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A Relation Theoretic Approach for ϕ-Fixed Point Result in Metric Space with an Application to an Integral Equation

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Abstract

In this paper, we prove the existence and uniqueness of ϕ-fixed point for (F, ϕ, θ)-contraction mapping in a complete metric space with a binary relation. Here the contractive condition is required to hold only for those elements that are related under the binary relation and not for the whole space. An application is given to show the usability of our result obtained.

Keywords: Binary relation, ϕ-fixed point, (F, ϕ, θ) contraction, Integral equation.
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1. Introduction and Preliminaries

In last few decades, the Banach contraction principle [3] has been extended using more general contractive condition on several spaces to obtain new fixed point results. Alam *et al.* [1] extended Banach contraction principle to a complete metric space with binary relation. For a nonempty set X, a binary relation $\mathcal{R}$ on X is a subset of $X \times X$. We write $(a, b) \in \mathcal{R}$ or $a \mathcal{R} b$ if an element $a \in X$ is related to $b \in X$ under relation $\mathcal{R}$. Following are some of the notations which we will use in this paper (see [1, 5, 7]):

- $F_T = \{s \in X : Ts = s\}$ = the set of all fixed points of T
- $Z_\phi = \{s \in X : \phi(s) = 0\}$ where $\phi : X \to [0, \infty)$ is a mapping
- $\Delta_X = \{s \in X : s \mathcal{R} s\}$ = the identity relation on X
- $X(T; \mathcal{R}) = \{s \in X : s \mathcal{R} Ts\}$

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- $\mathcal{R}^{-1} = \{(s,r) \in X \times X \ : \ r \ \mathcal{R} \ s\}$ = the inverse of $\mathcal{R}$
- $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$ = the symmetric closure of $\mathcal{R}$

Definition 1.1. [4] Let X be a nonempty set and $\phi : X \to [0,\infty)$ be a given function. An element $a \in X$ is called ϕ-fixed point of the mapping $T : X \to X$ if and only if a is both fixed point of T and zero of ϕ.

Definition 1.2. [4] Let (X,d) be a metric space and $\phi : X \to [0,\infty)$ be a given function. A mapping $T : X \to X$ is said to be a ϕ-Picard mapping if and only if

(i) T has unique ϕ-fixed point,
(ii) the sequence $\{T^n x\}$ converges for each $x \in X$, and the limit is ϕ-fixed point of T.

Definition 1.3. [4] Let (X,d) be a metric space and $\phi : X \to [0,\infty)$ be a given function. A mapping $T : X \to X$ is said to be a weakly ϕ-Picard mapping if and only if

(i) T has atleast one ϕ-fixed point,
(ii) the sequence $\{T^n x\}$ converges for each $x \in X$, and the limit is ϕ-fixed point of T.

Definition 1.4. [6] Let X be a nonempty set and T a self-mapping on X. A binary relation $\mathcal{R}$ defined on X is called T-closed if for any $x, y \in X$, $(x,y) \in \mathcal{R} \implies (Tx,Ty) \in \mathcal{R}$.

Definition 1.5. [1] Let $X \neq \emptyset$ and $\mathcal{R}$ a binary relation on X. For $x, y \in X$, a path of length $k \in \mathbb{N}$ in $\mathcal{R}$ from x to y is a finite sequence $x = z_0, z_1, z_2, \cdots, z_k = y \in X$ satisfying $(z_i, z_{i+1}) \in \mathcal{R}$ for each $i(0 \leq i \leq k1)$.

The set of all paths in $\mathcal{R}$ from x to y is denoted by $\Gamma(x,y,\mathcal{R})$. Jlelei *et al.* [4] introduced the class of functions denoted by $\mathcal{F}$ consisting of maps $F : [0,\infty)^3 \to [0,\infty)$ satisfying the following properties

(F1) $\max\{m,n\} \leq F(m,n,l)$ for all $m,n,l \in [0,\infty)$
(F2) $F(0,0,0) = 0$;
(F3) F is continuous.

Kumrod *et al.* [5] introduced the class of functions denoted by J consisting of maps $\theta : [0,\infty) \to [0,\infty)$ satisfying the following properties

(j1) θ is a nondecreasing function
(j2) θ is continuous
(j3) $\lim\limits_{n\to\infty} \theta^n(t) = 0$ for all $t \in (0,\infty)$
(j4) $\sum_{n=0}^{\infty} \theta^n(t) < \infty$ for all $t > 0$.

