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Fixed Point Results for Multivalued Operator in $G-$metric Space

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Abstract

In this paper, we shall give some results on fixed points of multivalued operator on $G-$metric spaces by using the method of Kikkawa [6]. Our results generalize and extend some old fixed point theorems to the multivalued case.

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1. Introduction and preliminaries

Mustafa and Sims [8] introduced the notion of G-metric space. Based on the notion of generalized metric space or $G-$metric space, many authors obtained some fixed point theorems for self mapping under some contractive conditions (e.g., [1, 9, 10, 11, 12]). Consistent with Mustafa and Sims [8], the following definitions and results will be needed in the sequel.

Definition 1.1. [8] Let X be a non empty set, $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G_1) G(x,y,z) = 0 if x = y = z,
(G_2) $0 <$ G(x,x,y) for all $x, y \in X$ with $x \neq y$,
(G_3) G(x,x,y) $\leq$ G(x,y,z) for all $x, y, z \in X$ with $x \neq y$,
(G_4) G(x,y,z) = G(x,z,y) = G(y,z,x) = ... (symmetry in all three variables),
(G_5) G(x,y,z) $\leq$ G(x,a,a) + G(a,y,z) for all $x, y, z, a \in X$ (rectangle inequality).

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Then the function G is called a generalized metric, or, more specially, a G- metric on X, and the pair (X, G) is called a G- metric space.

Definition 1.2. [8] Let (X, G) be a G- metric space, and let $\{x_n\}$ be a sequence of points of X, therefore, we say that $\{x_n\}$ is G- convergent to $x \in X$ if $lim_{n,m\to+\infty} G(x, x_n, x_m) = 0$, that is, for any $\epsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $lim_{n\to+\infty} x_n = x$.

Lemma 1.3. *[8] Let (X, G) be a G- metric space. The following statements are equivalent:*

1. *$\{x_n\}$ is G- convergent to x,*
2. *$G(x_n, x_n, x) \to 0$ as $n \to +\infty$,*
3. *$G(x_n, x, x) \to 0$ as $n \to +\infty$,*
4. *$G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$,*

Definition 1.4. [8] Let (X, G) be a G- metric space. A sequence $\{x_n\}$ is called a G- Cauchy sequence if, for any $\epsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Lemma 1.5. *[8] Let (X, G) be a G- metric space. The following statements are equivalent:*

1. *The sequence $\{x_n\}$ is G- Cauchy,*
2. *for any $\epsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \epsilon$ for all $m, n \geq N$.*

Definition 1.6. [8] A G- metric space (X, G) is called G-complete if every G- Cauchy sequence is G-convergent in (X, G).

Every G- metric on X defines a metric d_G on X given by

$$d_G = G(x, y, y) + G(y, x, x) \quad for \quad all \quad x, y \in X.$$

Lemma 1.7. *[8] If (X, G) is a G - metric space, then $G(x, y, y) = 2G(y, x, x)$ for all $x, y \in X$.*

Lemma 1.8. *[8] If (X, G) is a G- metric space, then $G(x, x, y) = G(x, x, z) + G(z, z, y)$ for all $x, y, z \in X$.*

Nadler [13] initiated the study of fixed points for multi-valued contraction mappings. There are many works about fixed point for multivalued mappings (cited in [7, 2, 3, 4, 5]) and weakly Picard maps (see in [15, 16, 17]).

We shall denote the set of all nonempty closed subset of X by $P_{cl}(X)$. Also, we shall denote the set of fixed points of a multifunction T by $Fix(T)$. Let X be a nonempty set and consider the space $\mathbb{R}^m_+$ endowed with the usual component-wise partial order. We denote by $M_{m,m}(\mathbb{R}^+)$ the set of all $m \times m$ matrices with positive elements and by I the identity $m \times m$ matrix. A matrix $\mathcal{A} \in M_{m,m}(\mathbb{R}^+)$ is said to be converges to zero whenever $\mathcal{A}^\setminus \to 0$.

Theorem 1.9. *[14] Let $\mathcal{A} \in M_{m,m}(\mathbb{R}^+)$. The following are equivalent:*

(i) $\mathcal{A}^n \to 0$.

(ii) The eigen values of $\mathcal{A}$ are in the open unit disc, i.e., $\mid \lambda \mid < 1$, for all $\lambda \in C$ with $det(\mathcal{A} - \lambda I) = 0$.

(iii) The matrix $(I - \mathcal{A})$ is non-singular and $(I - \mathcal{A})^{-1} = I + \mathcal{A} + \mathcal{A}^2 + ... + \mathcal{A}^n + ...$

(iv) The matrix $(I - \mathcal{A})$ is non-singular and $(I - \mathcal{A})^{-1}$ has non negative elements.

(v) $\mathcal{A}^n q \to 0$ and $q\mathcal{A}^n \to 0$, for all $q \in \mathbb{R}^m$.

By using Theorem 1.9(v), we have $-\mathcal{A}$ converges to zero whenever $\mathcal{A}$ is converges to zero. Again, Theorem 1.9 implies that $(I + \mathcal{A})$ is invertible and $(I + \mathcal{A})^{-1} \leq (I - \mathcal{A})^{-1}$.

2. Main Result

Theorem 2.1. *Let (X, G) be a complete $G-$ metric space, a matrix $\mathcal{A} \in M_{m,m}(\mathbb{R}^+)$ converges to zero and $T : X \times X \to P_{cl}(X)$ a multivalued operator. Suppose that for each $x, y, z, x', y', z' \in X$,*

$$(I + \mathcal{A}^{-1})[G(x, T(x, x'), T^2(x, x')) + G(x', T(x', x), T^2(x', x))] \leq (I - \mathcal{A}^{-1})[G(x, y, z) + G(x', y', z')]$$

implies that for each $u \in T(x, x'), u' \in T(x', x), v \in T(y, y'), v' \in T(y', y)$ there exist $w \in T(z, z'), w' \in T(z', z)$ such that

$$G(u, v, w) + G(u', v', w') \leq \mathcal{A}[G(x, y, z) + G(x', y', z')]. \tag{2.1}$$

Then T has a coupled fixed point.

