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# A Note on the Solutions of a Sturm-Liouville Differential Inclusion with "Maxima"

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**Abstract**

We consider a boundary value problem associated to a Sturm-Liouville differential inclusion with "maxima" and we prove a Filippov type existence result for this problem.

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## 1. Introduction

Differential equations with maximum have proved to be strong tools in the modelling of many physical problems: systems with automatic regulation, problems in control theory that correspond to the maximal deviation of the regulated quantity etc.. As a consequence there was an intensive development of the theory of differential equations with "maxima" [2, 3, 6, 7, 9–13] etc..

A classical example is the one of an electric generator ([2]). In this case the mechanism becomes active when the maximum voltage variation is reached in an interval of time. The equation describing the action of the regulator has the form

$$x'(t) = ax(t) + b \max_{s\in[t-h,t]} x(s) + f(t),$$

where $a, b$ are constants given by the system, $x(.)$ is the voltage and $f(.)$ is a perturbation given by the change of voltage.

In the theory of ordinary differential equations it is wellknown that any linear real second-order differential equation may be written in the self adjoint form $-(r(t)x')' + q(t)x = 0$. This equation together with boundary conditions of the form $a_1x(0) - a_2x'(0) = 0$, $b_1x(T) - b_2x'(T) = 0$ is called the Sturm–Liouville problem. This is the reason why differential inclusions of the form $(r(t)x')' \in F(t, x)$ are usually called Sturm-Liouville type differential inclusions, even if the boundary value problems associated are not as at the

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original Sturm–Liouville problem. Recent results on Sturm–Liouville differential inclusions may be found in [8].

In this paper we study the following problem

$$(p(t)x'(t))' \in F(t, x(t), \max_{s\in[a,t]} x(s), \max_{s\in[t,b]} x(s)) \; a.e. \; ([a, b]), \; x(a) = \alpha, x(b) = \beta, \tag{1.1}$$

where $p(.) : [a, b] \to \mathbf{R}$ is a continuous mapping, $\alpha, \beta \in \mathbf{R}$ and $F : [a, b] \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map.

Several existing results for problem (1.1) obtained using fixed point approaches may be found in our previous paper [4]. The aim of the present note is to show that Filippov's ideas ([5]) can be suitably adapted in order to obtain the existence of solutions of problem (1.1). We recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values Filippov's theorem ([5]) consists in proving the existence of a solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion. At the same time, it is known that concerning the existence of solutions for initial value problems or boundary value problems associated to differential inclusions, Filippov's type approach provides better results than the fixed point approach using Covitz-Nadler set-valued contraction principle.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main result.

## 2. Preliminaries

In this section we sum up some basic facts that we are going to use later.

Let $(X, d)$ be a metric space. The Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by $d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}$, $d^*(A, B) = \sup\{d(a, B); \, a \in A\}$, where $d(x, B) = \inf\{d(x, y); y \in B\}$. Let $I := [a, b]$ and denote by $\mathcal{L}(I)$ the $\sigma$-algebra of all Lebesgue measurable subsets of $I$. Denote by $\mathcal{P}(\mathbf{R})$ the family of all nonempty subsets of $\mathbf{R}$ and by $\mathcal{B}(\mathbf{R})$ the family of all Borel subsets of $\mathbf{R}$. For any subset $A \subset \mathbf{R}$ we denote by $\mathrm{cl}A$ the closure of $A$ and by $\overline{\mathrm{co}}(A)$ the closed convex hull of $A$.

As usual, we denote by $C(I, \mathbf{R})$ the Banach space of all continuous functions $x(.) : I \to \mathbf{R}$ endowed with the norm $|x|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I, \mathbf{R})$ the Banach space of all integrable functions $x(.) : I \to \mathbf{R}$ endowed with the norm $|x|_1 = \int_a^b |x(t)|\mathrm{d}t$. The Banach space of all absolutely continuous functions $x(.) : I \to \mathbf{R}$ will be denoted by $AC(I, \mathbf{R})$.

We recall, first, a selection result (e.g., [1]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem.