Note that if $\theta \in J$, then $\theta(t) < t$ for all $t > 0$. Further in [5] a new concept of a generalization of (F,ϕ)-contraction namely (F,ϕ,θ)-contraction mapping and some existence results of ϕ fixed point were proved.

Definition 1.6. [5] Let (X,d) be a metric space, $\phi : X \to [0,\infty)$ be a given function, $F \in \mathcal{F}$ and $\theta \in J$. The mapping $T : X \to X$ is said to be an (F,ϕ,θ)-contraction with respect to the metric d if and only if

$$F(d(Tx,Ty), \phi(Tx), \phi(Ty)) \leq \theta(F(d(x,y), \phi(x), \phi(y))) \tag{1.1}$$

for all $x, y \in X$.

Definition 1.7. [5] Let (X,d) be a metric space, $\phi : X \to [0,\infty)$ be a given function, $F \in \mathcal{F}$ and $\theta \in J$. The mapping $T : X \to X$ is said to be an (F,ϕ,θ)-weak contraction with respect to the metric d if and only if

$$\begin{aligned} F(d(Tx,Ty), \phi(Tx), \phi(Ty)) \ &\leq \ \theta(F(d(x,y), \phi(x), \phi(y))) + L[F(N(x,y), \phi(y), \phi(Tx)) \\ &\quad -F(0, \phi(y), \phi(Tx))] \end{aligned} \tag{1.2}$$

for all $x, y \in X$, where $N(x,y) = \min\{d(x,Tx), d(y,Ty), d(y,Tx)\}$ and $L \geq 0$.

In this work we derive the results of [5] on a complete metric space with a binary relation. Also we gave some examples where results of [5] are not applicable but existence of ϕ-fixed point can be assured by our results.

2. Main results

The following are our existence and uniqueness results for ϕ-fixed point.

Theorem 2.1. *Let (X, d) be a complete metric space, $T : X \to X$ be a mapping and $\mathcal{R}$ a T-closed binary relation on X. Let $\phi : X \to [0, \infty)$ be a lower semi-continuous function, $F \in \mathcal{F}$ and $\theta \in J$. Suppose $X(T; \mathcal{R}) \neq \emptyset$ and for all $(x, y) \in \mathcal{R} \cup \Delta_X$ the following holds*

$$F(d(Tx, Ty), \phi(Tx), \phi(Ty)) \leq \theta(F(d(x, y), \phi(x), \phi(y)). \tag{2.1}$$

Then every fixed point of T is a ϕ fixed point of T and T is a ϕ-Picard mapping. Further ϕ-fixed point is unique if $\Gamma(x, y, \mathcal{R}^s) \neq \emptyset$ for each $x, y \in X$.

Proof. Let a be the fixed point of T, then using (2.1) for $(a, a) \in \mathcal{R} \cup \Delta_X$ we have

$$F(0, \phi(a), \phi(a)) \leq \theta(F(0, \phi(a), \phi(a))).$$

So $F(0, \phi(a), \phi(a)) = 0$. Now by (F1), we have $\phi(a) \leq (F(0, \phi(a), \phi(a))$ Using (2.2) and (2.3), we get $\phi(a) = 0$ and thus $a \in Z_\phi$. Let $x_0 \in X(T, \mathcal{R})$, define $x_n = Tx_{n-1}, n \in \mathbb{N}$. Then as $\mathcal{R}$ is T-closed , we have

$$(x_1, x_2), (x_2, x_3), (x_3, x_4), \cdots (x_n, x_{n+1}), \cdots \in \mathcal{R},$$

therefore (2.1) holds for all $n \in \mathbb{N}$. Applying repeatedly (2.1) we obtain

$$F(d(T^{n+1}x, T^n x), \phi(T^{n+1}x), \phi(T^n x)) \leq \theta^n(F(d(Tx, x), \phi(Tx), \phi(x))) \tag{2.2}$$