Proof. For each $(x, x') \in X \times X$,

$$\begin{aligned} & (I + \mathcal{A}^{-1})[G(x, T(x, x'), T^2(x, x')) + G(x', T(x', x), T^2(x', x))] \\ \leq \; & (I - \mathcal{A}^{-1})[G(x, T(x, x'), T^2(x, x')) + G(x', T(x', x), T^2(x', x))]). \end{aligned}$$

Let $(x_0, x'_0) \in X \times X$ and take $x_1 \in T(x_0, x'_0), x'_1 \in T(x'_0, x_0), x_2 \in T(x_1, x'_1), x'_2 \in T(x'_1, x_1)$. If $x_0 = x_1 = x_2$ and $x'_0 = x'_1 = x'_2$ then (x_0, x'_0) is a coupled fixed point of T. Let any one of x_0, x_1, x_2 and x'_0, x'_1, x'_2 be not equal to other, from (2.1), there exist $x_3 \in T(x_2, x'_2), x'_3 \in T(x'_2, x_2)$ such that

$$G(x_1, x_2, x_3) + G(x'_1, x'_2, x'_3) \leq \mathcal{A}[G(x_0, x_1, x_2) + G(x'_0, x'_1, x'_2)]. \tag{2.2}$$

If $x_1 = x_2 = x_3$ and $x'_1 = x'_2 = x'_3$ then (x_1, x'_1) is a coupled fixed point of T. Let any one of x_1, x_2, x_3 and x'_1, x'_2, x'_3 be not equal to other, from (2.1) and (2.2), there exist $x_4 \in T(x_5, x'_5), x'_4 \in T(x'_5, x_5)$ such that

$$\begin{aligned} G(x_2, x_3, x_4) + G(x'_2, x'_3, x'_4) & \leq \mathcal{A}[G(x_1, x_2, x_3) + G(x'_1, x'_2, x'_3)] \\ & \leq \mathcal{A}^2[G(x_0, x_1, x_2) + G(x'_0, x'_1, x'_2)]. \end{aligned} \tag{2.3}$$

Now by induction, we construct sequences $\{x_n\}_{n \geq 0}$, $\{x'_n\}_{n \geq 0}$ in X such that $x_{n+1} \in T(x_n, x'_n), x'_{n+1} \in T(x'_n, x_n)$ and

$$G(x_n, x_{n+1}, x_{n+2}) + G(x'_n, x'_{n+1}, x'_{n+2}) \leq \mathcal{A}^n[G(x_0, x_1, x_2) + G(x'_0, x'_1, x'_2)] \tag{2.4}$$

for all $n \geq 0$. From Theorem 1.9, for all $m, n \in \mathbb{N}$, $n < m$ and by (G_3) and (G_5) we obtain

$$\begin{aligned} G(x_n, x_m, x_m) + G(x'_n, x'_m, x'_m) & \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) \\ & \quad + \cdots + G(x_{m-1}, x_m, x_m) + G(x'_n, x'_{n+1}, x'_{n+1}) + G(x'_{n+1}, x'_{n+2}, x'_{n+2}) \\ & \quad + G(x'_{n+2}, x'_{n+3}, x'_{n+3}) + \cdots + G(x'_{m-1}, x'_m, x'_m) \\ & \leq G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) + G(x_{n+2}, x_{n+3}, x_{n+4}) \\ & \quad + \cdots + G(x_{m-1}, x_m, x_{m+1}) + G(x'_n, x'_{n+1}, x'_{n+2}) + G(x'_{n+1}, x'_{n+2}, x'_{n+3}) \\ & \quad + G(x'_{n+2}, x'_{n+3}, x'_{n+4}) + \cdots + G(x'_{m-1}, x'_m, x'_{m+1}) \\ & \leq (\mathcal{A}^n + \mathcal{A}^{n+1} + \mathcal{A}^{n+2} + \cdots + \mathcal{A}^{m-1})[G(x_0, x_1, x_2) + G(x'_0, x'_1, x'_2)] \\ & \leq \mathcal{A}^n(I + \mathcal{A} + \mathcal{A}^2 + \cdots + \mathcal{A}^{m-1-n})[G(x_0, x_1, x_2) + G(x'_0, x'_1, x'_2)] \\ & \leq \mathcal{A}^n(I - \mathcal{A})^{-1}[G(x_0, x_1, x_2) + G(x'_0, x'_1, x'_2)], \end{aligned}$$

that is,
$[G(x_n, x_m, x_m) + G(x'_n, x'_m, x'_m)] \to 0$ as $n \to \infty$.

Hence $\{x_n\}_{n \geq 0}, \{x'_n\}_{n \geq 0}$ are Cauchy sequence in the complete $G-$ metric space (X, G). Choose $(x^*, x'^*) \in X \times X$ such that $x_n \to x^*$ and $x'_n \to x'^*$ as $n \to \infty$. We claim that $(x, x') \in (X \times X) \setminus (\{x^*\}, \{x'^*\})$,

$$[G(x^*, T(x, x'), T(x, x')) + G(x'^*, T(x, x'), T(x, x'))] \leq \mathcal{A}[G(x^*, x, x) + G(x'^*, x', x')]. \tag{2.5}$$

Let $(x, x') \in (X \times X) \setminus (\{x^*\}, \{x'^*\})$. Choose a natural number N such that

$$[G(x_n, x^*, x^*) + G(x'_n, x'^*, x'^*)] < \frac{1}{4}[G(x, x^*, x^*) + G(x', x'^*, x'^*)]$$

for all $n \geq N$. Hence, for each $n \geq N$ we have

$$\begin{aligned} G(x_n, T(x_n, x'_n), T(x_n, x'_n)) + G(x'_n, T(x_n, x'_n), T(x_n, x'_n)) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x'_n, x'_{n+1}, x'_{n+1}) \\ &\leq G(x_n, x^*, x^*) + G(x^*, x_{n+1}, x_{n+1}) \\ &\quad + G(x'_n, x'^*, x'^*) + G(x'^*, x'_{n+1}, x'_{n+1}) \\ &\leq G(x_n, x^*, x^*) + 2G(x_{n+1}, x^*, x^*) \\ &\quad + G(x'_n, x'^*, x'^*) + 2G(x'_{n+1}, x'^*, x'^*) \\ &\leq \frac{3}{4}[G(x, x^*, x^*) + G(x', x'^*, x'^*)] \\ &\leq G(x, x^*, x^*) + G(x', x'^*, x'^*) \\ &\quad - \frac{1}{4}[G(x, x^*, x^*) + G(x', x'^*, x'^*)] \\ &\leq G(x, x^*, x^*) + G(x', x'^*, x'^*) \\ &\quad - [G(x_n, x^*, x^*) + G(x'_n, x'^*, x'^*)] \\ G(x_n, T(x_n, x'_n), T(x_n, x'_n)) + G(x'_n, T(x_n, x'_n), T(x_n, x'_n)) &\leq G(x_n, x^*, x^*) + G(x'_n, x'^*, x'^*). \end{aligned}$$

