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Exact solutions of singular IVPs Lane-Emden equation

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Abstract

In the paper [A.M. Rismani, H. Monfared, Numerical solution of singular ivps of laneemden type using a modified Legendre spectral method. Applied Mathematical Modelling, 36 (2012), 4830-4836.] the authors state that exact solutions for the Lane-Emden nonlinear differential equation exists only for $m = 0, 1$ (linear cases) and 5 (nonlinear case). While here, we present real exact solutions for $m \in (3, \infty)$ and complex $m \in (-\infty, 3)\backslash\{1\}$. Some illustrated examples presented as well.

Keywords: Lane-Emden type equations, Singular IVPs.
2010 MSC: 34B30, 34L30.

1. Introduction and Preliminaries

Many problems of mathematical physics are as equations of the Lane-Emden type defined in the form

$$y'' + \frac{2}{t}y' + g(t)f(y) = h(t), \quad 0 < t < \infty. \tag{1.1}$$

subject to $y(0) = a$ and $y'(0) = 0$, where a is a constant and $f(y)$, $g(t)$ and $h(t)$ are given functions. For more detail of other extensions of analytic algorithm of Lane-Emden and Solutions of singular IVPs of Lane-Emden type by homotopy perturbation method refer to [2, 3, 4].

In this paper, we present real exact solutions for $m \in (3, \infty)$ and complex $m \in (-\infty, 3)\backslash\{1\}$, while authors in [1] had stated that exact solutions for the Lane-Emden nonlinear differential equation exist only for $m = 0, 1$ (linear cases) and 5 (nonlinear case).

In the sequel, we use the notation NDE for nonlinear differential equation.

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2. Main results

In this section, we obtain exact solutions for the NDE (1.1) for $m \in \mathbb{R}\backslash\{1,3\}$, when $g(t) = 1$, $h(t) = 0$ and $f(y) = y^m$; which is called Lane-Emden equation.

Theorem 2.1. *The $y = at^{\frac{2}{1-m}}$ is exact solution of the Lane-Emden equation*

$$y'' + \frac{2}{t}y' + y^m = 0, \tag{2.1}$$

where real exact solutions for $m \in (3,\infty)$, complex for $m \in (-\infty,3)\backslash\{1\}$; and $a^{m-1} = \frac{2m-6}{(1-m)^2}$.

Proof. It's easy to calculate if we pick $y := at^k$ for $a \neq 0, k \neq 0$; nontrivial and $k \neq -1$; so $y' = kat^{k-1}$ and $y'' = k(k-1)at^{k-2}$. Now replace them in the equation

$$\begin{aligned} y'' + \frac{2}{t}y' + y^m &= 0 \\ k(k-1)at^{k-2} + 2kat^{k-2} + a^m t^{mk} &= 0 \\ k(k-1) + 2k + a^{m-1}t^{mk-k+2} &= 0 \\ k(k+1) + a^{m-1}t^{mk-k+2} &= 0. \end{aligned}$$

Now if the above identities should be held, then we have to

$$\begin{cases} k(k+1) + a^{m-1} = 0, \\ mk - k + 2 = 0, \end{cases} = \begin{cases} a^{m-1} = \frac{2m-6}{(1-m)^2}, & \text{if we put } k \text{ in the following;} \\ k = \frac{2}{1-m}. \end{cases}$$

□

Example 2.2. Consider the nonlinear differential equations:

1. $y'' + \frac{2}{t}y' + y^{\frac{10}{3}} = 0;$
2. $y'' + \frac{2}{t}y' + y^2 = 0;$
3. $y'' + \frac{2}{t}y' + y^{-\sqrt{5}} = 0, \quad 0 < t < \infty;$

particular answers are respectively

1. $y = at^{\frac{-6}{7}}$, where $a^{\frac{7}{3}} = \frac{6}{49}$;
2. $y = -2t^{-2}$;
3. $y = at^{\frac{\sqrt{5}-1}{2}}$, where $a^{-\sqrt{5}-1} = -1$. In other word $t^{\frac{1}{\varphi}}$ is a answer, where φ is golden ratio; i.e., $\varphi^2 - \varphi - 1 = 0$ and $a^{\varphi} = -i$.