**Lemma 2.1.** *Consider $X$ a separable Banach space, $B$ is the closed unit ball in $X$, $H : I \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g : I \to X, L : I \to \mathbf{R}_+$ are measurable functions. If*

$$H(t) \cap (g(t) + L(t)B) \neq \emptyset \quad a.e.(I),$$

*then the set-valued map $t \to H(t) \cap (g(t) + L(t)B)$ has a measurable selection.*

Let $I(.) : \mathbf{R} \to \mathcal{P}(\mathbf{R})$ a set-valued map with compact convex values defined by $I(t) = [a(t), b(t)]$, where $a(.), b(.) : \mathbf{R} \to \mathbf{R}$ are continuous functions with $a(t) \le b(t)$ $\forall t \in \mathbf{R}$. For $x(.) : \mathbf{R} \to \mathbf{R}$ continuous we define $(\max_I)(t) = \max_{s\in I(t)} x(s)$. Therefore, $\max_I : C(\mathbf{R}, \mathbf{R}) \to C(\mathbf{R}, \mathbf{R})$ is an operator whose properties are summarized in the next lemma proved in [11].

**Lemma 2.2.** *If $x(.), y(.) \in C(\mathbf{R}, \mathbf{R})$, then one has*

*i)* $|\max_{s\in I(t)} x(s) - \max_{s\in I(t)} y(s)| \le \max_{s\in I(t)} |x(s) - y(s)|$ $\forall t \in \mathbf{R}$.

*ii)* $\max_{t\in K} |\max_{s\in I(t)} x(s) - \max_{s\in I(t)} y(s)| \le \max_{s\in \cup_{t\in K} I(t)} |x(s) - y(s)|$ $\forall t \in \mathbf{R}$.

*Remark* 2.3. We recall that if $f(.) \in L^1([a,b],\mathbf{R})$ then the solution $x(.) \in C^2([a,b],\mathbf{R})$ of problem $(p(t)x'(t))' = f(t), \quad t \in [a,b]$ with boundary conditions $x(a) = \alpha, x(b) = \beta$ is given by

$$x(t) = Q(t) - \int_a^b G(t,s)f(s)ds, \quad t \in [a,b],$$

where $S(t,\sigma) := \int_\sigma^t \frac{1}{p(s)}ds$, $t,\sigma \in [a,b]$, $Q(t) = \frac{S(t,a)}{S(b,a)}(\beta - \alpha)$, $G(t,\sigma) = \frac{S(t,a)S(b,\sigma) - S(t,\sigma)S(b,a)\chi_{[0,t]}(\sigma)}{S(b,a)}$ and $\chi_U(.)$ is the characteristic function of the set $U$.

In what follows we assume that $p(.) : [a,b] \to (0,\infty)$ is a continuous function such that $|S(t,\sigma)| \le m_0$ $\forall t,\sigma \in [a,b]$. Denote $m_1 := \sup_{t\in[a,b]} |Q(t)|$ and $M_1 := \sup_{t,\sigma\in[a,b]} |G(t,\sigma)|$.

Denote also $Q_1(t) = \frac{S(t,a)}{S(b,a)}(\beta_1 - \alpha_1)$, $\alpha_1, \beta_1 \in \mathbf{R}$. Obviously, $|Q(t) - Q_1(t)| \le \frac{m_0}{|S(b,a)|}(|\beta - \beta_1| + |\alpha - \alpha_1|)$ $\forall t \in [a,b]$.

## 3. The main results

In order to obtain our existence result for problem (1.1) we introduce the following hypothesis on $F$.

**Hypothesis 3.1.** i) $F : I \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and for every $x,y,z \in \mathbf{R}$, $F(.,x,y,z)$ is measurable.

ii) There exists $l_1, l_2, l_3 \in L^1(I,\mathbf{R}_+)$ such that for almost all $t \in I$,

$$d_H(F(t,x_1,y_1,z_1), F(t,x_2,y_2,z_2)) \le l_1(t)|x_1 - x_2| + l_2(t)|y_1 - y_2| + l_3(t)|z_1 - z_2|$$

$\forall\ x_1,x_2,y_1,y_2,z_1,z_2 \in \mathbf{R}$.