Thus

$$\begin{aligned} & (I + \mathcal{A})^{-1}[G(x_n, T(x_n, x'_n), T(x_n, x'_n)) + G(x'_n, T(x_n, x'_n), T(x_n, x'_n))] \\ \leq\ & (I - \mathcal{A})^{-1}[G(x_n, T(x_n, x'_n), T(x_n, x'_n)) + G(x'_n, T(x_n, x'_n), T(x_n, x'_n))] \\ \leq\ & (I - \mathcal{A})^{-1}[G(x_n, x, x) + G(x'_n, x', x')] \end{aligned}$$

for $n \geq N$.
Since $x_{n+1} \in T(x_n, x'_n)$, $x'_{n+1} \in T(x'_n, x_n)$, by using (2.1), for each $n \geq N$ there exist $u_n \in T(x, x')$ and $u'_n \in T(x', x)$ such that

$$G(u_n, x_{n+1}, x_{n+1}) + G(u'_n, x'_{n+1}, x'_{n+1}) \leq \mathcal{A}[G(x_n, x, x) + G(x'_n, x', x')].$$

Hence

$$G(x_{n+1}, T(x, x'), T(x, x')) + G(x'_{n+1}, T(x', x), T(x', x)) \leq \mathcal{A}[G(x_n, x, x) + G(x'_n, x', x')]$$

and so

$$\lim_{n\to\infty}[G(x_{n+1}, T(x, x'), T(x, x')) + G(x'_{n+1}, T(x', x), T(x', x))] \leq \lim_{n\to\infty}[G(x_n, x, x) + G(x'_n, x', x')].$$

Thus

$$G(x^*, T(x, x'), T(x, x')) + G(x'^*, T(x', x), T(x', x)) \leq \mathcal{A}[G(x^*, x, x) + G(x'^*, x', x')]$$

for all $(x, x') \in (X \times X) \setminus (\{x^*\}, \{x'^*\})$.

Now we show that for each $(x, x') \in X \times X$ and $u \in T(x, x')$, $u' \in T(x', x)$ there exist $v \in T(x^*, x'^*)$, $v' \in T(x'^*, x^*)$ such that

$$G(u, v, v) + G(u', v', v') \leq \mathcal{A}[G(x, x^*, x^*) + G(x', x'^*, x'^*)].$$

If $x_n \to x^*$ and $x'_n \to x'^*$ we have nothing to prove. Let $x \neq x^*$ and $x' \neq x'^*$. By definition of $G(x^*, T(x,x'), T(x,x'))$, $G(x'^*, T(x',x), T(x',x))$ and for each $n \geq 1$ there exist $y_n \in T(x,x')$ and $y'_n \in T(x',x)$ such that

$$\begin{aligned} G(x^*, y_n, y_n) + G(x'^*, y'_n, y'_n) &\leq G(x^*, T(x,x'), T(x,x')) + G(x'^*, T(x',x), T(x',x)) \\ &\quad + \frac{1}{n}[G(x, x^*, x^*) + G(x', x'^*, x'^*)]. \end{aligned}$$

Hence we have

$$\begin{aligned} G(x, T(x,x'), T(x,x')) + G(x', T(x',x), T(x',x)) &\leq G(x, y_n, y_n) + G(x', y'_n, y'_n) \\ &\leq G(x, x^*, x^*) + G(x^*, y_n, y_n) \\ &\quad + G(x', x'^*, x'^*) + G(x'^*, y'_n, y'_n) \\ &\leq G(x, x^*, x^*) + G(x^*, T(x,x'), T(x,x')) + \frac{1}{n}G(x, x^*, x^*) \\ &\quad + G(x', x'^*, x'^*) + G(x'^*, T(x',x), T(x',x)) + \frac{1}{n}G(x', x'^*, x \end{aligned}$$

From (2.5),

$$\begin{aligned} (I+\mathcal{A})^{-1}[G(x, T(x,x'), T(x,x')) + G(x', T(x',x), T(x',x))] &\leq G(x, x^*, x^*) + \frac{1}{n}(I+\mathcal{A})^{-1}G(x, x^*, x^*) \\ &\quad + G(x', x'^*, x'^*) + \frac{1}{n}(I+\mathcal{A})^{-1}G(x', x'^*, x'^*) \end{aligned}$$

for all $n \geq 1$.

Thus

$$\begin{aligned} (I+\mathcal{A})^{-1}[G(x, T(x,x'), T(x,x')) + G(x', T(x',x), T(x',x))] &\leq G(x, x^*, x^*) + G(x', x'^*, x'^*) \\ &\leq (I-\mathcal{A})^{-1}[G(x, x^*, x^*) + G(x', x'^*, x'^*)]. \end{aligned}$$

Now by using (2.1), for each $u \in T(x,x')$, $u' \in T(x',x)$, there exist $v \in T(x^*, x'^*)$, $v' \in T(x'^*, x^*)$ such that

$$G(u, v, v) + G(u', v', v') \leq \mathcal{A}[G(x, x^*, x^*) + G(x', x'^*, x'^*)].$$

Since $x_{n+1} \in T(x_n, x'_n)$ and $x'_{n+1} \in T(x'_n, x_n)$ for all $n \geq 1$, there exist $v_n \in T(x^*, x'^*)$ and $v'_n \in T(x'^*, x^*)$ such that

$$G(v, x_n, x_{n+1}) + G(v', x'_n, x'_{n+1}) \leq \mathcal{A}[G(x_n, x^*, x^*) + G(x'_n, x'^*, x'^*)].$$

Hence

$$\begin{aligned} G(v_n, x^*, x^*) + G(v'_n, x'^*, x'^*) &\leq G(v_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x^*, x^*) \\ &\quad + G(v'_n, x'_{n+1}, x'_{n+1}) + G(x'_{n+1}, x'^*, x'^*) \\ &\leq \mathcal{A}G(x_n, x^*, x^*) + G(x_{n+1}, x^*, x^*) \\ &\quad + \mathcal{A}G(x'_n, x'^*, x'^*) + G(x'_{n+1}, x'^*, x'^*) \end{aligned}$$

for all $n \geq 1$. Therefore $v_n \to x^*$ and $v'_n \to x'^*$.

