Available online at http://www.cna-journal.com
Commun. Nonlinear Anal. 7(1) (2019), 1–16
Research Article

Communications in Nonlinear Analysis

Journal Homepage: www.cna-journal.com

PPF Dependent Fixed Points Of Generalized Contractions Via C_G-Simulation Functions

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Abstract

In this paper, we introduce the notion of generalized $Z_{G,\alpha,\mu,\eta,\varphi}-$contraction with respect to the C_G-simulation function introduced by Liu, Ansari, Chandok and Radenović[20] and prove the existence of PPF dependent fixed points in Banach spaces. We draw some corollaries and an example is provided to illustrate our main result.

Keywords: $\alpha-$ admissible, $\mu-$ subadmissible, $C-$ class function, Razumikhin class, PPF dependent fixed point, simulation function, C_G-simulation function.
2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries

Banach contraction principle is one of the famous and basic fundemental result in fixed point theory. Due to its significance, many authors generalized and extended the Banach contraction principle by introducing new functions like $\alpha-$admissible mapping, $C-$class function, simulation function etc., for more details we refer [1, 2, 7, 18, 23].

Throughout this paper, we denote the real line by $\mathbb{R}$, $\mathbb{R}^+ = [0, \infty)$, and $\mathbb{N}$ is the set of all natural numbers, $\mathbb{Z}$ is the set of intergers.

In 2013, Karapınar, Kumam and Salimi[18] introduced the notion of triangular $\alpha-$admissible mappings as follows.

Definition 1.1. [18] Let T be a self mapping on X and let $\alpha : X \times X \to \mathbb{R}^+$ be a function. Then T is said to be a *triangular* $\alpha-$*admissible* mapping if for any $x, y, z \in X$,

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Received 2019-08-11

$\alpha(x,y) \geq 1 \implies \alpha(Tx,Ty) \geq 1$ and
$\alpha(x,z) \geq 1,\ \alpha(z,y) \geq 1 \implies \alpha(x,y) \geq 1.$

In 2014, Ansari[1] introduced the concept of $C-$class function and many authors [2, 20] extended and generalized various fixed point results of a selfmap satisfying certain inequality involving $C-$class function in complete metric spaces.

Definition 1.2. [1] A mapping $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called a *$C-$class function* if it is continuous and for any $s,t \in \mathbb{R}^+$, the function G satisfies the following conditions:
(i) $G(s,t) \leq s$ and
(ii) $G(s,t) = s$ implies that either $s = 0$ or $t = 0$.
The family of all $C-$class functions is denoted by Δ.

Example 1.3. [1] The following functions belong to Δ.
(i) $G(s,t) = s - t$ for all $s,t \in \mathbb{R}^+$.
(ii) $G(s,t) = ks$ for all $s,t \in \mathbb{R}^+$ where $0 < k < 1$.
(iii) $G(s,t) = \frac{s}{(1+t)^r}$ for all $s,t \in \mathbb{R}^+$ where $r \in \mathbb{R}^+$.
(iv) $G(s,t) = s\beta(s)$ for all $s,t \in \mathbb{R}^+$ where $\beta : \mathbb{R}^+ \to [0,1)$ is continuous.
(v) $G(s,t) = s - \phi(s)$ for all $s,t \in \mathbb{R}^+$ where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $\phi(t) = 0$ if and only if $t = 0$.
(vi) $G(s,t) = sh(s,t)$ for all $s,t \in \mathbb{R}^+$ where $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous such that $h(s,t) < 1$ for all $s,t \in \mathbb{R}^+$.

In 2015, Khojasteh, Shukla and Radenović[14] introduced the notion of simulation function and proved the existence of fixed points of Z_H-contractions in complete metric spaces. Later, many authors extended and generalized the simulation function by using different types of functions, for more details we refer [17, 21, 22].

Definition 1.4. [14] A function $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is said to be a *simulation function* if it satisfies the following conditions:
(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$ for all $t,s > 0$;
(ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n > 0$, then $\limsup\limits_{n\to\infty} \zeta(t_n, s_n) < 0$.

We denote the set of all simulation functions in the sense of Definition 1.4 by Z_H.

Example 1.5. [14, 17] Let $\phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function with $\phi_i(t) = 0$ if and only if $t = 0$ for $i = 1,2,3$. Then the following functions $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ belong to Z_H.
(i) $\zeta(t,s) = \frac{s}{s+1} - t$ for all $t,s \in \mathbb{R}^+$.
(ii) $\zeta(t,s) = \lambda s - t$ for all $t,s \in \mathbb{R}^+$ and $0 < \lambda < 1$.
(iii) $\zeta(t,s) = \phi_1(s) - \phi_2(t)$ for all $t,s \in \mathbb{R}^+$, where $\phi_1(t) < t \leq \phi_2(t)$ for all $t > 0$.
(iv) $\zeta(t,s) = s - \phi_3(s) - t$ for all $t,s, \in \mathbb{R}^+$.

Definition 1.6. [14] Let (X,d) be a metric space, $T : X \to X$ be a mapping and $\zeta \in Z_H$. Then T is called a *Z_H-contraction with respect to ζ* if

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0 \tag{1.1}$$

for any $x,y \in X$.

Theorem 1.7. [14] *Let (X,d) be a complete metric space and $T : X \to X$ be a Z_H-contraction with respect to ζ. Then T has a unique fixed point u in X and for every $x_0 \in X$ the Picard sequence $\{x_n\}$ where $x_n = Tx_{n-1}$ for any $n \in \mathbb{N}$ converges to the fixed point of T.*

In 2015, Nastasi and Vetro[3] proved the existence of fixed points in complete metric spaces by using simulation functions and a lowersemicontinuous function.

Theorem 1.8. [3] *Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping. Suppose that there exist a simulation function $\zeta \in Z_H$ and a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ such that*

$$\zeta(d(Tx,Ty) + \varphi(Tx) + \varphi(Ty), d(x,y) + \varphi(x) + \varphi(y)) \geq 0 \tag{1.2}$$

for any $x, y \in X$. Then T has a unique fixed point $u \in X$ such that $\varphi(u) = 0$.

In 2018, Cho[11] introduced the notion of generalized weakly contractive mappings in metric spaces and proved the existence of its fixed points in complete metric spaces.

Definition 1.9. [11] Let (X,d) be a metric space, T a self-mapping of X. Then T is called a generalized weakly contractive mapping if

$$\psi(d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)) \leq \psi(m(x,y,d,T,\varphi)) - \phi(l(x,y,d,T,\varphi)) \tag{1.3}$$

for any $x, y \in X$, where
(i) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and $\psi(t) = 0 \iff t = 0$,
(ii) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semicontinuous function and $\phi(t) = 0 \iff t = 0$,
(iii) $m(x,y,d,T,\varphi) = \max\{d(x,y) + \varphi(x) + \varphi(y), d(x,Tx) + \varphi(x) + \varphi(Tx), d(y,Ty) + \varphi(y) + \varphi(Ty), \frac{1}{2}[d(x,Ty) + \varphi(x) + \varphi(Ty) + d(y,Tx) + \varphi(y) + \varphi(Ty)]\}$,
(iv) $l(x,y,d,T,\varphi) = \max\{d(x,y) + \varphi(x) + \varphi(y),\ d(y,Ty) + \varphi(y) + \varphi(Ty)\}$ and
(v) $\varphi : X \to \mathbb{R}^+$ is a lower semicontinuous function.

Theorem 1.10. [11] *Let X be a complete metric space. If T is a generalized weakly contractive mapping, then there exists a unique $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$.*

In 2018, Liu, Ansari, Chandok and Radenović[20] generalized the simulation function introduced by Khojasteh, Shukla and Radenović[14] by using $C-$class functions with C_G property.

Definition 1.11. [20] A mapping $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ has the *property* C_G if there exists an $C_G \geq 0$ such that
(i) $G(s,t) > C_G$ implies $s > t$, and
(ii) $G(t,t) \leq C_G$ for all $s, t \in \mathbb{R}^+$.

Example 1.12. [20] The following functions $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are functions of Δ that are from Definition 1.2 and having the property C_G. For all $s, t \in \mathbb{R}^+$,
(i) $G(s,t) = s - t, C_G = r, r \in \mathbb{R}^+$,
(ii) $G(s,t) = s - \frac{(2+t)t}{1+t}, C_G = 0$,
(iii) $G(s,t) = \frac{s}{1+kt}, k \geq 1, C_G = \frac{r}{1+k}, r \geq 2$.

Definition 1.13. [20]A function $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is said to be a C_G-*simulation function* if it satisfies the following conditions:
(ζ_4) $\zeta(0,0) = 0$;
(ζ_5) $\zeta(t,s) < G(s,t)$ for all $t, s > 0$; here $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is an element of Δ which has property C_G;
(ζ_6) if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ then $\limsup_{n\to\infty} \zeta(t_n, s_n) < C_G$.

We denote the set of all C_G-simulation functions by Z_G.

Example 1.14. [20] The following functions ζ belong to Z_G.
(i) Let $k \in \mathbb{R}$ be such that $k < 1$ and $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be the function defined by $\zeta(t,s) = kG(s,t) - t$, here $C_G = 0$.
(ii) Let $k \in \mathbb{R}$ be such that $k < 1$ and let $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be the function defined by $\zeta(t,s) = kG(s,t)$, here $C_G = 1$.