Then again by (F1)

$$\max\{d(T^{n+1}x, T^n x), \phi(T^{n+1}x\} \leq F(d(T^{n+1}x, T^n x), \phi(T^{n+1}x), \phi(T^n x)) \leq \theta^n F(d(Tx, x), \phi(Tx), \phi(x))$$

and so

$$d(T^{n+1}x, T^n x) \leq \theta^n(F(d(Tx, x), \phi(Tx), \phi(x)))$$

holds for all $n \in \mathbb{N}$. For $m, n \in \mathbb{N}, m > n$, by triangular inequality we have

$$\begin{aligned} d(T^n x, T^m x) &\leq d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) + \cdots + d(T^{m-1}x, T^m x) \\ &= \theta^n(F(d(Tx, x), \phi(Tx), \phi(x))) + \theta^{n+1}(F(d(Tx, x), \phi(Tx), \phi(x))) \\ &\quad + \cdots + \theta^{m-1}(F(d(Tx, x), \phi(Tx), \phi(x))) \\ &= S_{m-1} - S_{n-1} \text{ where } S_n = \text{nth partial sum} \\ &\to 0 \text{ as } m, n \to \infty \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence. As X is complete there exists $t \in X$ such that $x_n \to t$. Now by (2.2) and (F1)

$$\phi(T^{n+1}x) \leq \theta^n F(d(Tx, x), \phi(Tx), \phi(x)).$$

then as $n \to \infty$ we get $\phi(T^{n+1}x) \to 0$. Now lower semi continuity of ϕ gives us

$$0 \leq \phi(t) \leq \lim_{n\to\infty} \inf \phi(T^{n+1}x) = 0. \tag{2.3}$$

Case 1 : If T is continuous then $Tx_n \to Tt$ which than from uniqueness of limit implies t is a fixed point.
Case 2 : If $\mathcal{R}$ is d-self closed then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k}, t] \in \mathcal{R}$. Then from (contraction)

$$F(d(Tx_{n_k}, Tt), \phi(Tx_{n_k}), \phi(Tt)) \leq \theta(F(d(x_{n_k}, t), \phi(x_{n_k}), \phi(t))) \tag{2.4}$$

Taking limit as $k \to \infty$ in (2.4) and using continuity of F we obtain $F(d(t,Tt),0,\phi(Tt)) \leq \theta(F(0,0,0)) = 0$. This further from (F1) implies that $d(t,Tt) = 0$. Hence from Case 1, Case 2 and (2.3) we obtain t is a ϕ-fixed point of T. Let $r \neq t$ be another ϕ- fixed point of T. By assumption (e), there exists a path of length k in R^s from t to r so that $z_0 = t, z_k = r, [z_i, z_{i+1}] \in \mathcal{R}$ for each i; $0 \leq i \leq k-1$. Again as $\mathcal{R}$ is T closed, using (2.1) we have

$$\begin{aligned}
F(d(Tt,Tr),\phi(Tt),\phi(Tr)) &= F(d(T^{n+1}z_0,T^{n+1}z_k),\phi(T^{n+1}z_0),\phi(TrT^{n+1}z_k)) \\
&\leq \theta(F(d(T^n z_0,T^n z_k),\phi(T^n z_0),\phi(T^n z_k)) \\
&\leq \theta^2(F(d(T^{n-1}z_0,T^{n-1}z_k),\phi(T^{n-1}z_0),\phi(T^{n-1}z_k)) \\
&\vdots \\
&\leq \theta^{n+1}(F(d(z_0,z_k),\phi(z_0),\phi(z_k)) \\
&= \theta^{n+1}(F(d(t,r),\phi(t),\phi(r)) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

Thus, we obtain $F(d(Tt,Tr),0,0) = 0$ and hence we get $t = r$. □

Example 2.2. Let $X = \{1,3,7,8,9\}$ with usual metric. Define a binary relation

$$\mathcal{R} = \{(1,1),(3,3),(7,7),(8,8),(9,9),(1,3),(1,7),(1,9),(3,7),(3,9),(7,9)\}$$

on X and the mapping $T : X \to X$ defined by

$$T(1) = 1, T(3) = 1, T(7) = 3, T(8) = 7, T(9) = 7,$$

then $\mathcal{R}$ is T closed. Let $\phi : X \to [0,\infty)$ be defined by

$$\phi(x) = x - 1, x \neq 8 \ \text{ and } \phi(8) = 8,$$

the function $F : [0,\infty)^3 \to [0,\infty)$ defined by $F(a,b,c) = a+b+c$ and the function $\theta : [0,\infty) \to [0,\infty)$ defined by $\theta(t) = \frac{3}{4}t$. Here $T : X \to X$ is not an (F,ϕ,θ)-contraction as

$$F(d(Tx,Ty),\phi(Tx),\phi(Ty)) \leq \theta(F(d(x,y),\phi(x),\phi(y)))$$

does not hold for $x = 1, y = 8$. But it holds for all $(x,y) \in \mathcal{R}$. Thus all conditions of Theorem 2.1 are satisfied and T has a ϕ-fixed point in X.