Since $v_n \in T(x^*, x'^*)$ and $v'_n \in T(x'^*, x^*)$ for all $n \geq 1$ and $T(x^*, x'^*)$ is a closed subset of $X \times X$, $x^* \in T(x^*, x'^*)$ and $x'^* \in T(x'^*, x^*)$. □

Theorem 2.2. *Let (X, G) be a complete $G-$ metric space, a matrix $\mathcal{A} \in M_{m,m}(\mathbb{R}^+)$ converges to zero and $T : X \times X \to P_{cl}(X)$ a multivalued operator. Suppose that for each $x, y, z, x', y', z' \in X$,*

$$(I+\mathcal{A}^{-1})\max\{G(x, T(x,x'), T^2(x,x')), G(x', T(x',x), T^2(x',x))\} \leq (I-\mathcal{A}^{-1})\max\{G(x,y,z), G(x',y',z')\}$$

implies that for each $u \in T(x,x'), u' \in T(x',x), v \in T(y,y'), v' \in T(y',y)$ *there exist* $w \in T(z,z'), w' \in T(z',z)$ *such that*

$$\max\{G(u,v,w), G(u',v',w')\} \leq \mathcal{A} \max \left\{ \begin{array}{l} G(x,y,z), G(x,T(x,x'),T(x,x')), \\ G(y,T(y,y'),T(y,y')), G(z,T(z,z'),T(z,z')), \\ G(x',y',z'), G(x',T(x',x),T(x',x)), \\ G(y',T(y',y),T(y',y)), G(z',T(z',z),T(z',z)) \end{array} \right\}. \tag{2.6}$$

Then T *has a coupled fixed point.*

Proof. For each $(x,x') \in X \times X$,

$$\begin{aligned} & (I+\mathcal{A}^{-1})[G(x,T(x,x'),T^2(x,x')) + G(x',T(x',x),T^2(x',x))] \\ \leq \; & (I-\mathcal{A}^{-1})[G(x,T(x,x'),T^2(x,x')) + G(x',T(x',x),T^2(x',x))]). \end{aligned}$$

Let $(x_0,x_0') \in X \times X$ and take $x_1 \in T(x_0,x_0'), x_1' \in T(x_0',x_0), x_2 \in T(x_1,x_1'), x_2' \in T(x_1',x_1)$. If $x_0 = x_1 = x_2$ and $x_0' = x_1' = x_2'$ then (x_0,x_0') is a coupled fixed point of T. Let any one of x_0,x_1,x_2 and x_0',x_1',x_2' be not equal to other. From (2.6) there exist $x_3 \in T(x_2,x_2'), x_3' \in T(x_2',x_2)$ such that

$$\max\{G(x_1,x_2,x_3), G(x_1',x_2',x_3')\} \leq \mathcal{A} \max \left\{ \begin{array}{l} G(x_0,x_1,x_2), G(x_0,T(x_0,x_0'),T(x_0,x_0')), \\ G(x_1,T(x_1,x_1'),T(x_1,x_1')), G(x_2,T(x_2,x_2'),T(x_2,x_2')), \\ G(x_0',x_1',x_2'), G(x_0',T(x_0',x_0),T(x_0',x_0)), \\ G(x_1',T(x_1',x_1),T(x_1',x_1)), G(x_2',T(x_2',x_2),T(x_2',x_2)) \end{array} \right\}$$

$$\max\{G(x_1,x_2,x_3), G(x_1',x_2',x_3')\} \leq \mathcal{A} \max \left\{ \begin{array}{l} G(x_0,x_1,x_2), G(x_0,x_1,x_1), G(x_1,x_2,x_2), G(x_2,x_3,x_3) \\ G(x_0',x_1',x_2'), G(x_0',x_1',x_1'), G(x_1',x_2',x_2'), G(x_2',x_3',x_3') \end{array} \right\} \tag{2.7}$$

$$\max\{G(x_1,x_2,x_3), G(x_1',x_2',x_3')\} \leq \mathcal{A} \max\{G(x_0,x_1,x_2), G(x_0',x_1',x_2')\}. \tag{2.8}$$

If $x_1 = x_2 = x_3$ and $x_1' = x_2' = x_3'$ then (x_1,x_1') is a coupled fixed point of T. Let any one of x_1,x_2,x_3 and x_1',x_2',x_3' be not equal to other, from (2.1) and (2.8) there exist $x_4 \in T(x_5,x_5'), x_4' \in T(x_5',x_5)$ such that

$$\begin{aligned}\max\{G(x_2,x_3,x_4),G(x'_2,x'_3,x'_4)\} &\leq \mathcal{A}\max\left\{\begin{array}{l}G(x_1,x_2,x_3),G(x_1,T(x_1,x'_1),T(x_1,x'_1)),\\ G(x_2,T(x_2,x'_2),T(x_2,x'_2)),G(x_3,T(x_3,x'_3),T(x_3,x'_3))\\ G(x'_1,x'_2,x'_3),G(x'_1,T(x'_1,x_1),T(x'_1,x_1)),\\ G(x'_2,T(x'_2,x_2),T(x'_2,x_2)),G(x'_3,T(x'_3,x_3),T(x'_3,x_3))\end{array}\right\}\\ &\leq \mathcal{A}\max\left\{\begin{array}{l}G(x_1,x_2,x_3),G(x_1,x_2,x_2),G(x_2,x_3,x_3),G(x_3,x_4,x_4)\\ G(x'_1,x'_2,x'_3),G(x'_1,x'_2,x'_2),G(x'_2,x'_3,x'_3),G(x'_3,x'_4,x'_4)\end{array}\right\}\\ &\leq \mathcal{A}\max\{G(x_1,x_2,x_3),G(x'_1,x'_2,x'_3)\}\\ \max\{G(x_2,x_3,x_4),G(x'_2,x'_3,x'_4)\} &\leq \mathcal{A}^2\max\{G(x_0,x_1,x_2),G(x'_0,x'_1,x'_2)\}. \qquad (2.9)\end{aligned}$$

Now by induction we construct sequences $\{x_n\}_{n\geq 0}$, $\{x'_n\}_{n\geq 0}$ in X such that $x_{n+1}\in T(x_n,x'_n)$, $x'_{n+1}\in T(x'_n,x_n)$ and

$$\max\{G(x_n,x_{n+1},x_{n+2}),G(x'_n,x'_{n+1},x'_{n+2})\}\leq \mathcal{A}^n\max\{G(x_0,x_1,x_2),G(x'_0,x'_1,x'_2)\} \qquad (2.10)$$

which gives

$$G(x_n,x_{n+1},x_{n+2})\leq \mathcal{A}^nG(x_0,x_1,x_2) \qquad (2.11)$$

and

$$G(x'_n,x'_{n+1},x'_{n+2})\leq \mathcal{A}^nGG(x'_0,x'_1,x'_2) \qquad (2.12)$$

for all $n\geq 0$.