(iii) We define $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $\zeta(t,s) = \lambda s - t$, where $\lambda \in (0,1)$ and $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $G(s,t) = s - t$ for any $s, t \in \mathbb{R}^+$. Clearly $\zeta(0,0) = 0$ and $G \in \Delta$ with $C_G = 0$.
Clearly $\zeta(t,s) = \lambda s - t < s - t = G(s,t)$ and hence ζ satisfies (ζ_5).
If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n = k > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\limsup\limits_{n\to\infty} \zeta(t_n, s_n) = \limsup\limits_{n\to\infty}(\lambda s_n - t_n) = \lambda k - k = (\lambda - 1)k < 0$.
Therefore ζ satisfies (ζ_6) and hence $\zeta \in Z_G$.

In 1977, Bernfeld, Lakshmikantham and Reddy[9] introduced the concept of fixed point for mappings that have different domains and ranges which is called PPF (Past, Present and Future) dependent fixed point, for more details we refer [5, 6, 8, 12, 13, 16, 19].

Let $(E, ||.||_E)$ be a Banach space and we denote it simply by E. Let $I = [a,b] \subseteq \mathbb{R}$ and $E_0 = C(I,E)$, the set of all continuous functions on I equipped with the supremum norm $||.||_{E_0}$ and we define it by $||\phi||_{E_0} = \sup\limits_{a \le t \le b} ||\phi(t)||_E$ for $\phi \in E_0$.

For a fixed $c \in I$, the Razumikhin class R_c of functions in E_0 is defined by $R_c = \left\{\phi \in E_0 / \, ||\phi||_{E_0} = ||\phi(c)||_E\right\}$. Clearly every constant function from I to E belongs to R_c so that R_c is a non-empty subset of E_0 .

Definition 1.15. [9] Let R_c be the Razumikhin class of continuous functions in E_0. We say that
(i) the class R_c is *algebraically closed with respect to the difference* if $\phi - \psi \in R_c$ whenever $\phi, \psi \in R_c$.
(ii) the class R_c is *topologically closed* if it is closed with respect to the topology on E_0 by the norm $||.||_{E_0}$.

The Razumikhin class of functions R_c has the following properties.

Theorem 1.16. [4] *Let R_c be the Razumikhin class of functions in E_0. Then*
(i) *for any $\phi \in R_c$ and $\alpha \in \mathbb{R}$, we have $\alpha\phi \in R_c$.*
(ii) *the Razumikhin class R_c is topologically closed with respect to the norm defined on E_0.*
(iii) $\bigcap\limits_{c \in [a,b]} R_c = \{\phi \in E_0 / \phi : I \to E \text{ is constant}\}$.

Definition 1.17. [9] Let $T : E_0 \to E$ be a mapping. A function $\phi \in E_0$ is said to be a *PPF dependent fixed point* of T if $T\phi = \phi(c)$ for some $c \in I$.

Definition 1.18. [9] Let $T : E_0 \to E$ be a mapping. Then T is called a *Banach type contraction* if there exists $k \in [0,1)$ such that $||T\phi - T\psi||_E \le k \, ||\phi - \psi||_{E_0}$ for all $\phi, \psi \in E_0$.

Theorem 1.19. [9] *Let $T : E_0 \to E$ be a Banach type contraction. Let R_c be algebraically closed with respect to the difference and topologically closed. Then T has a unique PPF dependent fixed point in R_c.*

Definition 1.20. Let $c \in I$. Let $T : E_0 \to E$ and $\alpha : E \times E \to \mathbb{R}^+$ be two functions. Then T is said to be an α_c-*admissible* mapping if

$$\alpha(\phi(c), \psi(c)) \ge 1 \implies \alpha(T\phi, T\psi) \ge 1 \tag{1.4}$$

for any $\phi, \psi \in E_0$.

Definition 1.21. Let $c \in I$. Let $T : E_0 \to E$ and $\mu : E \times E \to (0,\infty)$ be two functions. Then T is said to be a μ_c-*subadmissible* mapping if

$$\mu(\phi(c), \psi(c)) \le 1 \implies \mu(T\phi, T\psi) \le 1 \tag{1.5}$$

for any $\phi, \psi \in E_0$.

In 2014, Ciric, Alsulami, Salimi and Vetro[10] introduced the concept of triangular α_c-admissible mapping with respect to μ_c as follows.

Definition 1.22. [10] Let $c \in I$ and $T : E_0 \to E$. Let $\alpha, \mu : E \times E \to \mathbb{R}^+$ be two functions. Then T is said to be a *triangular* α_c-*admissible mapping with respect to* μ_c if

$$\begin{cases} \text{(i) } \alpha(\phi(c), \psi(c)) \geq \mu(\phi(c), \psi(c)) \implies \alpha(T\phi, T\psi) \geq \mu(T\phi, T\psi) \\ \qquad\qquad\qquad\qquad \text{and} \\ \text{(ii) } \alpha(\phi(c), \psi(c)) \geq \mu(\phi(c), \psi(c)),\ \alpha(\psi(c), \varphi(c)) \geq \mu(\psi(c), \varphi(c)) \\ \qquad \implies \alpha(\phi(c), \varphi(c)) \geq \mu(\phi(c), \varphi(c)) \end{cases} \tag{1.6}$$

for any $\phi, \psi, \varphi \in E_0$.

Note that if $\mu(x, y) = 1$ for any $x, y \in E$, then we say that T *is a triangular* α_c- *admissible* mapping and if $\alpha(x, y) = 1$ for any $x, y \in E$, then we say that T is a *triangular* μ_c-*subadmissible* mapping.

Lemma 1.23. [10] *Let T be a triangular α_c-admissible mapping with respect to μ_c. We define the sequence $\{\phi_n\}$ by $T\phi_n = \phi_{n+1}(c)$ for all $n \in \mathbb{N} \cup \{0\}$, where $\phi_0 \in R_c$ is such that $\alpha(\phi_0(c), T\phi_0) \geq \mu(\phi_0(c), T\phi_0)$. Then $\alpha(\phi_m(c), \phi_n(c)) \geq \mu(\phi_m(c), \phi_n(c))$ for all $m, n \in \mathbb{N}$ with $m < n$.*

Remark 1.24. If $\mu(x, y) = 1$ for any $x, y \in E$ in Lemma 1.23, we get the following lemma.

Lemma 1.25. *Let T be a triangular α_c-admissible mapping. We define the sequence $\{\phi_n\}$ by $T\phi_n = \phi_{n+1}(c)$ for all $n \in \mathbb{N} \cup \{0\}$, where $\phi_0 \in R_c$ is such that $\alpha(\phi_0(c), T\phi_0) \geq 1$. Then $\alpha(\phi_m(c), \phi_n(c)) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$.*

Remark 1.26. If $\alpha(x, y) = 1$ for any $x, y \in E$ in Lemma 1.23, we get the following lemma.

Lemma 1.27. *Let T be a triangular μ_c-subadmissible mapping. We define the sequence $\{\phi_n\}$ by $T\phi_n = \phi_{n+1}(c)$ for all $n \in \mathbb{N} \cup \{0\}$, where $\phi_0 \in R_c$ is such that $\mu(\phi_0(c), T\phi_0) \leq 1$. Then $\mu(\phi_m(c), \phi_n(c)) \leq 1$ for all $m, n \in \mathbb{N}$ with $m < n$.*

The following lemma is useful to prove our main result.

Lemma 1.28. [6] *Let $\{\phi_n\}$ be a sequence in E_0 such that $||\phi_n - \phi_{n+1}||_{E_0} \to 0$ as $n \to \infty$. If $\{\phi_n\}$ is not a Cauchy sequence, then there exists an $\epsilon > 0$ and two subsequences $\{\phi_{m_k}\}$ and $\{\phi_{n_k}\}$ of $\{\phi_n\}$ with $m_k > n_k > k$ such that $||\phi_{n_k} - \phi_{m_k}||_{E_0} \geq \epsilon$, $||\phi_{n_k} - \phi_{m_k - 1}||_{E_0} < \epsilon$ and*
i) $\lim\limits_{k\to\infty} ||\phi_{n_k} - \phi_{m_k+1}||_{E_0} = \epsilon$, ii) $\lim\limits_{k\to\infty} ||\phi_{n_k+1} - \phi_{m_k}||_{E_0} = \epsilon$,
iii) $\lim\limits_{k\to\infty} ||\phi_{n_k} - \phi_{m_k}||_{E_0} = \epsilon$, iv) $\lim\limits_{k\to\infty} ||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0} = \epsilon$.

In Section 2, we introduce the notion of generalized $Z_{G,\alpha,\mu,\eta,\varphi}-$contraction with respect to the C_G-simulation function and prove the existence and uniqueness of PPF dependent fixed points of generalized $Z_{G,\alpha,\mu,\eta,\varphi}-$contraction with respect to the C_G-simulation function in Banach spaces. In Section 3, we draw some corollaries and an example is provided to illustrate our main result.