Theorem 2.3. *Let (X,d) be a complete metric space, $T : X \to X$ be a mapping and $\mathcal{R}$ a T-closed binary relation on X. Let $\phi : X \to [0,\infty)$ be a lower semi-continuous function, $F \in \mathcal{F}$ and $\theta \in J$. Suppose $X(T;\mathcal{R}) \neq \emptyset$ and for all $(x,y) \in \mathcal{R} \cup \Delta_X$ the following holds*

$$\begin{aligned}
F(d(Tx,Ty),\phi(Tx),\phi(Ty)) \leq\ & \theta(F(d(x,y),\phi(x),\phi(y)) + L[F(N(x,y),\phi(y),\phi(Tx)) \\
& -F(0,\phi(y),\phi(Tx))]
\end{aligned} \tag{2.5}$$

where $N(x,y) = \min\{d(x,Tx), d(y,Ty), d(y,Tx)\}$ and $L \geq 0$ holds for all $(x,y) \in \mathcal{R} \cup \Delta_X$. Then every fixed point of T is a ϕ fixed point of T and T is a wekly ϕ-Picard mapping. Further ϕ-fixed point is unique if $\Gamma(x,y,\mathcal{R}^s) \neq \emptyset$ for each $x,y \in X$.

Proof. Let $a \in X$ be such that $a \in F_T$. Then using (2.5) for $(a,a) \in \mathcal{R} \cup \Delta_X$ we have

$$F(0,\phi(a),\phi(a)) \leq \theta(F(0,\phi(a),\phi(a))) + L[F(0,\phi(a),\phi(a)) - F(0,\phi(a),\phi(a))] = \theta(F(0,\phi(a),\phi(a))).$$

So $F(0,\phi(a),\phi(a)) = 0$. From (F1), we have $\phi(a) \leq \theta(F(0,\phi(a),\phi(a)) + L[F(0,\phi(a),\phi(a)) - F(0,\phi(a),\phi(a))]$. Hence we get $\phi(a) = 0$ and thus $a \in Z_\phi$.
Let $x_0 \in X(T,\mathcal{R})$, define $x_n = Tx_{n-1}, n \in \mathbb{N}$. Then as $\mathcal{R}$ is T-closed , we have

$$(x_1,x_2),(x_2,x_3),(x_3,x_4),\cdots(x_n,x_{n+1}),\cdots \in \mathcal{R},$$

therefore (2.5) holds for all $n \in \mathbb{N}$. By (2.5)

$$\begin{aligned} & F(d(T^{n+1}x, T^n x), \phi(T^{n+1}x), \phi(T^n x)) \\ \leq\; & \theta(F(d(T^n x, T^{n-1}x), \phi(T^n x), \phi(T^{n-1}x)) + L[F(0, \phi(T^n x), \phi(T^{n+1}x)) \\ & -F(0, \phi(T^n x), \phi(T^{n+1}x))] \\ =\; & \theta(F(d(T^n x, T^{n-1}x), \phi(T^n x), \phi(T^{n-1}x)) \end{aligned}$$

Applying this process for each $n \in \mathbb{N}$ we obtain

$$F(d(T^{n+1}x, T^n x), \phi(T^{n+1}x), \phi(T^n x)) \leq \theta^n(F(d(Tx, x), \phi(Tx), \phi(x)) \tag{2.6}$$

Rest of the proof can be proceeded in exactly similar manner as in above result. □