From Theorem 1.9, for all $m,n\in N$, $n<m$ and by (G_3) and (G_5) we obtain

$$\begin{aligned}\max\{G(x_n,x_m,x_m),G(x'_n,x'_m,x'_m)\} &\leq (\mathcal{A}^n+\mathcal{A}^{n+1}+\mathcal{A}^{n+2}+\cdots+\mathcal{A}^{m-1})\max\{G(x_0,x_1,x_2),G(x'_0,x'_1,x'_2)\}\\ &\leq \mathcal{A}^n(I+\mathcal{A}+\mathcal{A}^2+\cdots+\mathcal{A}^{m-1-n})\max\{G(x_0,x_1,x_2),G(x'_0,x'_1,x'_2)\}\\ &\leq \mathcal{A}^n(I-\mathcal{A})^{-1}\max\{G(x_0,x_1,x_2),G(x'_0,x'_1,x'_2)\}\end{aligned}$$

that is, $\max\{G(x_0,x_1,x_2),G(x'_0,x'_1,x'_2)\}\to 0$ as $n\to\infty$.

Hence $\{x_n\}_{n\geq 0},\{x'_n\}_{n\geq 0}$ are Cauchy sequence in the complete $G-$ metric space (X,G). Choose $(x^*,x'^*)\in X\times X$ such that $x_n\to x^*$ and $x'_n\to x'^*$ as $n\to\infty$. We claim that $(x,x')\in(X\times X)\setminus(\{x^*\},\{x'^*\})$,

$$\max\{G(x^*,T(x,x'),T(x,x')),G(x'^*,T(x,x'),T(x,x'))\}\leq\mathcal{A}\max\{G(x^*,x,x),G(x'^*,x',x')\}. \qquad (2.13)$$

Let $(x,x')\in(X\times X)\setminus(\{x^*\},\{x'^*\})$. Choose a natural number N such that

$$\max\{G(x_n,x^*,x^*),G(x'_n,x'^*,x'^*)\}<\frac{1}{4}\max\{G(x,x^*,x^*),G(x',x'^*,x'^*)\}$$

for all $n\geq N$. Hence, for each $n\geq N$ we have

$$\begin{aligned}\max\{G(x_n,T(x_n,x'_n),T(x_n,x'_n)),G(x'_n,T(x_n,x'_n),T(x_n,x'_n))\} &\leq \max\{G(x,x^*,x^*),G(x',x'^*,x'^*)\}\\ &\quad -\max\{G(x_n,x^*,x^*),G(x'_n,x'^*,x'^*)\}\\ \max\{G(x_n,T(x_n,x'_n),T(x_n,x'_n)),G(x'_n,T(x_n,x'_n),T(x_n,x'_n))\} &\leq \max\{G(x_n,x^*,x^*),G(x'_n,x'^*,x'^*)\}.\end{aligned}$$

Thus

$$\begin{aligned} &(I+\mathcal{A})^{-1}\max\{G(x_n,T(x_n,x'_n),T(x_n,x'_n)),G(x'_n,T(x_n,x'_n),T(x_n,x'_n))\} \\ \leq\ &(I-\mathcal{A})^{-1}\max\{G(x_n,T(x_n,x'_n),T(x_n,x'_n)),G(x'_n,T(x_n,x'_n),T(x_n,x'_n))\} \\ \leq\ &(I-\mathcal{A})^{-1}\max\{G(x_n,x,x),G(x'_n,x',x')\} \end{aligned}$$

for $n \geq N$. Since $x_{n+1} \in T(x_n, x'_n)$, $x'_{n+1} \in T(x'_n, x_n)$, by using (2.1) for each $n \geq N$ there exist $u_n \in T(x, x')$ and $u'_n \in T(x', x)$ such that

$$G(u_n,x_{n+1},x_{n+1})+G(u'_n,x'_{n+1},x'_{n+1}) \leq \mathcal{A}[G(x_n,x,x)+G(x'_n,x',x')].$$

Hence

$$\max\{G(x_{n+1},T(x,x'),T(x,x')),G(x'_{n+1},T(x',x),T(x',x))\} \leq \mathcal{A}\max\{G(x_n,x,x),G(x'_n,x',x')\}$$

and so

$$\lim_{n\to\infty}\max\{G(x_{n+1},T(x,x'),T(x,x')),G(x'_{n+1},T(x',x),T(x',x))\} \leq \lim_{n\to\infty}\max\{G(x_n,x,x),G(x'_n,x',x')\}.$$

Thus

$$\max\{G(x^*,T(x,x'),T(x,x')),G(x'^*,T(x',x),T(x',x))\} \leq \mathcal{A}\max\{G(x^*,x,x),G(x'^*,x',x')\}$$

for all $(x, x') \in (X \times X) \setminus (\{x^*\}, \{x'^*\})$.

Now we show that for each $(x, x') \in X \times X$ and $u \in T(x, x')$, $u' \in T(x', x)$ there exist $v \in T(x^*, x'^*)$, $v' \in T(x'^*, x^*)$ such that

$$\max\{G(u,v,v),G(u',v',v')\} \leq \mathcal{A}\max\{G(x,x^*,x^*),G(x',x'^*,x'^*)\}.$$

If $x_n \to x^*$ and $x'_n \to x'^*$ we have nothing to prove. Let $x \neq x^*$ and $x' \neq x'^*$. By definition of $G(x^*, T(x, x'), T(x, x'))$, $G(x'^*, T(x', x), T(x', x))$ and for each $n \geq 1$ there exist $y_n \in T(x, x')$ and $y'_n \in T(x', x)$ such that

$$\begin{aligned} \max\{G(x^*,y_n,y_n),G(x'^*,y'_n,y'_n)\} \leq\ &\max\{G(x^*,T(x,x'),T(x,x')),G(x'^*,T(x',x),T(x',x)\}) \\ &+\frac{1}{n}\max\{G(x,x^*,x^*),G(x',x'^*,x'^*)\}. \end{aligned}$$

Hence from 2.13, we have

$$\begin{aligned} (I+\mathcal{A})^{-1}\max\{G(x,T(x,x'),T(x,x')),G(x',T(x',x),T(x',x))\} \leq\ &\max\{G(x,x^*,x^*),G(x',x'^*,x'^*)\} \\ &+\frac{1}{n}(I+\mathcal{A})^{-1}\max\{G(x',x'^*,x'^*),G(x,x^*, \end{aligned}$$

for all $n \geq 1$. Thus

$$\begin{aligned} (I+\mathcal{A})^{-1}\max\{G(x,T(x,x'),T(x,x')),G(x',T(x',x),T(x',x))\} \leq\ &\max\{G(x,x^*,x^*),G(x',x'^*,x'^*)\} \\ \leq\ &(I-\mathcal{A})^{-1}\max\{G(x,x^*,x^*),G(x',x'^*,x'^*) \end{aligned}$$