2. Existence of PPF dependent fixed points

We denote
$\Psi = \{\eta \mid \eta : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is continuous, nondecreasing and } \eta(t) = 0 \iff t = 0\}$.

Definition 2.1. Let $c \in I$. Let $T : E_0 \to E$ be a function and $\zeta \in Z_G$. If there exist $\alpha : E \times E \to \mathbb{R}^+$, $\mu : E \times E \to (0, \infty)$, $\eta \in \Psi$ and a lower semicontinuous function $\varphi : E \to \mathbb{R}^+$ such that

$$\zeta(\alpha(\phi(c), \psi(c))\eta(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)),\ \mu(\phi(c), \psi(c))\eta(M(\phi, \psi))) \geq C_G \tag{2.1}$$

for any $\phi, \psi \in E_0$, where
$M(\phi, \psi) = \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi),$
$||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi), \frac{1}{2}[||\phi(c) - T\psi||_E + \varphi(\phi(c)) + \varphi(T\psi) + ||\psi(c) - T\phi||_E + \varphi(\psi(c)) + \varphi(T\phi)]\}$,
then we say that T is a *generalized* $Z_{G,\alpha,\mu,\eta,\varphi}-$*contraction with respect to* ζ.

Remark 2.2. (i) If $\varphi(x) = 0$ for any $x \in E$ in the inequality (2.1) then T is called a *generalized $Z_{G,\alpha,\mu,\eta}-$contraction with respect to ζ.*

(ii) If $\varphi(x) = 0, \mu(x, y) = 1 = \alpha(x, y)$ for any $x, y \in E$ in the inequality (2.1) then T is called a *generalized $Z_{G,\eta}-$contraction with respect to ζ.*

(iii) If $\varphi(x) = 0, \mu(x, y) = 1 = \alpha(x, y)$ for any $x, y \in E$ and $\eta(t) = t$ for any $t \in \mathbb{R}^+$ in the inequality (2.1) then T is called a *generalized Z_G-contraction with respect to ζ.*

Theorem 2.3. *Let $c \in I$. Let $T : E_0 \to E$ be a function satisfying the following conditions:*

(i) *T is a generalized $Z_{G,\alpha,\mu,\eta,\varphi}-$contraction with respect to ζ,*

(ii) *T is a triangular α_c-admissible mapping and triangular μ_c-subadmissible mapping,*

(iii) *R_c is algebraically closed with respect to the difference,*

(iv) *if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to \infty, \alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ and $\mu(\phi_n(c), \phi_{n+1}(c)) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$, then $\alpha(\phi_n(c), \phi(c)) \geq 1$ and $\mu(\phi_n(c), \phi(c)) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$ and*

(v) *there exists $\phi_0 \in R_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$ and $\mu(\phi_0(c), T\phi_0) \leq 1$.*

Then T has a PPF dependent fixed point $\phi^* \in R_c$ such that $\varphi(\phi^*(c)) = 0$.

Proof. From (v) we have $\phi_0 \in R_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$ and $\mu(\phi_0(c), T\phi_0) \leq 1$.
Let $\{\phi_n\}$ be a sequence in R_c defined by

$$T\phi_n = \phi_{n+1}(c) \tag{2.2}$$

for any $n = 0, 1, 2, 3...$.
Since R_c is algebraically closed with respect to the difference, we have

$$||\phi_{n+1} - \phi_n||_{E_0} = ||\phi_{n+1}(c) - \phi_n(c)||_E \tag{2.3}$$

for any $n = 0, 1, 2, 3...$.
Since T is traingular α_c-admissible and triangular μ_c-subadmissible mappings, by Lemma 1.25 and Lemma 1.27 we have

$$\alpha(\phi_m(c), \phi_n(c)) \geq 1 \quad \text{and} \quad \mu(\phi_m(c), \phi_n(c)) \leq 1 \tag{2.4}$$

for any $m, n \in \mathbb{N}$ with $m < n$.
If there exists $n \in \mathbb{N} \cup \{0\}$ such that $\phi_n = \phi_{n+1}$ then $T\phi_n = \phi_{n+1}(c) = \phi_n(c)$ and hence $\phi_n \in R_c$ is a PPF dependent fixed point of T. Suppose that $\phi_n \neq \phi_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$.
We consider

$$\begin{aligned} M(\phi_n, \phi_{n+1}) = \max\{&||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)), ||\phi_n(c) - T\phi_n||_E + \varphi(\phi_n(c)) + \varphi(T\phi_n), \\ &||\phi_{n+1}(c) - T\phi_{n+1}||_E + \varphi(\phi_{n+1}(c)) + \varphi(T\phi_{n+1}), \\ &\tfrac{1}{2}[||\phi_n(c) - T\phi_{n+1}||_E + \varphi(\phi_n(c)) + \varphi(T\phi_{n+1}) + ||\phi_{n+1}(c) - T\phi_n||_E + \varphi(\phi_{n+1}(c)) + \varphi(T\phi_n)]\} \\ = \max\{&||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)), ||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)), \\ &||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c)), \\ &\tfrac{1}{2}[||\phi_n - \phi_{n+2}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+2}(c)) + ||\phi_{n+1} - \phi_{n+1}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+1}(c))]\} \\ = \max\{&||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)), ||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))\}. \end{aligned}$$

Suppose that
$M(\phi_n, \phi_{n+1}) = ||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))$.
Since $\phi_{n+1} \neq \phi_{n+2}$, we have $||\phi_{n+1} - \phi_{n+2}||_{E_0} > 0$ and hence $||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c)) > 0$.
Therefore
$\eta(M(\phi_n, \phi_{n+1})) = \eta(||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))) > 0$.
Clearly

$$\begin{gathered} \alpha(\phi_n(c), \phi_{n+1}(c))\eta(||\phi_{n+1} - \phi_{n+2}||_{E_0} + \varphi(\phi_{n+1}(c)) + \varphi(\phi_{n+2}(c))) > 0 \\ \text{and} \\ \mu(\phi_n(c), \phi_{n+1}(c))\eta(M(\phi_n, \phi_{n+1})) > 0. \end{gathered} \tag{2.5}$$

From (2.1), we have
$C_G \leq \zeta(\alpha(\phi_n(c), \phi_{n+1}(c))\eta(||T\phi_n - T\phi_{n+1}||_E + \varphi(T\phi_n) + \varphi(T\phi_{n+1})), \mu(\phi_n(c), \phi_{n+1}(c))\eta(M(\phi_n, \phi_{n+1})))$

$$= \zeta(\alpha(\phi_n(c),\phi_{n+1}(c))\eta(||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c))),\\ \mu(\phi_n(c),\phi_{n+1}(c))\eta(||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c))))\\ < G(\mu(\phi_n(c),\phi_{n+1}(c))\eta(||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c))),\\ \alpha(\phi_n(c),\phi_{n+1}(c))\eta(||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c)))).$$
(by (2.5) and (ζ_5))

Now by the property C_G, we get
$\mu(\phi_n(c),\phi_{n+1}(c))\eta(||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c)))$
$> \alpha(\phi_n(c),\phi_{n+1}(c))\eta(||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c)))$.
Clearly
$$\eta(||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c)))\\ \geq \mu(\phi_n(c),\phi_{n+1}(c))\eta(||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c)))\\ > \alpha(\phi_n(c),\phi_{n+1}(c))\eta(||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c)))\\ \geq \eta(||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c))),$$
a contradiction.
Therefore
$M(\phi_n,\phi_{n+1}) = ||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c))$ and hence
$||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c)) > ||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c))$.
Let $d_n = ||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c))$.
Then the sequence $\{d_n\}$ is a decreasing sequence and hence convergent.
Let $\lim_{n\to\infty} d_n = k$ (say). Suppose that $k>0$.
Since $\phi_n \neq \phi_{n+1}$ we have $d_n = ||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c)) > 0$ and which implies that $\eta(d_n) = \eta(||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c))) > 0$. Clearly $\mu(\phi_n(c),\phi_{n+1}(c))\eta(d_n) > 0$.
From (2.1), we have

$$C_G \leq \zeta(\alpha(\phi_n(c),\phi_{n+1}(c))\eta(||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c))),\\ \mu(\phi_n(c),\phi_{n+1}(c))\eta(||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c)))) \tag{2.6}$$