**Theorem 3.2.** *Assume that Hypothesis 3.1 is satisfied and $M_1(|l_1|_1 + |l_2|_1 + |l_3|_1) < 1$. Let $y(.) \in C(I,\mathbf{R})$ with $y(a) = \alpha_1, y(b) = \beta_1$ be such that there exists $q(.) \in L^1(I,\mathbf{R}_+)$ verifying $d((p(t)y'(t))', F(t,y(t), \max_{s\in[a,t]} y(s), \max_{s\in[t,b]} y(s))) \le q(t)$ a.e. $(I)$.*

*Then there exists $x(.)$ a solution of problem (1.1) satisfying*

$$|x - y|_C \le \frac{1}{1 - M_1(|l_1|_1 + |l_2|_1 + |l_3|_1)}\left[\frac{m_0}{|S(b,a)|}(|\beta - \beta_1| + |\alpha - \alpha_1|) + |q|_1\right]. \tag{3.1}$$

*Proof.* We set $x_0(.) = y(.)$, $f_0(.) = (p(.)y'(.))'$.

The set-valued map $t \to F(t, x_0(t), \max_{s\in[a,t]} x_0(s), \max_{s\in[t,b]} x_0(s))$ is measurable with closed values and

$$F(t, x_0(t), \max_{s\in[a,t]} x_0(s), \max_{s\in[t,b]} x_0(s)) \cap \{(p(t)x_0'(t))' + q(t)[-1,1]\} \ne \emptyset \quad a.e.\ (I).$$

It follows from Lemma 2.1 that there exists a measurable function $f_1(.)$ such that $f_1(t) \in F(t, x_0(t), \max_{s\in[a,t]} x_0(s), \max_{s\in[t,b]} x_0(s))$ a.e. $(I)$ and, for almost all $t \in I$, $|f_1(t) - (p(t)y'(t))'| \le q(t)$. Define $x_1(t) = Q(t) - \int_a^b G(t,s)f_1(s)ds$, $t \in I$ and for all $t \in I$ one has

$$|x_1(t) - y(t)| \le \frac{m_0}{|S(b,a)|}(|\beta - \beta_1| + |\alpha - \alpha_1|) + M_1|q|_1.$$

Thus $|x_1 - y|_C \le \frac{m_0}{|S(b,a)|}(|\beta - \beta_1| + |\alpha - \alpha_1|) + M_1|q|_1$.

The set-valued map $t \to F(t, x_0(t), \max_{s\in[a,t]} x_0(s), \max_{s\in[t,b]} x_0(s))$ is measurable. Moreover, the map $t \to l_1(t)|x_1(t) - x_0(t)| + l_2(t)|\max_{s\in[a,t]} x_1(s) - \max_{s\in[a,t]} x_0(s)| + l_3(t)|\max_{s\in[t,b]} x_1(s) - \max_{s\in[t,b]} x_0(s)|$ is measurable. By the lipschitzianity of $F(t,.,.,.)$ we have that for almost all $t \in I$

$$d(f_1(t), F(t, x_1(t), \max_{s\in[a,t]} x_1(s), \max_{s\in[t,b]} x_1(s))) \le d_H(F(t, x_0(t), \max_{s\in[a,t]} x_0(s), \max_{s\in[t,b]} x_0(s)), F(t, x_1(t), \max_{s\in[a,t]} x_1(s),$$

$$\max_{s\in[t,b]} x_1(s))) \leq l_1(t)|x_1(t)-x_0(t)| + l_2(t)|\max_{s\in[a,t]} x_0(s) - \max_{s\in[a,t]} x_1(s)| + l_3(t)|\max_{s\in[t,b]} x_0(s) - \max_{s\in[t,b]} x_1(s)|.$$

Therefore,

$$F(t, x_1(t), \max_{s\in[a,t]} x_1(s), \max_{s\in[t,b]} x_1(s)) \cap \{f_1(t) + (l_1(t)|x_1(t)-x_0(t)|+ \\ l_2(t)|\max_{s\in[0,t]} x_1(s) - \max_{s\in[0,t]} x_0(s)| + l_3(t)|\max_{s\in[t,b]} x_0(s) - \max_{s\in[t,b]} x_1(s)|)[-1,1]\} \neq \emptyset.$$