Now by using (2.1) for each $u \in T(x, x')$, $u' \in T(x', x)$ there exist $v \in T(x^*, x'^*)$, $v' \in T(x'^*, x^*)$ such that

$$\max\{G(u,v,v),G(u',v',v')\} \leq \mathcal{A}\max\{G(x,x^*,x^*),G(x',x'^*,x'^*)\}.$$

Since $x_{n+1} \in T(x_n, x'_n)$ and $x'_{n+1} \in T(x'_n, x_n)$ for all $n \geq 1$, there exist $v_n \in T(x^*, x'^*)$ and $v'_n \in T(x'^*, x^*)$ such that

$$\max\{G(v, x_n, x_{n+1}), G(v', x'_n, x'_{n+1})\} \leq \mathcal{A} \max\{G(x_n, x^*, x^*), G(x'_n, x'^*, x'^*)\}.$$

Hence

$$\begin{aligned} \max\{G(v_n, x^*, x^*), G(v'_n, x'^*, x'^*)\} &\leq \max\{G(v_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x^*, x^*), G(v'_n, x'_{n+1}, x'_{n+1}) + G(x'_{n+1}, \\ &\leq \mathcal{A} \max\{G(x_n, x^*, x^*) + G(x_{n+1}, x^*, x^*), G(x'_n, x'^*, x'^*) + G(x'_{n+1}, x'^*, x'^* \end{aligned}$$

for all $n \geq 1$. Therefore $v_n \to x^*$ and $v'_n \to x'^*$.

Since $v_n \in T(x^*, x'^*)$ and $v'_n \in T(x'^*, x^*)$ for all $n \geq 1$ and $T(x^*, x'^*)$ is a closed subset of $X \times X$, $x^* \in T(x^*, x'^*)$ and $x'^* \in T(x'^*, x^*)$, that is, $G(x_n, x_m, x_m) \to 0$ as $n \to \infty$.

Hence $\{x_n\}_{n \geq 0}$ is Cauchy sequence in the complete $G-$ metric space (X, G). Choose $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. We claim that $x \in X \setminus \{x^*\}$,

□

Theorem 2.3. *Let (X,G) be a complete $G-$ metric space, a matrix $\mathcal{A} \in M_{m,m}(\mathbb{R}^+)$ converges to zero and $T : X \times X \to P_{cl}(X)$ a multivalued operator and $F : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ an increasing sublinear continuous function such that $F(0) = 0$ and $F(t) > 0$ for all $t = (t_i)_{i=1}^m \in \mathbb{R}^m_+$ where*

$$\mathbb{R}^m_+ = \{(t_1, ..., t_m) : t_i > 0, \quad for \quad i = 1, 2, 3, ..., m\}. \tag{2.14}$$

Suppose that for each $x, y, z, x', y', z' \in X$,
$(I + \mathcal{A}^{-1})F(\max\{G(u, v, w), G(u', v', w')\}) \leq (I - \mathcal{A}^{-1})F(\max\{G(x, y, z), G(x', y', z')\})$
implies that for each $u \in T(x, x'), u' \in T(x', x), v \in T(y, y'), v' \in T(y', y)$ there exist $w \in T(z, z'), w' \in T(z', z)$ such that

$$F(\max\{G(u, v, w), G(u', v', w')\}) \leq \mathcal{A}F\left(\max\left\{\begin{array}{l} G(x, y, z), G(x, T(x, x'), T(x, x')), \\ G(y, T(y, y'), T(y, y')), G(z, T(z, z'), T(z, z')), \\ G(x', y', z'), G(x', T(x', x), T(x', x)), \\ G(y', T(y', y), T(y', y)), G(z', T(z', z), T(z', z)) \end{array}\right\}\right) \tag{2.15}$$

Then T has a coupled fixed point.

Proof. For each $(x, x') \in X \times X$,

$$\begin{aligned} &(I + \mathcal{A}^{-1})F(\max\{G(x, T(x, x'), T^2(x, x')), G(x', T(x', x), T^2(x', x))\}) \leq \\ &\quad (I - \mathcal{A}^{-1})F(\max\{G(x, T(x, x'), T^2(x, x')), G(x', T(x', x), T^2(x', x))\}). \end{aligned} \tag{2.16}$$

Let $(x_0, x'_0) \in X \times X$ and take $x_1 \in T(x_0, x'_0), x'_1 \in T(x'_0, x_0), x_2 \in T(x_1, x'_1), x'_2 \in T(x'_1, x_1)$. If $x_0 = x_1 = x_2$ and $x'_0 = x'_1 = x'_2$ then (x_0, x'_0) is a coupled fixed point of T. Let any one of x_0, x_1, x_2 and x'_0, x'_1, x'_2 be not equal to other. From (2.6), there exist $x_3 \in T(x_2, x'_2), x'_3 \in T(x'_2, x_2)$ such that

$$F(\max\{G(x_1, x_2, x_3), G(x'_1, x'_2, x'_3)\}) \leq \mathcal{A}F\left(\max\left\{\begin{array}{l} G(x_0, x_1, x_2), G(x_0, T(x_0, x'_0), T(x_0, x'_0)), \\ G(x_1, T(x_1, x'_1), T(x_1, x'_1)), G(x_2, T(x_2, x'_2), T(x_2, x'_2)), \\ G(x'_0, x'_1, x'_2), G(x'_0, T(x'_0, x_0), T(x'_0, x_0)), \\ G(x'_1, T(x'_1, x_1), T(x'_1, x_1)), G(x'_2, T(x'_2, x_2), T(x'_2, x_2)) \end{array}\right\}\right)$$

If $x_1 = x_2 = x_3$ and $x'_1 = x'_2 = x'_3$ then (x_1, x'_1) is a coupled fixed point of T. Let any one of x_1, x_2, x_3 and x'_1, x'_2, x'_3 be not equal to other, from (2.1) and (2.8) there exist $x_4 \in T(x_5, x'_5), x'_4 \in T(x'_5, x_5)$ such that

$$\begin{aligned} F(\max\{G(x_2, x_3, x_4), G(x'_2, x'_3, x'_4)\}) &\leq \mathcal{A}F(\max\{G(x_1, x_2, x_3), G(x'_1, x'_2, x'_3)\}) \\ &\leq \mathcal{A}^2F(\max\{G(x_0, x_1, x_2), G(x'_0, x'_1, x'_2)\}). \end{aligned} \tag{2.18}$$