$< G(\mu(\phi_n(c),\phi_{n+1}(c))\eta(d_n),\alpha(\phi_n(c),\phi_{n+1}(c))\eta(d_{n+1}))$. (by(2.5) and ($\zeta_5$))
Now by the property C_G, we get that $\mu(\phi_n(c),\phi_{n+1}(c))\eta(d_n) > \alpha(\phi_n(c),\phi_{n+1}(c))\eta(d_{n+1})$.
Clearly $\eta(d_n) \geq \mu(\phi_n(c),\phi_{n+1}(c))\eta(d_n) > \alpha(\phi_n(c),\phi_{n+1}(c))\eta(d_{n+1}) \geq \eta(d_{n+1})$.
On applying limits as $n\to\infty$, we get that
$\lim_{n\to\infty}\mu(\phi_n(c),\phi_{n+1}(c))\eta(d_n) = \lim_{n\to\infty}\alpha(\phi_n(c),\phi_{n+1}(c))\eta(d_{n+1}) = \eta(k) > 0$.
On applying limit superior to (2.6), we get that
$$C_G \leq \limsup_{n\to\infty}\zeta(\alpha(\phi_n(c),\phi_{n+1}(c))\eta(||\phi_{n+1}-\phi_{n+2}||_{E_0}+\varphi(\phi_{n+1}(c))+\varphi(\phi_{n+2}(c))),\\ \mu(\phi_n(c),\phi_{n+1}(c))\eta(||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c))))\\ < C_G,$$
a contadiction.
Therefore $k=0$ and hence $\lim_{n\to\infty}[||\phi_n-\phi_{n+1}||_{E_0}+\varphi(\phi_n(c))+\varphi(\phi_{n+1}(c))] = 0$.
That is
$$\lim_{n\to\infty}||\phi_n-\phi_{n+1}||_{E_0} = 0 \text{ and } \lim_{n\to\infty}\varphi(\phi_n(c)) = 0. \tag{2.7}$$
We now show that the sequence $\{\phi_n\}$ is a Cauchy sequence in R_c.
Suppose that the sequence $\{\phi_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon>0$ and two subsequences $\{\phi_{m_k}\}$ and $\{\phi_{n_k}\}$ of $\{\phi_n\}$ with $m_k > n_k > k$ such that $||\phi_{n_k}-\phi_{m_k}||_{E_0} \geq \epsilon$, $||\phi_{n_k}-\phi_{m_k-1}||_{E_0} < \epsilon$ and by Lemma 1.28 we get
$$\lim_{k\to\infty}||\phi_{n_k}-\phi_{m_k}||_{E_0} = \epsilon \tag{2.8}$$
and $\lim_{k\to\infty}||\phi_{n_k}-\phi_{m_k+1}||_{E_0} = \epsilon = \lim_{k\to\infty}||\phi_{n_k+1}-\phi_{m_k}||_{E_0} = \lim_{k\to\infty}||\phi_{n_k+1}-\phi_{m_k+1}||_{E_0}$.
Therefore $\lim_{k\to\infty} d_{n_k m_k} = \lim_{k\to\infty}[||\phi_{n_k}-\phi_{m_k}||_{E_0}+\varphi(\phi_{n_k}(c))+\varphi(\phi_{m_k}(c))] = \epsilon$ and

$\lim_{k\to\infty} d_{n_{k+1}m_{k+1}} = \lim_{k\to\infty} [||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0} + \varphi(\phi_{n_k+1}(c)) + \varphi(\phi_{m_k+1}(c))] = \epsilon.$
Since η is continuous, we get that

$$\lim_{k\to\infty} \eta(d_{n_{k+1}m_{k+1}}) = \eta(\epsilon) > 0. \tag{2.9}$$

We consider

$$\begin{aligned} M(\phi_{n_k},\phi_{m_k}) = \max\{&||\phi_{n_k} - \phi_{m_k}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{m_k}(c)), ||\phi_{n_k}(c) - T\phi_{n_k}||_E + \varphi(\phi_{n_k}(c)) + \varphi(T\phi_{n_k}),\\ &||\phi_{m_k}(c) - T\phi_{m_k}||_E + \varphi(\phi_{m_k}(c)) + \varphi(T\phi_{m_k}),\\ &\tfrac{1}{2}[||\phi_{n_k}(c) - T\phi_{m_k}||_E + \varphi(\phi_{n_k}(c)) + \varphi(T\phi_{m_k}) + ||\phi_{m_k}(c) - T\phi_{n_k}||_E + \varphi(\phi_{m_k}(c)) + \varphi(T\phi_{n_k})]\}\\ = \max\{&||\phi_{n_k} - \phi_{m_k}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{m_k}(c)), ||\phi_{n_k} - \phi_{n_k+1}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{n_k+1}(c)),\\ &||\phi_{m_k} - \phi_{m_k+1}||_{E_0} + \varphi(\phi_{m_k}(c)) + \varphi(\phi_{m_k+1}(c)),\\ &\tfrac{1}{2}[||\phi_{n_k} - \phi_{m_k+1}||_{E_0} + \varphi(\phi_{n_k}(c)) + \varphi(\phi_{m_k+1}(c)) + ||\phi_{m_k} - \phi_{n_k+1}||_{E_0} + \varphi(\phi_{m_k}(c)) + \varphi(\phi_{n_k+1}(c))]\}\\ = \max\{&d_{n_km_k}, d_{n_kn_{k+1}}, d_{m_km_{k+1}}, \tfrac{1}{2}[d_{n_km_{k+1}} + d_{m_kn_{k+1}}]\}. \end{aligned}$$

On applying limits as $k \to \infty$, we get that $\lim_{k\to\infty} M(\phi_{n_k},\phi_{m_k}) = \epsilon$.
Since η is continuous, we get that

$$\lim_{k\to\infty} \eta(M(\phi_{n_k},\phi_{m_k})) = \eta(\epsilon) > 0. \tag{2.10}$$

From (2.9) and (2.10), there exists $k_1 \in \mathbb{N}$ such that

$$\begin{gathered} \eta(M(\phi_{n_k},\phi_{m_k})) > \tfrac{\eta(\epsilon)}{2} > 0\\ \text{and}\\ \eta(d_{n_{k+1}m_{k+1}}) > \tfrac{\eta(\epsilon)}{2} > 0 \end{gathered} \tag{2.11}$$

for any $k \geq k_1$.
From (2.4), we have

$$\alpha(\phi_{n_k}(c),\phi_{m_k}(c))\eta(d_{n_{k+1}m_{k+1}}) \geq \eta(d_{n_{k+1}m_{k+1}}) > 0 \tag{2.12}$$

for any $k \geq k_1$.
Clearly

$$\mu(\phi_{n_k}(c),\phi_{m_k}(c))\eta(M(\phi_{n_k},\phi_{m_k})) > 0 \tag{2.13}$$

for any $k \geq k_1$.
For any $k \geq k_1$, from (2.1) we have

$$\begin{aligned} C_G &\leq \zeta(\alpha(\phi_{n_k}(c),\phi_{m_k}(c))\eta(||T\phi_{n_k} - T\phi_{m_k}||_E + \varphi(T\phi_{n_k}) + \varphi(T\phi_{m_k})), \mu(\phi_{n_k}(c),\phi_{m_k}(c))\eta(M(\phi_{n_k},\phi_{m_k})))\\ &= \zeta(\alpha(\phi_{n_k}(c),\phi_{m_k}(c))\eta(||\phi_{n_k+1} - \phi_{m_k+1}||_{E_0} + \varphi(\phi_{n_k+1}(c)) + \varphi(\phi_{m_k+1}(c))), \mu(\phi_{n_k}(c),\phi_{m_k}(c))\eta(M(\phi_{n_k},\phi_{m_k})))\\ &= \zeta(\alpha(\phi_{n_k}(c),\phi_{m_k}(c))\eta(d_{n_{k+1}m_{k+1}}), \mu(\phi_{n_k}(c),\phi_{m_k}(c))\eta(M(\phi_{n_k},\phi_{m_k}))) \qquad (2.14)\\ &< G(\mu(\phi_{n_k}(c),\phi_{m_k}(c))\eta(M(\phi_{n_k},\phi_{m_k})), \alpha(\phi_{n_k}(c),\phi_{m_k}(c))\eta(d_{n_{k+1}m_{k+1}})). \text{ (by (2.12),(2.13) and } (\zeta_5)) \end{aligned}$$

Now by the property C_G, we have

$$\mu(\phi_{n_k}(c),\phi_{m_k}(c))\eta(M(\phi_{n_k},\phi_{m_k})) > \alpha(\phi_{n_k}(c),\phi_{m_k}(c))\eta(d_{n_{k+1}m_{k+1}}). \tag{2.15}$$

Clearly

$$\begin{aligned} \eta(M(\phi_{n_k},\phi_{m_k})) &\geq \mu(\phi_{n_k}(c),\phi_{m_k}(c))\eta(M(\phi_{n_k},\phi_{m_k}))\\ &> \alpha(\phi_{n_k}(c),\phi_{m_k}(c))\eta(d_{n_{k+1}m_{k+1}}) \text{ (by(2.15))}\\ &\geq \eta(d_{n_{k+1}m_{k+1}}). \end{aligned}$$

On applying limits as $k \to \infty$, we get that

$$\lim_{k\to\infty} \mu(\phi_{n_k}(c),\phi_{m_k}(c))\eta(M(\phi_{n_k},\phi_{m_k})) = \lim_{k\to\infty} \alpha(\phi_{n_k}(c),\phi_{m_k}(c))\eta(d_{n_{k+1}m_{k+1}}) = \eta(\epsilon) > 0. \tag{2.16}$$