More general above Lane-Emden's equation is:

Theorem 2.3. *The $y = a(b+t)^{\frac{2}{1-m}}$ is exact solution of the*

$$y'' + \frac{2}{t+b}y' + y^m = 0, \tag{2.2}$$

where real exact solutions for $m \in (3,\infty)$, complex for $m \in (-\infty,3)\backslash\{1\}$; and $a^{m-1} = \frac{2m-6}{(1-m)^2}$, and b a constant.

The equation (2.1) is linear for $m = 0$ and 1, and nonlinear otherwise. Exact solutions exist only for $m = 0$, 1 and 5 that given in Bender [5] are respectively

$$y = 1 - \frac{t^2}{6}, \quad y = \frac{\sin t}{t} \quad \text{and} \quad y = \left(1 + \frac{t^2}{3}\right)^{\frac{-1}{2}}.$$

Theorem 2.4. *If y be a particular solution of (1.1). Then $u = y + \frac{1}{t}$ is a solution for*

$$u'' + \frac{2}{t}u' + g(t)f\left(u - \frac{1}{t}\right) = h(t), \quad 0 < t < \infty. \tag{2.3}$$

Example 2.5. Consider the nonlinear differential equation

$$y'' + \frac{2}{t}y' + y^4 = 0. \tag{2.4}$$

A particular answer shall be $y = \sqrt[3]{\frac{2}{9t^2}}$. The $y = \sqrt[3]{\frac{2}{9t^2}} + \frac{k}{t}$, is a solution for the following NDE:

$$y'' + \frac{2}{t}y' + \left(y - \frac{k}{t}\right)^4 = 0,$$

for some constant k, when we apply several times Theorem 2.4.

Example 2.6. In the [1], there exists another particular solution as follows $y = \left(1 + \frac{t^2}{3}\right)^{\frac{-1}{2}}$, for the following NDE

$$y'' + \frac{2}{t}y' + y^5 = 0, \quad 0 < t < \infty. \tag{2.5}$$

subject to $y(0) = 1; \quad y'(0) = 0$.

Here we state certain another answers to the NDE with $m = 5$.

Example 2.7. By the Theorem 2.1, we obtain the solutions for (2.5)

$$y = at^{\frac{-1}{2}}, \quad a^4 = \frac{1}{4}.$$

In other word $y = \frac{1}{\sqrt{2t}}$, subject to $y(0^+) = +\infty$, $y'(0^+) = -\infty$.
Another answers are: $y = a(1 \pm t^2)^{\frac{-1}{2}} = \frac{a}{\sqrt{1 \pm t^2}}$, with $a^4 = 3$, where $y(0) = \sqrt[4]{3}$ and $y'(0) = 0$.

Example 2.8. When $g(t) = 1$ and $h(t) = -\frac{(1+t^2)}{2}$. We have a particular solution $y = \left(1 + \frac{t^2}{2}\right)^{\frac{-1}{2}}$ for the following NDE:

$$y'' + \frac{2}{t}y' + y^3 = -\frac{1+t^2}{2}, \quad 0 < t < \infty, \tag{2.6}$$

subject to $y(0) = 1; \quad y'(0) = 0$.

3. Numerical solutions by spectral method with Legendre points

The following answer for the equation (1.1) is as follows:

$$y(t) = a + \int_0^t \left(\frac{s^2}{t} - s\right)(sg(s)f(y(s)) - h(s))ds. \tag{3.1}$$

Euler's equation, $ty'' + 2y' = 0$ implies $y_h = c_1 + c_2/t$; and by initial conditions $y_h = a$. Wronskian will be $w\left(1, \frac{1}{t}\right) = -\frac{1}{t^2}$. Therefore

$$\begin{aligned} y &= a + \int_0^t \left[\left(-s^2 g(s)f(y(s)) + sh(s)\right) + \left(\frac{s^3}{t}g(s)f(y(s)) + \frac{s^2}{t}h(s)\right)\right] ds. \\ y(t) &= a + \int_0^t \left(\frac{s^2}{t} - s\right)(sg(s)f(y(s)) - h(s))ds. \end{aligned}$$

So the Numerical solutions by spectral method with Legendre points are the same way which stated in [1]. Also the relations (2.5), (2.7) and (2.8) on the paper should be corrected and recalculated in Matlab software package to the right form as equation (3.1). The rest of calculating is according to algorithm and method in [1].

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