Now by induction we construct sequences $\{x_n\}_{n\geq 0}$, $\{x'_n\}_{n\geq 0}$ in X such that $x_{n+1} \in T(x_n, x'_n)$, $x'_{n+1} \in T(x'_n, x_n)$ and

$$F(\max\{G(x_n, x_{n+1}, x_{n+2}), G(x'_n, x'_{n+1}, x'_{n+2})\}) \leq \mathcal{A}^nF(\max\{G(x_0, x_1, x_2), G(x'_0, x'_1, x'_2)\}) \tag{2.19}$$

for all $n \geq 0$. Since $\mathcal{A}$ converges to zero,

$$F(\max\{G(x_n, x_{n+1}, x_{n+2}), G(x'_n, x'_{n+1}, x'_{n+2})\}) \to 0.$$

We claim that

$$\max\{G(x_n, x_{n+1}, x_{n+2}), G(x'_n, x'_{n+1}, x'_{n+2})\} \to 0.$$

If

$$\max\{G(x_n, x_{n+1}, x_{n+2}), G(x'_n, x'_{n+1}, x'_{n+2})\} \to 0$$

is not true, then there exists $\gamma \in \mathbb{R}^m_+$ such that for each $k > 0$ there is an integer number $n_k \geq k$ such that

$$\max\{G(x_{n_k}, x_{n_k+1}, x_{n_k+2}), G(x'_{n_k}, x'_{n_k+1}, x'_{n_k+2})\} \geq \gamma.$$

Hence,

$$0 < F(\gamma) \leq F(\max\{G(x_n, x_{n+1}, x_{n+2}), G(x'_n, x'_{n+1}, x'_{n+2})\}) \to 0.$$

This contradiction shows that

$$\max\{G(x_n, x_{n+1}, x_{n+2}), G(x'_n, x'_{n+1}, x'_{n+2})\} \to 0.$$

Now, from sublinearity of F Theorem (1.9), for all $m, n \in N$, $n < m$ and by (G_3) and (G_5) we obtain

$$F(\max\{G(x_n, x_m, x_m), G(x'_n, x'_m, x'_m)\}) \leq \mathcal{A}^n(I-\mathcal{A})^{-1}F(\max\{G(x_0, x_1, x_2), G(x'_0, x'_1, x'_2)\})$$

that is

$$F(\max\{G(x_n, x_m, x_m), G(x'_n, x'_m, x'_m)\}) \to 0$$

as $n \to \infty$ and so

$$\max\{G(x_n, x_m, x_m), G(x'_n, x'_m, x'_m)\} \to 0.$$

If $x_n \to x^*$ and $x'_n \to x'^*$ we have nothing to prove. Let $x \neq x^*$ and $x' \neq x'^*$. By definition of $G(x^*, T(x, x'), T(x, x'))$, $G(x'^*, T(x', x), T(x', x))$ and for each $n \geq 1$ there exist $y_n \in T(x, x')$ and $y'_n \in T(x', x)$ such that

$$\begin{aligned}\max\{G(x^*, y_n, y_n), G(x'^*, y'_n, y'_n)\} &\leq \max\{G(x^*, T(x, x'), T(x, x')), G(x'^*, T(x', x), T(x', x)\}) \\ &\quad + \frac{1}{n}\max\{G(x, x^*, x^*), G(x', x'^*, x'^*)\}.\end{aligned}$$

Hence from (2.13), we have

$$\begin{aligned}(I+\mathcal{A})^{-1}\max\{G(x, T(x, x'), T(x, x')), G(x', T(x', x), T(x', x))\} &\leq \max\{G(x, x^*, x^*), G(x', x'^*, x'^*)\} \\ &\quad + \frac{1}{n}(I+\mathcal{A})^{-1}\max\{G(x', x'^*, x'^*), G(x, x^*,\end{aligned}$$

for all $n \geq 1$. Thus

$$\begin{aligned}(I+\mathcal{A})^{-1}\max\{G(x, T(x, x'), T(x, x')), G(x', T(x', x), T(x', x))\} &\leq \max\{G(x, x^*, x^*), G(x', x'^*, x'^*)\} \\ &\leq (I-\mathcal{A})^{-1}\max\{G(x, x^*, x^*), G(x', x'^*, x'^*)\end{aligned}$$

Now by using (2.1) for each $u \in T(x, x')$, $u' \in T(x', x)$ there exist $v \in T(x^*, x'^*)$, $v' \in T(x'^*, x^*)$ such that

$$\max\{G(u, v, v), G(u', v', v')\} \leq \mathcal{A}\max\{G(x, x^*, x^*), G(x', x'^*, x'^*)\}.$$

Since $x_{n+1} \in T(x_n, x'_n)$ and $x'_{n+1} \in T(x'_n, x_n)$ for all $n \geq 1$, there exist $v_n \in T(x^*, x'^*)$ and $v'_n \in T(x'^*, x^*)$ such that

$$\max\{G(v, x_n, x_{n+1}), G(v', x'_n, x'_{n+1})\} \leq \mathcal{A}\max\{G(x_n, x^*, x^*), G(x'_n, x'^*, x'^*)\}.$$

Hence

$$\begin{aligned}\max\{G(v_n,x^*,x^*),G(v'_n,x'^*,x'^*)\} &\leq \max\{G(v_n,x_{n+1},x_{n+1})+G(x_{n+1},x^*,x^*),G(v'_n,x'_{n+1},x'_{n+1})+G(x'_{n+1},\\ &\leq \mathcal{A}\max\{G(x_n,x^*,x^*)+G(x_{n+1},x^*,x^*),G(x'_n,x'^*,x'^*)+G(x'_{n+1},x'^*,x'^*\end{aligned}$$

for all $n \geq 1$. Therefore $v_n \to x^*$ and $v'_n \to x'^*$.

Since $v_n \in T(x^*,x'^*)$ and $v'_n \in T(x'^*,x^*)$ for all $n \geq 1$ and $T(x^*,x'^*)$ is a closed subset of $X \times X$, so $x^* \in T(x^*,x'^*)$ and $x'^* \in T(x'^*,x^*)$. That is $G(x_n,x_m,x_m) \to 0$ as $n \to \infty$.