On applying limit superior as $k \to \infty$ to (2.14), by (2.15) ,(2.16) and (ζ_6) we get
$C_G \leq \limsup_{k\to\infty} \zeta(\alpha(\phi_{n_k}(c), \phi_{m_k}(c))\eta(d_{n_{k+1}m_{k+1}}), \mu(\phi_{n_k}(c), \phi_{m_k}(c))\eta(M(\phi_{n_k}, \phi_{m_k}))) < C_G,$
a contradiction.
Therefore the sequence $\{\phi_n\}$ is a Cauchy sequence in R_c. Since E_0 is complete, there exists $\phi^* \in E_0$ such that $\phi_n \to \phi^*$. Since R_c is topologically closed, we have $\phi^* \in R_c$. Clearly $||\phi^*||_{E_0} = ||\phi^*(c)||_E$. (since $\phi^* \in R_c$) Since φ is lower semicontinuous function, we have $\varphi(\phi^*(c)) \leq \liminf_{n\to\infty} \varphi(\phi_n(c)) = 0$ and hence $\varphi(\phi^*(c)) = 0$.
We now show that $T\phi^* = \phi^*(c)$. Suppose that $T\phi^* \neq \phi^*(c)$.
From (2.4) we have $\alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ and $\mu(\phi_n(c), \phi_{n+1}(c)) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$.
From (iv) we get that $\alpha(\phi_n(c), \phi^*(c)) \geq 1$ and $\mu(\phi_n(c), \phi^*(c)) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$.
We consider

$$\begin{aligned}
M(\phi_n, \phi^*) = \max\{&||\phi_n - \phi^*||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi^*(c)), ||\phi_n(c) - T\phi_n||_E + \varphi(\phi_n(c)) + \varphi(T\phi_n),\\
&||\phi^*(c) - T\phi^*||_E + \varphi(\phi^*(c)) + \varphi(T\phi^*),\\
&\tfrac{1}{2}[||\phi_n(c) - T\phi^*||_E + \varphi(\phi_n(c)) + \varphi(T\phi^*) + ||\phi^*(c) - T\phi_n||_E + \varphi(\phi^*(c)) + \varphi(T\phi_n)]\}\\
= \max\{&||\phi_n - \phi^*||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi^*(c)), ||\phi_n - \phi_{n+1}||_{E_0} + \varphi(\phi_n(c)) + \varphi(\phi_{n+1}(c)),\\
&||\phi^*(c) - T\phi^*||_E + \varphi(\phi^*(c)) + \varphi(T\phi^*),\\
&\tfrac{1}{2}[||\phi_n(c) - T\phi^*||_E + \varphi(\phi_n(c)) + \varphi(T\phi^*) + ||\phi^* - \phi_{n+1}||_{E_0} + \varphi(\phi^*(c)) + \varphi(\phi_{n+1}(c))]\}.
\end{aligned}$$

If $M(\phi_n, \phi^*) = 0$ then $T\phi^* = \phi^*(c)$, a contradiction.
Therefore $M(\phi_n, \phi^*) > 0$ and hence $\eta(M(\phi_n, \phi^*)) > 0$.
Clearly

$$\mu(\phi_n(c), \phi^*(c))\eta(M(\phi_n, \phi^*)) > 0. \tag{2.17}$$

On applying limits as $n \to \infty$ to $M(\phi_n, \phi^*)$, we get that $\lim_{n\to\infty} M(\phi_n, \phi^*) = ||\phi^*(c) - T\phi^*||_E + \varphi(T\phi^*)$.
Since η is continuous, we get that $\lim_{n\to\infty} \eta(M(\phi_n, \phi^*)) = \eta(||\phi^*(c) - T\phi^*||_E + \varphi(T\phi^*)) > 0$. (since $T\phi^* \neq \phi^*(c)$)
If $||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*) = 0$, then $\phi_{n+1}(c) = T\phi_n = T\phi^*$.
On applying limits as $n \to \infty$, we get $\phi^*(c) = T\phi^*$, a contradiction.
Therefore $||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*) > 0$ and hence $\eta(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)) > 0$.
Clearly

$$\alpha(\phi_n(c), \phi^*(c))\eta(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)) > 0. \tag{2.18}$$

From (2.1) we have

$$\begin{aligned}
C_G &\leq \zeta(\alpha(\phi_n(c), \phi^*(c))\eta(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)), \mu(\phi_n(c), \phi^*(c))\eta(M(\phi_n, \phi^*)))\\
&< G(\mu(\phi_n(c), \phi^*(c))\eta(M(\phi_n, \phi^*)), \alpha(\phi_n(c), \phi^*(c))\eta(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*))).
\end{aligned}$$

Now by the property C_G, we get that

$$\mu(\phi_n(c), \phi^*(c))\eta(M(\phi_n, \phi^*)) > \alpha(\phi_n(c), \phi^*(c))\eta(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)). \tag{2.19}$$

Clearly

$$\begin{aligned}
\eta(M(\phi_n, \phi^*)) &\geq \mu(\phi_n(c), \phi^*(c))\eta(M(\phi_n, \phi^*))\\
&> \alpha(\phi_n(c), \phi^*(c))\eta(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*))\\
&\geq \eta(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*))\\
&= \eta(||\phi_{n+1}(c) - T\phi^*||_E + \varphi(\phi_{n+1}(c)) + \varphi(T\phi^*)).
\end{aligned}$$

On applying limits as $n \to \infty$, we get

$$\begin{aligned}
&\lim_{n\to\infty} \alpha(\phi_n(c), \phi^*(c))\eta(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*))\\
&\quad = \lim_{n\to\infty} \mu(\phi_n(c), \phi^*(c))\eta(M(\phi_n, \phi^*)) = \eta(||\phi^*(c) - T\phi^*||_E + \varphi(T\phi^*)) > 0.
\end{aligned}$$

From (2.1) we have
$C_G \leq \zeta(\alpha(\phi_n(c), \phi^*(c))\eta(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)), \mu(\phi_n(c), \phi^*(c))\eta(M(\phi_n, \phi^*)))$.
On applying limit superior as $n \to \infty$, by (ζ_6) we get that

$$\begin{aligned}
C_G &\leq \limsup_{n\to\infty} \zeta(\alpha(\phi_n(c), \phi^*(c))\eta(||T\phi_n - T\phi^*||_E + \varphi(T\phi_n) + \varphi(T\phi^*)), \mu(\phi_n(c), \phi^*(c))\eta(M(\phi_n, \phi^*)))\\
&< C_G,
\end{aligned}$$

a contradiction.
Therefore $T\phi^* = \phi^*(c)$ and hence $\phi^* \in R_c$ is a PPF dependent fixed point of T such that $\varphi(\phi^*(c)) = 0$. □

Theorem 2.4. *In addition to the assumptions of Theorem 2.3 assume the following.*
If $\alpha(x, y) \geq 1,\ \mu(x, y) \leq 1$ for any $x, y \in E$ and T is one-one then T has a unique PPF dependent fixed point in R_c.

Proof. By Theorem 2.3, T has a PPF dependent fixed point $\phi^* \in R_c$ such that $\varphi(\phi^*(c)) = 0$.
We now show that T has a unique PPF dependent fixed point in R_c.
Let $\phi, \psi \in R_c$ be two PPF dependent fixed points of T such that $\varphi(\phi(c)) = 0$ and $\varphi(\psi(c)) = 0$.
Then we get $T\phi = \phi(c)$ and $T\psi = \psi(c)$. Since R_c is algebraically closed with respect to the difference, we have $||\phi - \psi||_{E_0} = ||\phi(c) - \psi(c)||_E$. Suppose that $\phi \neq \psi$.
If $||T\phi - T\psi||_E = 0$ then $T\phi = T\psi$. Since T is one-one we have $\phi = \psi$, a contradiction.
Therefore $||T\phi - T\psi||_E \neq 0$ and hence $||T\phi - T\psi||_E > 0$.
Clearly $\eta(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)) = \eta(||T\phi - T\psi||_E + \varphi(\phi(c)) + \varphi(\psi(c))) = \eta(||T\phi - T\psi||_E) > 0$ and hence $\alpha(\phi(c), \psi(c))\eta(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)) > 0$.
We consider

$$\begin{aligned} M(\phi, \psi) &= \max\{||\phi - \psi||_{E_0} + \varphi(\phi(c)) + \varphi(\psi(c)), ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi), \\ &\qquad ||\psi(c) - T\psi||_E + \varphi(\psi(c)) + \varphi(T\psi), \\ &\qquad \tfrac{1}{2}[||\phi(c) - T\psi||_E + \varphi(\phi(c)) + \varphi(T\psi) + ||\psi(c) - T\phi||_E + \varphi(\psi(c)) + \varphi(T\phi)]\} \\ &= \max\{||\phi - \psi||_{E_0}, \tfrac{||\phi(c)-\psi(c)||_E + ||\psi(c)-\phi(c)||_E}{2}\} \\ &= \max\{||\phi - \psi||_{E_0}, ||\phi - \psi||_{E_0}\} = ||\phi - \psi||_{E_0} \end{aligned}$$

and hence $\mu(\phi(c), \psi(c))\eta(M(\phi, \psi)) > 0$. (since $\phi \neq \psi$)
From (2.1), we get that

$$\begin{aligned} C_G &\leq \zeta(\alpha(\phi(c), \psi(c))\eta(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)), \mu(\phi(c), \psi(c))\eta(M(\phi, \psi))) \\ &< G(\mu(\phi(c), \psi(c))\eta(M(\phi, \psi)), \alpha(\phi(c), \psi(c))\eta(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi))). \end{aligned}$$