Example 2.4. Let $X = \mathbb{N}$ with usual metric. Define a binary relation

$$\mathcal{R} = \{(1,1), (3,3), (1,3), (1,7), (1,9), (3,7), (3,9), (7,9)\}$$

on X and the mapping $T : X \to X$ defined by

$$T(7) = 3, T(8) = 7, T(9) = 7 \text{ and } T(x) = 1 \;\text{ otherwise},$$

then $\mathcal{R}$ is T closed. Let $\phi : X \to [0, \infty)$ be defined by

$$\phi(x) = x - 1, x \neq 8 \;\text{ and } \phi(8) = 8,$$

the function $F : [0,\infty)^3 \to [0,\infty)$ defined by $F(a,b,c) = a + b + c$ and the function $\theta : [0,\infty) \to [0,\infty)$ defined by $\theta(t) = \frac{3}{4}t$. Here $T : X \to X$ does not satisfy

$$\begin{aligned} F(d(Tx, Ty), \phi(Tx), \phi(Ty)) \;\leq\; & \theta(F(d(x,y), \phi(x), \phi(y)) + L[F(N(x,y), \phi(y), \phi(Tx)) \\ & -F(0, \phi(y), \phi(Tx))] \end{aligned}$$

for $x = 1, y = 8$. But it holds for all $(x, y) \in \mathcal{R}$ with $L = 1$ Thus all conditions of Theorem 2.3 are satisfied and T has a ϕ-fixed point in X.

3. An Application

In this section using Theorem 2.1 we will prove existence of solution of the integral equation

$$z(t) = \phi(t) + \int_0^1 r(t, z(s))ds, \tag{3.1}$$

where $r : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $\phi : [0,1] \to \mathbb{R}$ are continuous. Let $X = C([0,1], \mathbb{R})$ and d be the metric on it given as

$$d(z_1, z_2) = \|z_1 - z_2\|_\infty = \max_{s \in [0,1]} |z_1(s) - z_2(s)| \;,$$

which makes X complete. Let the operator $T : X \to X$ be defined as

$$Tz(t) = \phi(t) + \int_0^1 r(t, z(s))ds. \tag{3.2}$$

To solve equation (3.1), it simply means to find fixed point of equation (3.2).

Theorem 3.1. *Suppose the following conditions are satisfied*

C_1 *: there exists $\lambda \in (0,1)$ such that if $z_1(s) \leq z_2(s)$ holds for all $s \in [0,1]$ then*

$$0 \leq |r(t, z_1(s)) - r(t, z_2(s))| \leq \lambda \; |z_1(s) - z_2(s)|, \tag{3.3}$$

C_2 : *$r(t, z_1(s)) \geq r(t, z_2(s))$ whenever $z_1(s) \geq z_2(s)$ holds,*

C_3 : *there exists $z_0 \in X$ for which $z_0(t) \leq \phi(t) + \int_0^1 r(t, z_0(s))ds$.*

Then (3.1) has a solution.

Proof. We define a binary relation $\mathcal{R}$ on X as follows : for $x, y \in X$, $x \mathcal{R} y$ if and only if $x(t) \leq y(t)$ for all $t \in [0, 1]$. By condition C_1 we have

$$\begin{aligned} |Tz_1(t) - Tz_2(t)| &= \left| \int_0^1 r(t, z_1(s)) \, ds - \int_0^1 r(t, z_2(s)) \, ds \right| \\ &= \left| \int_0^1 (r(t, z_1(s)) - r(t, z_2(s))) \, ds \right| \\ &\leq \int_0^1 |r(t, z_1(s)) - r(t, z_2(s))| \, ds \\ &\leq \int_0^1 \lambda \, |z_1(s) - z_2(s)| \, ds \\ &\leq \lambda \, \|z_1 - z_2\|_\infty \end{aligned} \tag{3.4}$$

$$\text{This implies that } \|Tz_1 - Tz_2\|_\infty \leq \lambda \, \|z_1 - z_2\|_\infty \quad i.e., \; d(Tz_1, Tz_2) \leq \lambda \, d(z_1, z_2). \tag{3.5}$$

Now we define $F : [0, \infty)^3 \to [0, \infty)$ as $F(p, q, r) = p + q + r$ where $p, q, r \in [0, \infty)$, $\phi : X \to [0, \infty)$ as $\phi(v) = 0$ for all $v \in X$ and $\theta : [0, \infty) \to [0, \infty)$ as $\theta(t) = \lambda t$ where $0 < \lambda < 1$. Then from equation (3.5) we get $F(d(Tz_1, Tz_2), \phi(Tz_1), \phi(Tz_2)) \leq \theta(F(d(z_1, z_2), \phi(z_1), \phi(z_2)))$ which means T is an (F, ϕ, θ) contraction. Further by conditions (C_2) and (C_3) it can be shown that $X(T, \mathcal{R})$ is nonempty and $\mathcal{R}$ is T closed. Hence all conditions of Theorem 2.1 are satisfied and as a result T has a fixed point. □

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