Hence $\{x_n\}_{n\geq 0}, \{x'_n\}_{n\geq 0}$ are Cauchy sequence in the complete $G-$ metric space (X,G). Choose $(x^*,x'^*) \in X \times X$ such that $x_n \to x^*$ and $x'_n \to x'^*$ as $n \to \infty$. We claim that $(x,x') \in (X \times X) \setminus (\{x^*\},\{x'^*\})$,

$$F(\max\{G(x^*,T(x,x'),T(x,x')),G(x'^*,T(x,x'),T(x,x'))\}) \leq \mathcal{A}F(\max\{G(x^*,x,x),G(x'^*,x',x')\}) \tag{2.20}$$

Let $(x,x') \in (X \times X) \setminus (\{x^*\},\{x'^*\})$. Choose a natural number N such that

$$\max\{G(x_n,x^*,x^*),G(x'_n,x'^*,x'^*)\} < \frac{1}{4}\max\{G(x,x^*,x^*),G(x',x'^*,x'^*)\}$$

for all $n \geq N$. Hence, for each $n \geq N$ we have

$$F(\max\{G(x_n,T(x_n,x'_n),T(x_n,x'_n)),G(x'_n,T(x_n,x'_n),T(x_n,x'_n))\}) \leq F(\max\{G(x_n,x^*,x^*),G(x'_n,x'^*,x'^*)\}).$$

Thus

$$\begin{aligned}&(I+\mathcal{A})^{-1}F\left(\max\{G(x_n,T(x_n,x'_n),T(x_n,x'_n)),G(x'_n,T(x_n,x'_n),T(x_n,x'_n))\}\right)\\ \leq\ &(I-\mathcal{A})^{-1}F\left(\max\{G(x_n,T(x_n,x'_n),T(x_n,x'_n)),G(x'_n,T(x_n,x'_n),T(x_n,x'_n))\}\right)\\ \leq\ &(I-\mathcal{A})^{-1}F\left(\max\{G(x_n,x,x),G(x'_n,x',x')\}\right)\end{aligned}$$

for $n \geq N$. Since $x_{n+1} \in T(x_n,x'_n)$, $x'_{n+1} \in T(x'_n,x_n)$, by using (2.1) for each $n \geq N$ there exist $u_n \in T(x,x')$ and $u'_n \in T(x',x)$ such that

$$F(\max\{G(u_n,x_{n+1},x_{n+1}),G(u'_n,x'_{n+1},x'_{n+1})\}) \leq \mathcal{A}F(\max\{G(x_n,x,x),G(x'_n,x',x')\}).$$

Hence

$$F(\max\{G(x_{n+1},T(x,x'),T(x,x')),G(x'_{n+1},T(x',x),T(x',x))\}) \leq \mathcal{A}F(\max\{G(x_n,x,x),G(x'_n,x',x')\})$$

and so

$$\lim_{n\to\infty} F(\max\{G(x_{n+1},T(x,x'),T(x,x')),G(x'_{n+1},T(x',x),T(x',x))\}) \leq \lim_{n\to\infty} F(\max\{G(x_n,x,x),G(x'_n,x',x')\}).$$

Thus $F(\max\{G(x^*,T(x,x'),T(x,x')),G(x'^*,T(x',x),T(x',x))\}) \leq \mathcal{A}F(G(\max\{G(x^*,x,x),G(x'^*,x',x')\})$ for all $(x,x') \in (X \times X) \setminus (\{x^*\},\{x'^*\})$.

Now we show that for each $(x,x') \in X \times X$ and $u \in T(x,x')$, $u' \in T(x',x)$ there exist $v \in T(x^*,x'^*)$, $v' \in T(x'^*,x^*)$ such that

$$F(\max\{G(u,v,v),G(u',v',v')\}) \leq \mathcal{A}F(\max\{G(x,x^*,x^*),G(x',x'^*,x'^*)\}).$$

From (2.20),

$$\begin{aligned}
&(I+\mathcal{A})^{-1}F(\max\{G(x,T(x,x'),T(x,x')),G(x',T(x',x),T(x',x))\}) \\
\leq\ & F(\max\{G(x,T(x,x'),T(x,x')),G(x',T(x',x),T(x',x))\}) \\
&+\frac{1}{n}(I+\mathcal{A})^{-1}F(\max\{G(x,T(x,x'),T(x,x')),G(x',T(x',x),T(x',x))\})
\end{aligned}$$

for all $n \geq 1$.

Thus

$$\begin{aligned}
&(I+\mathcal{A})^{-1}F(\max\{G(x,T(x,x'),T(x,x')),G(x',T(x',x),T(x',x))\}) \\
\leq\ & F(\max\{G(x,x^*,x^*),G(x',x'^*,x'^*)\}) \\
\leq\ & (I-\mathcal{A})^{-1}F(\max\{G(x,x^*,x^*),G(x',x'^*,x'^*)\}).
\end{aligned}$$

Now by using (2.15), for each $u \in T(x,x')$, $u' \in T(x',x)$ there exist $v \in T(x^*,x'^*)$, $v' \in T(x'^*,x^*)$ such that

$$F(\max\{G(u,v,v),G(u',v',v')\}) \leq \mathcal{A}F(\max\{G(x,x^*,x^*),G(x',x'^*,x'^*)\}).$$

Since $x_{n+1} \in T(x_n,x'_n)$, $x'_{n+1} \in T(x'_n,x_n)$ for all $n \geq 1$, there exist $v_n \in T(x^*,x'^*)$, $v'_n \in T(x'^*,x^*)$ such that

$$F(\max\{G(v,x_n,x_{n+1}),G(v',x'_n,x'_{n+1})\}) \leq \mathcal{A}F(\max\{G(x,x^*,x^*),G(x',x'^*,x'^*)\}).$$

Hence

$$\begin{aligned}
F(\max\{G(v_n,x^*,x^*),G(v'_n,x'^*,x'^*)\}) &\leq F(\max\{G(v_n,x_{n+1},x_{n+1}),G(v'_n,x'_{n+1},x'_{n+1})\}) \\
&\quad +F(\max\{G(x_{n+1},x^*,x^*),G(x'_{n+1},x'^*,x'^*)\}) \\
&\leq \mathcal{A}F(\max\{G(x_n,x^*,x^*),G(x'_n,x'^*,x'^*)\}) \\
&\quad +F(\max\{G(x_{n+1},x^*,x^*),G(x'_{n+1},x'^*,x'^*)\})
\end{aligned}$$

for all $n \geq 1$. Therefore $v_n \to x^*$ and $v'_n \to x'^*$.

Since $v \in T(x^*,x'^*)$, $v' \in T(x'^*,x^*)$ for all $n \geq 1$ and $T(x^*,x'^*)$ is a closed subset of $X \times X$, $x^* \in T(x^*,x'^*)$ and $x'^* \in T(x'^*,x^*)$.

□

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