By the property C_G, we get that

$$\mu(\phi(c), \psi(c))\eta(M(\phi, \psi)) > \alpha(\phi(c), \psi(c))\eta(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)). \tag{2.20}$$

Clearly

$$\begin{aligned} \eta(||\phi - \psi||_{E_0}) &= \eta(M(\phi, \psi)) \\ &\geq \mu(\phi(c), \psi(c))\eta(M(\phi, \psi)) \\ &> \alpha(\phi(c), \psi(c))\eta(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi) \text{ (by (2.20))} \\ &\geq \eta(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)) \\ &= \eta(||T\phi - T\psi||_E) = \eta(||\phi(c) - \psi(c)||_E) \\ &= \eta(||\phi - \psi||_{E_0}), \end{aligned}$$

a contradiction.
Therefore $\phi = \psi$ and hence T has a unique PPF dependent fixed point in R_c. □

3. Corollaries and Examples

Corollary 3.1. *Let $c \in I$. Let $T : E_0 \to E$ be a function satisfying the following conditions:*

(i) *T is a generalized $Z_{G,\alpha,\mu,\eta}-$contraction with respect to ζ,*
(ii) *T is a triangular α_c-admissible mapping and triangular μ_c-subadmissible mapping,*
(iii) *R_c is algebraically closed with respect to the difference,*
(iv) *if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n \to \phi$ as $n \to \infty, \alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ and $\mu(\phi_n(c), \phi_{n+1}(c)) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$ then $\alpha(\phi_n(c), \phi(c)) \geq 1$ and $\mu(\phi_n(c), \phi(c)) \leq 1$ for any $n \in \mathbb{N} \cup \{0\}$ and*
(v) *there exists $\phi_0 \in R_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$ and $\mu(\phi_0(c), T\phi_0) \leq 1$.*

Then T has a PPF dependent fixed point in R_c.

Proof. By taking $\varphi(x) = 0$ for any $x \in E$ in Theorem 2.3 we obtain the desired result. □

Remark 3.2. In addition to the hypotheses of Corollary 3.1 assume the following.
If $\alpha(x,y)\geq 1, \mu(x,y)\leq 1$ for any $x,y\in E$ and T is one-one then T has a unique PPF dependent fixed point in R_c.

By choosing $\alpha(x,y)=1=\mu(x,y)$ for any $x,y\in E$ in Corollary 3.1 we get the following corollary.

Corollary 3.3. *Let $c\in I$. Let $T:E_0\to E$ be a function satisfying the following conditions:*
(i) *T is a generalized $Z_{G,\eta}-$contraction with respect to ζ and*
(ii) *R_c is algebraically closed with respect to the difference.*
Then T has a PPF dependent fixed point in R_c.

By choosing $\eta(t)=t$ for any $t\in\mathbb{R}^+$ in Corollary 3.3 we get the following corollary.

Corollary 3.4. *Let $c\in I$. Let $T:E_0\to E$ be a function satisfying the following conditions:*
(i) *T is a generalized Z_G-contraction with respect to ζ and*
(ii) *R_c is algebraically closed with respect to the difference.*
Then T has a PPF dependent fixed point in R_c.

By choosing $\alpha(x,y)=1=\mu(x,y)$ for any $x,y\in E$, $\eta(t)=t$ for any $t\in\mathbb{R}^+$ and $C_G=0$ in Theorem 2.3 we get the following corollary.

Corollary 3.5. *Let $c\in I$ and $\zeta\in Z_G$. Let $T:E_0\to E$ be a function satisfying the following conditions:*
(i) *if there exists a lower semicontinuous function $\varphi:E\to\mathbb{R}^+$ such that*
$$\zeta(||T\phi-T\psi||_E+\varphi(T\phi)+\varphi(T\psi),M(\phi,\psi))\geq 0$$
for any $\phi,\psi\in E_0$, where
$$\begin{aligned}M(\phi,\psi)=\max\{&||\phi-\psi||_{E_0}+\varphi(\phi(c))+\varphi(\psi(c)),||\phi(c)-T\phi||_E+\varphi(\phi(c))+\varphi(T\phi),\\&||\psi(c)-T\psi||_E+\varphi(\psi(c))+\varphi(T\psi),\\&\tfrac{1}{2}[||\phi(c)-T\psi||_E+\varphi(\phi(c))+\varphi(T\psi)+||\psi(c)-T\phi||_E+\varphi(\psi(c))+\varphi(T\phi)]\}\end{aligned}$$ *and*
(ii) *R_c is algebraically closed with respect to the difference.*
Then T has a PPF dependent fixed point in R_c.

By choosing $\mu(x,y)=1$ for any $x,y\in E$, $\eta(t)=t$ for any $t\in\mathbb{R}^+$ and $C_G=0$ in Corollary 3.1 we get the following corollary.

Corollary 3.6. *Let $c\in I$ and $\zeta\in Z_G$. Let $T:E_0\to E$ be a function satisfying the following conditions:*
(i) *if there exists $\alpha:E\times E\to\mathbb{R}^+$ such that*
$$\zeta(\alpha(\phi(c),\psi(c))||T\phi-T\psi||_E,M(\phi,\psi))\geq 0$$
for any $\phi,\psi\in E_0$, where
$$M(\phi,\psi)=\max\{||\phi-\psi||_{E_0},||\phi(c)-T\phi||_E,||\psi(c)-T\psi||_E,\tfrac{1}{2}[||\phi(c)-T\psi||_E+||\psi(c)-T\phi||_E]\},$$
(ii) *T a triangular α_c-admissible mapping,*
(iii) *R_c is algebraically closed with respect to the difference,*
(iv) *if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n\to\phi$ as $n\to\infty$ and $\alpha(\phi_n(c),\phi_{n+1}(c))\geq 1$ for any $n\in\mathbb{N}\cup\{0\}$ then $\alpha(\phi_n(c),\phi(c))\geq 1$ for any $n\in\mathbb{N}\cup\{0\}$ and*
(v) *there exists $\phi_0\in R_c$ such that $\alpha(\phi_0(c),T\phi_0)\geq 1$.*
Then T has a PPF dependent fixed point in R_c.
Moreover, if $\alpha(x,y)\geq 1$ for any $x,y\in E$ and T is one-one then T has a unique fixed point in R_c.

By choosing $\alpha(x,y)=1$ for any $x,y\in E$ in Corollary 3.6 we get the following corollary.

Corollary 3.7. *Let $c\in I$ and $\zeta\in Z_G$. Let $T:E_0\to E$ be a function satisfying the following conditions:*
(i) $\zeta(||T\phi-T\psi||_E,M(\phi,\psi))\geq 0$
for any $\phi,\psi\in E_0$, where
$$M(\phi,\psi)=\max\{||\phi-\psi||_{E_0},||\phi(c)-T\phi||_E,||\psi(c)-T\psi||_E,\tfrac{1}{2}[||\phi(c)-T\psi||_E+||\psi(c)-T\phi||_E]\}$$ *and*
(ii) *R_c is algebraically closed with respect to the difference.*
Then T has a PPF dependent fixed point in R_c.

Remark 3.8. In addition to the hypotheses of Corollary 3.3 (Corollary 3.4, Corollary 3.5, Corollary 3.7) assume the following.
If T is one-one then T has a unique PPF dependent fixed point in R_c.

By choosing $\zeta(t,s)=\lambda s-t, G(s,t)=s-t$ for any $s,t\in\mathbb{R}^+, C_G=0$ and $\lambda\in(0,1)$ in Theorem 2.3 we get the following corollary.

Corollary 3.9. *Let $c\in I$. Let $T:E_0\to E$ be a function satisfying the following conditions:*

(i) *if there exist $\alpha:E\times E\to\mathbb{R}^+,\ \mu:E\times E\to(0,\infty),\eta\in\Psi,\lambda\in(0,1)$ and a lower semicontinuous function $\varphi:E\to\mathbb{R}^+$ such that*

$$\alpha(\phi(c),\psi(c))\eta(||T\phi-T\psi||_E+\varphi(T\phi)+\varphi(T\psi))\le\lambda\mu(\phi(c),\psi(c))\eta(M(\phi,\psi)) \tag{3.1}$$

for any $\phi,\psi\in E_0$, where
$M(\phi,\psi)=\max\{||\phi-\psi||_{E_0}+\varphi(\phi(c))+\varphi(\psi(c)),||\phi(c)-T\phi||_E+\varphi(\phi(c))+\varphi(T\phi),$
$||\psi(c)-T\psi||_E+\varphi(\psi(c))+\varphi(T\psi),$
$\frac{1}{2}[||\phi(c)-T\psi||_E+\varphi(\phi(c))+\varphi(T\psi)+||\psi(c)-T\phi||_E+\varphi(\psi(c))+\varphi(T\phi)]\}$,

(ii) *T is a triangular α_c-admissible mapping and triangular μ_c-subadmissible mapping,*

(iii) *R_c is algebraically closed with respect to the difference,*

(iv) *if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n\to\phi$ as $n\to\infty,\alpha(\phi_n(c),\phi_{n+1}(c))\ge1$ and $\mu(\phi_n(c),\phi_{n+1}(c))\le1$ for any $n\in\mathbb{N}\cup\{0\}$ then $\alpha(\phi_n(c),\phi(c))\ge1$ and $\mu(\phi_n(c),\phi(c))\le1$ for any $n\in\mathbb{N}\cup\{0\}$ and*

(v) *there exists $\phi_0\in R_c$ such that $\alpha(\phi_0(c),T\phi_0)\ge1$ and $\mu(\phi_0(c),T\phi_0)\le1$.*

Then T has a PPF dependent fixed point $\phi^*\in R_c$ such that $\varphi(\phi^*(c))=0$.

