2}, x_{n+3}, x_{n+3}) + ... + G^Q(x_{m-1}, x_m, x_m) \\ &\preceq [q^n + q^{n+1} + q^{n+2} + ... + q^{m-1}]G^Q(x_0, x_1, x_1) \\ &\preceq q^n[1 + q + q^2 + ... + q^{m-n-1}]G^Q(x_0, x_1, x_1) \\ &\preceq [\frac{q^n}{1-q}]G^Q(x_0, x_1, x_1). \end{aligned}$$

Taking the limit of $QG_b(x_n, x_m, x_m)$ as $n, m \to \infty$, we have

$$\lim_{n,m\to\infty} |G^Q(x_n, x_m, x_m)| = \lim_{n,m\to\infty} [\frac{q^n}{1-q}]|G^Q(x_0, x_1, x_1)| = 0. \tag{3.18}$$

For $n, m, l \in N$

$$G^Q(x_n, x_m, x_l) \preceq G^Q(x_n, x_m, x_m) + G^Q(x_m, x_m, x_l). \tag{3.19}$$

Taking the limit of $G^Q(x_n, x_m, x_l)$ as $n, m, l \to \infty$,
we have

$$G^Q(x_n, x_m, x_l) \preceq \lim_{n,m,l\to\infty} |G^Q(x_n, x_m, x_m) + G^Q(x_m, x_m, x_l)| = 0. \tag{3.20}$$

So, x_n is a G^Q-Cauchy Sequence.
By completeness of (X, G^Q), there exist $u \in X$ such that $\{x_n\}$ is G^Q-convergent to u.
Suppose $Tu \neq u$

$$G^Q(x_n, Tu, Tu) \preceq k[G^Q(x_{n-1}, Tu, Tu) + G^Q(u, x_n, x_n) + G^Q(u, x_n, x_n)]. \tag{3.21}$$

Taking the limit as n $\to \infty$, we get

$$G^Q(u, Tu, Tu) \preceq kG^Q(u, Tu, Tu). \tag{3.22}$$

Hence,

$$G^Q(u, Tu, Tu) \preceq 0. \tag{3.23}$$

This is a contradiction. So, $Tu = u$.
To show the uniqueness, suppose $v \neq u$ is such that $Tv = v$, then

$$G^Q(Tu, Tv, Tv) \preceq k[G^Q(u, Tv, Tv) + 2G^Q(v, Tu, Tu)]. \tag{3.24}$$

Since $Tu = u$ and $Tv = v$, we have

$$G^Q(u, v, v) \preceq 0. \tag{3.25}$$

which implies that $v = u$

Remark 4. If $\forall q = a + bi + cj + dk \in H$, $b = c = d = 0$ in Theorem 2.7, Chatterjea's fixed point theorem in a real valued G-metric space is obtained. Setting $d(x, y) = G(x, y, y)$, the theorem reduces to Chatterjea's fixed point theorem in [4].

Corollary 2.8. Let X be a complete G^Q - metric space and $T : X \to X$ a map for which there exist the real numbers a, b, c satisfying $0 \leq a < 1$, $b \leq \frac{1}{2}$ and $c < \frac{1}{2}$ such that for each pair $x, y, z \in X$ at least one of the following is true.

$$\begin{aligned}
(G^QZ_1) \;\; G^Q(Tx, Ty, Tz) &\preceq aG^Q(x, y, z) \\
(G^QZ_2) \;\; G^Q(Tx, Ty, Tz) &\preceq b[G^Q(x, Tx, Tx) + G^Q(y, Ty, Ty) + G^Q(z, Tz, Tz)] \\
(G^QZ_3) \;\; G^Q(Tx, Ty, Tz) &\preceq c[G^Q(x, Ty, Ty) + G^Q(y, Tx, Tx) + G^Q(z, Tx, Tx)].
\end{aligned}$$

Then T has a unique fixed point.

References

[1] A. E. Ahmed, A. J. Asad, S. Omran, *Fixed point theorems in quaternion-valued metric spaces*, Abstract and Applied Analysis, *2014* (2014), Article ID 258985.

[2] A. Akbar, F. Brian, M. Khan, *Common fixed point theorem in complex valued metric spaces*, Numerical Functional Analysis and Optimizaton, *32*(3) (2011), 245–255.

[3] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fundamenta Mathematicae: (1922), 133–181.

[4] S. K. Chatterjea, *Fixed point theorems*, C. R. Acad. Bulgare Sci., *25* (1972), 727–730.

[5] K. S. Eke, J. O. Olaleru, *Some fixed point results on ordered G - partial metric spaces*, I CASTOR Journal of Mathematical Sciences, *7*(1) (2013), 65-78.

[6] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., *10* (1968), 71–76.

[7] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Analysis, *7*(2) (2006), 289–297.