By choosing $\eta(t)=t,\ t\in\mathbb{R}^+$ in Corollary 3.9 we get the following corollary.

Corollary 3.10. *Let $c\in I$. Let $T:E_0\to E$ be a function satisfying the following conditions:*

(i) *if there exist $\alpha:E\times E\to\mathbb{R}^+,\ \mu:E\times E\to(0,\infty),\lambda\in(0,1)$ and a lower semicontinuous function $\varphi:E\to\mathbb{R}^+$ such that*

$$\alpha(\phi(c),\psi(c))(||T\phi-T\psi||_E+\varphi(T\phi)+\varphi(T\psi))\le\lambda\mu(\phi(c),\psi(c))M(\phi,\psi) \tag{3.2}$$

for any $\phi,\psi\in E_0$, where
$M(\phi,\psi)=\max\{||\phi-\psi||_{E_0}+\varphi(\phi(c))+\varphi(\psi(c)),||\phi(c)-T\phi||_E+\varphi(\phi(c))+\varphi(T\phi),$
$||\psi(c)-T\psi||_E+\varphi(\psi(c))+\varphi(T\psi),$
$\frac{1}{2}[||\phi(c)-T\psi||_E+\varphi(\phi(c))+\varphi(T\psi)+||\psi(c)-T\phi||_E+\varphi(\psi(c))+\varphi(T\phi)]\}$,

(ii) *T is a triangular α_c-admissible mapping and triangular μ_c-subadmissible mapping,*

(iii) *R_c is algebraically closed with respect to the difference,*

(iv) *if $\{\phi_n\}$ is a sequence in E_0 such that $\phi_n\to\phi$ as $n\to\infty,\alpha(\phi_n(c),\phi_{n+1}(c))\ge1$ and $\mu(\phi_n(c),\phi_{n+1}(c))\le1$ for any $n\in\mathbb{N}\cup\{0\}$ then $\alpha(\phi_n(c),\phi(c))\ge1$ and $\mu(\phi_n(c),\phi(c))\le1$ for any $n\in\mathbb{N}\cup\{0\}$ and*

(v) *there exists $\phi_0\in R_c$ such that $\alpha(\phi_0(c),T\phi_0)\ge1$ and $\mu(\phi_0(c),T\phi_0)\le1$.*

Then T has a PPF dependent fixed point $\phi^*\in R_c$ such that $\varphi(\phi^*(c))=0$.

By choosing $\varphi(x)=0$ for any $x\in E$ in Corollay 3.10 we get the following corollary.

Corollary 3.11. *Let $c\in I$. Let $T:E_0\to E$ be a function satisfying the following conditions:*

(i) *if there exist $\alpha:E\times E\to\mathbb{R}^+,\ \mu:E\times E\to(0,\infty)$ and $\lambda\in(0,1)$ such that*

$$\alpha(\phi(c),\psi(c))||T\phi-T\psi||_E\le\lambda\mu(\phi(c),\psi(c))M(\phi,\psi) \tag{3.3}$$

for any $\phi,\psi\in E_0$, where
$M(\phi,\psi)=\max\{||\phi-\psi||_{E_0},||\phi(c)-T\phi||_E,||\psi(c)-T\psi||_E,\frac{1}{2}[||\phi(c)-T\psi||_E+||\psi(c)-T\phi||_E]\}$,

(ii) *T is a triangular α_c-admissible mapping and triangular μ_c-subadmissible mapping,*

(iii) *R_c is algebraically closed with respect to the difference,*

(iv) *if* $\{\phi_n\}$ *is a sequence in* E_0 *such that* $\phi_n \to \phi$ *as* $n \to \infty, \alpha(\phi_n(c), \phi_{n+1}(c)) \geq 1$ *and* $\mu(\phi_n(c), \phi_{n+1}(c)) \leq 1$ *for any* $n \in \mathbb{N} \cup \{0\}$ *then* $\alpha(\phi_n(c), \phi(c)) \geq 1$ *and* $\mu(\phi_n(c), \phi(c)) \leq 1$ *for any* $n \in \mathbb{N} \cup \{0\}$ *and*

(v) *there exists* $\phi_0 \in R_c$ *such that* $\alpha(\phi_0(c), T\phi_0) \geq 1$ *and* $\mu(\phi_0(c), T\phi_0) \leq 1$.

Then T *has a PPF dependent fixed point in* R_c.

Remark 3.12. In addition to the hypotheses of Corollary 3.9 (Corollary 3.10, Corollary 3.11) assume the following.
If $\alpha(x, y) \geq 1, \mu(x, y) \leq 1$ for any $x, y \in E$ and T is one-one then T has a unique PPF dependent fixed point in R_c.

By choosing $\alpha(x, y) = 1 = \mu(x, y)$ for any $x, y \in E$ in Corollary 3.11 we get the following corollary.

Corollary 3.13. *Let* $c \in I$. *Let* $T : E_0 \to E$ *be a function satisfying the following conditions:*

(i) *if there exists* $\lambda \in (0, 1)$ *such that* $||T\phi - T\psi||_E \leq \lambda M(\phi, \psi)$ *for any* $\phi, \psi \in E_0$, *where* $M(\phi, \psi) = \max\{||\phi - \psi||_{E_0}, ||\phi(c) - T\phi||_E, ||\psi(c) - T\psi||_E, \frac{1}{2}[||\phi(c) - T\psi||_E + ||\psi(c) - T\phi||_E]\}$ *and*

(ii) R_c *is algebraically closed with respect to the difference.*

Then T *has a PPF dependent fixed point in* R_c. *Moreover, if* T *is one-one then* T *has a unique PPF dependent fixed point in* R_c.

The following is an example in support of Theorem 2.3. Further, this example illustrates that if T is not one-one then T may have more than one fixed point.

Example 3.14. Let $E = \mathbb{R},\ c = 1 \in I = [\frac{1}{2}, 2] \subseteq \mathbb{R},\ E_0 = C(I, E)$.
We define $T : E_0 \to E,\ \alpha : E \times E \to \mathbb{R}^+,\ \mu : E \times E \to (0, \infty)$ by

$$T\phi = \begin{cases} -2 & \text{if} \quad \phi(c) < 0 \\ \frac{\phi(c)}{11-2\phi(c)} & \text{if} \quad 0 \leq \phi(c) < 2 \\ \frac{1}{2} & \text{if} \quad \phi(c) \geq 2, \end{cases}$$

$$\alpha(x, y) = \begin{cases} 1 & \text{if} \quad x \geq y \\ 0 & \text{if} \quad x < y, \end{cases}$$

and

$$\mu(x, y) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if} \quad x \geq y \\ 2 & \text{if} \quad x < y. \end{cases}$$

We first prove that T is an α_c-admissible mapping.
For any $\phi, \psi \in E_0$, we suppose that $\alpha(\phi(c), \psi(c)) \geq 1$. From the definition of α, we get $\phi(c) \geq \psi(c)$.
Case (i): Suppose that $0 \leq \phi(c), \psi(c) < 2$.
Clearly $11 - 2\phi(c) \leq 11 - 2\psi(c)$ and which implies that $\frac{1}{11-2\phi(c)} \geq \frac{1}{11-2\psi(c)}$.
Therefore $T\phi \geq T\psi$ and hence $\alpha(T\phi, T\psi) \geq 1$.
Case (ii): Suppose that $\phi(c), \psi(c) \geq 2$.
Clearly $T\phi = \frac{1}{2} = T\psi$ and which implies that $\alpha(T\phi, T\psi) \geq 1$.
Case (iii): Suppose that $\phi(c), \psi(c) < 0$.
Clearly $T\phi = -2 = T\psi$ and which implies that $\alpha(T\phi, T\psi) \geq 1$.
Case (iv): Suppose that $0 \leq \phi(c) < 2$ and $\psi(c) < 0$.
Since $\phi(c) \leq \frac{22}{3}$ we have $T\phi = \frac{\phi(c)}{11-2\phi(c)} \geq -2 = T\psi$ and which implies that $\alpha(T\phi, T\psi) \geq 1$.
Case (v): Suppose that $\phi(c) \geq 2$ and $\psi(c) < 0$.
Clearly $T\phi = \frac{1}{2} > -2 = T\psi$ and which implies that $\alpha(T\phi, T\psi) \geq 1$.
Case (vi): Suppose that $\phi(c) \geq 2$ and $0 \leq \psi(c) < 2$.
Since $\psi(c) \leq \frac{11}{4}$ we have $T\phi = \frac{1}{2} \geq \frac{\psi(c)}{11-2\psi(c)} = T\psi$ and which implies that $\alpha(T\phi, T\psi) \geq 1$.
From the above cases, we get that T is an α_c-admissible mapping.

For any $\phi, \psi, \gamma \in E_0$, we suppose that $\alpha(\phi(c), \psi(c)) \geq 1$ and $\alpha(\psi(c), \gamma(c)) \geq 1$.
From the definition of α, we get $\phi(c) \geq \psi(c) \geq \gamma(c)$. Therefore $\phi(c) \geq \gamma(c)$ and hence $\alpha(\phi(c), \gamma(c)) \geq 1$.
Therefore T is a traingular α_c-admissible mapping.
Similarly, we can prove that T is a triangular μ_c-subadmissible mapping.
Let $\lambda = \frac{1}{\sqrt{2}}$. Then $\lambda \in (0, 1)$.
We define $\varphi : E \to \mathbb{R}^+$ by

$$\varphi(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ x & \text{if} \quad 0 \leq x < 2 \\ 0 & \text{if} \quad x \geq 2. \end{cases}$$

Clearly φ is a lower semicontinuous function.
Let $\phi, \psi \in E_0$.
If $\phi(c) < \psi(c)$ then from the definition of α, the inequality (3.2) trivially holds.
Without loss of generality, we assume that $\phi(c) \geq \psi(c)$.
From the definition of α, we get $T\phi \geq T\psi$.
We consider
$||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi) \leq T\phi - T\psi + T\phi + T\psi = 2\, T\phi$.
Therefore

$$\alpha(\phi(c), \psi(c))(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)) \leq 2\, T\phi. \tag{3.4}$$

Case (i): Suppose that $T\phi = \phi(c)$.
If $\phi \in R_c$ then ϕ is a PPF dependent fixed point of T and hence the result holds.
Let us suppose $\phi \notin R_c$.
We define $\psi_1 : I \to E$ by $\psi_1(x) = \phi(c),\ x \in I$. Clearly $\psi_1 \in R_c$.
From the definition of T, we have

$$T\psi_1 = \begin{cases} -2 & \text{if} \quad \psi_1(c) \leq 0 \\ \frac{\psi_1(c)}{11 - 2\psi_1(c)} & \text{if} \quad 0 \leq \psi_1(c) < 2 \\ \frac{1}{2} & \text{if} \quad \psi_1(c) \geq 2. \end{cases}$$

That is

$$T\psi_1 = \begin{cases} -2 & \text{if} \quad \phi(c) \leq 0 \\ \frac{\phi(c)}{11 - 2\phi(c)} & \text{if} \quad 0 \leq \phi(c) < 2 \\ \frac{1}{2} & \text{if} \quad \phi(c) \geq 2. \end{cases}$$

Therefore $T\psi_1 = T\phi = \phi(c) = \psi_1(c)$.
Hence ψ_1 is a PPF dependent fixed point of T in R_c and the result follows.
Case (ii): Suppose that $\phi(c) < T\phi$.
Clearly $\phi(c) < -2$ and hence $T\phi = -2$.
We consider

$$\begin{aligned} M(\phi, \psi) &\geq ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi) \\ &= T\phi - \phi(c) = -2 - \phi(c) \text{ and hence} \end{aligned}$$

$$\begin{aligned} \lambda\, \mu(\phi(c), \psi(c))M(\phi, \psi) &\geq \frac{-2-\phi(c)}{2} \geq -4 \qquad (\text{since } \phi(c) \leq 6) \\ &= 2 \times -2 = 2\, T\phi \\ &\geq \alpha(\phi(c), \psi(c))(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi)). \text{ (by (3.4))} \end{aligned}$$

Therefore the inequality (3.2) is holds.
Case (iii): Suppose that $\phi(c) > T\phi$.
Sub-case (i): Suppose that $-2 < \phi(c) < 0$.
Clearly $T\phi = -2$.
We consider

$$\begin{aligned} M(\phi, \psi) &\geq ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi) \\ &= \phi(c) - T\phi = \phi(c) + 2 \text{ and hence} \end{aligned}$$

$\lambda\ \mu(\phi(c),\psi(c))M(\phi,\psi) \geq \frac{\phi(c)+2}{2} \geq -4$ (since $\phi(c) \geq -10$)
$= 2 \times -2 = 2\ T\phi$
$\geq \alpha(\phi(c),\psi(c))(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi))$. (by (3.4))
Therefore the inequality (3.2) is holds.
Sub-case (ii): Suppose that $0 < \phi(c) < 2$.
Clearly $T\phi = \frac{\phi(c)}{11-2\phi(c)}$.
We consider
$M(\phi,\psi) \geq ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi)$
$= \phi(c) - T\phi + \phi(c) + T\phi = 2\ \phi(c)$ and hence
$\lambda\ \mu(\phi(c),\psi(c))M(\phi,\psi) \geq \phi(c) \geq 2\ T\phi$ (since $\phi(c) \leq \frac{9}{2}$)
$\geq \alpha(\phi(c),\psi(c))(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi))$. (by (3.4))
Therefore the inequality (3.2) is holds.
Sub-case (iii): Suppose that $\phi(c) \geq 2$.
Clearly $T\phi = \frac{1}{2}$.
We consider
$M(\phi,\psi) \geq ||\phi(c) - T\phi||_E + \varphi(\phi(c)) + \varphi(T\phi)$
$= \phi(c) - T\phi + 0 + T\phi = \phi(c)$ and hence
$\lambda\ \mu(\phi(c),\psi(c))M(\phi,\psi) \geq \frac{\phi(c)}{2} \geq 2\ T\phi$ (since $\phi(c) \geq 2$)
$\geq \alpha(\phi(c),\psi(c))(||T\phi - T\psi||_E + \varphi(T\phi) + \varphi(T\psi))$. (by (3.4))
Therefore the inequality (3.2) is holds.
Let $\{\phi_n\}$ be a sequence in E_0 such that $\alpha(\phi_n(c),\phi_{n+1}(c)) \geq 1$ and $\mu(\phi_n(c),\phi_{n+1}(c)) \leq 1$ for any $n \in \mathbb{N}\cup\{0\}$.
Then from the definition of α and μ, we have $\phi_n(c) \geq \phi_{n+1}(c)$ for any $n \in \mathbb{N}\cup\{0\}$ and hence convergent.
Since $\mathbb{R}$ is complete, there exists $r \in \mathbb{R}$ such that $\phi_n(c) \to r$ as $n \to \infty$.
We define $\gamma : I \to E$ by $\gamma(x) = r, x \in I$. Then $\gamma \in R_c$ and $\gamma(c) = r$.
Therefore $\phi_n(c) \to \gamma(c)$ as $n \to \infty$. Clearly $\phi_n(c) \geq \gamma(c)$ for any $n \in \mathbb{N}\cup\{0\}$.
From the definition of α and μ, we get $\alpha(\phi_n(c),\gamma(c)) \geq 1$ and $\mu(\phi_n(c),\gamma(c)) \leq 1$ for any $n \in \mathbb{N}\cup\{0\}$.
Therefore the condition (iv) is satisfied.
For any $n \in \mathbb{R}$, we define $\phi_n : I \to E$ by

$$\phi_n(x) = \begin{cases} nx^2 & \text{if } x \in [\frac{1}{2}, 1] \\ \frac{n}{x^2} & \text{if } x \in [1, 2]. \end{cases}$$

Clearly $\phi_n \in E_0, ||\phi_n||_{E_0} = ||\phi_n(c)||_E$ and hence $\phi_n \in R_c$ for any $n \in \mathbb{R}$.
Let $\mathrm{F}_0 = \{\phi_n \mid n \in \mathbb{R}\}$. Then $\mathrm{F}_0 \subseteq R_c$ and F_0 is algebraically closed with respect to the difference.
Clearly $\phi_{\frac{1}{4}}(c) \geq T\phi_{\frac{1}{4}}$ and hence $\alpha(\phi_{\frac{1}{4}}(c), T\phi_{\frac{1}{4}}) \geq 1$ and $\mu(\phi_{\frac{1}{4}}(c), T\phi_{\frac{1}{4}}) \leq 1$.
Therefore the condition (v) is satisfied.

Therefore T satisfies all the hypotheses of Corollary 3.10 which in turn T satisfies all the hypotheses of Theorem 2.3 with $\zeta(t,s) = \lambda s - t, G(s,t) = s - t, \eta(t) = t$ for any $s,t \in \mathbb{R}^+, C_G = 0$ and $\lambda = \frac{1}{\sqrt{2}} \in (0,1)$. Here we observe that $\phi_0, \phi_{-2} \in R_c$ are two PPF dependent fixed points of T such that $\varphi(\phi_0(c)) = 0 = \varphi(\phi_{-2}(c))$.

Further we note that T is not one-one. For, we define $\gamma_1, \gamma_2 : I \to E$ by $\gamma_1(x) = 3x$ and $\gamma_2(x) = 4x$ for any $x \in I$. Clearly $\gamma_1, \gamma_2 \in E_0$ and $\gamma_1(c) = 3 \geq 2, \gamma_2(c) = 4 \geq 2$. By definition of T, we get $T\gamma_1 = \frac{1}{2} = T\gamma_2$, but $\gamma_1 \neq \gamma_2$.

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