From Lemma 2.1 we deduce the existence of a measurable function $f_2(.)$ such that $f_2(t) \in F(t, x_1(t), \max_{s\in[a,t]} x_1(s), \max_{s\in[t,b]} x_1(s))$ a.e. $(I)$ and for almost all $t \in I$

$$|f_1(t) - f_2(t)| \leq d(f_1(t), F(t, x_1(t), \max_{s\in[a,t]} x_1(s), \max_{s\in[t,b]} x_1(s))) \leq d_H(F(t, x_0(t), \max_{s\in[a,t]} x_0(s), \\ \max_{s\in[t,b]} x_0(s)), F(t, x_1(t), \max_{s\in[a,t]} x_1(s), \max_{s\in[t,b]} x_1(s)) \leq l_1(t)|x_1(t)-x_0(t)| + l_2(t)|\max_{s\in[a,t]} x_0(s) \\ - \max_{s\in[a,t]} x_1(s)| + l_3(t)|\max_{s\in[t,b]} x_0(s) - \max_{s\in[t,b]} x_1(s)|.$$

Define $x_2(t) = Q(t) - \int_a^b G(t,s) f_2(s) ds$, $t \in I$ and one has

$$|x_1(t) - x_2(t)| \leq \int_a^b |f_1(s) - f_2(s)| ds \leq M_1 \int_a^b [l_1(s) + l_2(s) + l_3(s)] |x_1 - x_0|_C ds \\ \leq M_1(|l_1|_1 + |l_2|_1 + |l_3|_1)[\tfrac{m_0}{|S(b,a)|}(|\beta - \beta_1| + |\alpha - \alpha_1|) + |q|_1].$$

Assume that for some $n \geq 1$ we have constructed $(x_i(.))_{i=1}^n$ with $x_n$ satisfying

$$|x_n - x_{n-1}|_C \leq [M_1(|l_1|_1 + |l_2|_1 + |l_3|_1)]^{n-1}[\frac{m_0}{|S(b,a)|}(|\beta - \beta_1| + |\alpha - \alpha_1|) + |q|_1].$$

The set-valued map $t \to F(t, x_n(t), \max_{s\in[a,t]} x_n(s), \max_{s\in[t,b]} x_n(s))$ is measurable. At the same time, the map $t \to l_1(t)|x_n(t) - x_{n-1}(t)| + l_2(t) \cdot |\max_{s\in[a,t]} x_n(s) - \max_{s\in[a,t]} x_{n-1}(s)| + l_3(t)|\max_{s\in[t,b]} x_n(s) - \max_{s\in[t,b]} x_{n-1}(s)|$ is measurable. As before, by the lipschitzianity of $F(t,.,.,.)$ we have that for almost all $t \in I$

$$F(t, x_n(t), \max_{s\in[a,t]} x_n(s), \max_{s\in[t,b]} x_n(s)) \cap \{f_n(t) + (l_1(t)|x_n(t) - x_{n-1}(t)|+ \\ l_2(t)|\max_{s\in[0,t]} x_n(s) - \max_{s\in[0,t]} x_{n-1}(s)| + l_3(t)|\max_{s\in[t,b]} x_n(s) - \max_{s\in[t,b]} x_{n-1}(s)|)[-1,1]\} \neq \emptyset.$$

Using again Lemma 2.1 we deduce the existence of a measurable function $f_{n+1}(.)$ such that $f_{n+1}(t) \in F(t, x_n(t), \max_{s\in[a,t]} x_n(s), \max_{s\in[t,b]} x_n(s))$ a.e. $(I)$ and for almost all $t \in I$

$$|f_{n+1}(t) - f_n(t)| \leq d(f_{n+1}(t), F(t, x_{n+1}(t), \max_{s\in[a,t]} x_{n+1}(s), \max_{s\in[t,b]} x_{n+1}(s))) \leq d_H(F(t, x_n(t), \\ \max_{s\in[a,t]} x_n(s), \max_{s\in[t,b]} x_n(s)), F(t, x_{n+1}(t), \max_{s\in[a,t]} x_{n+1}(s), \max_{s\in[t,b]} x_{n+1}(s)) \leq l_1(t)|x_{n+1}(t) - x_n(t)| \\ + l_2(t)|\max_{s\in[a,t]} x_{n+1}(s) - \max_{s\in[a,t]} x_n(s)| + l_3(t)|\max_{s\in[t,b]} x_{n+1}(s) - \max_{s\in[t,b]} x_n(s)|.$$

Define

$$x_{n+1}(t) = Q(t) - \int_a^b G(t,s) f_{n+1}(s) ds, \quad t \in I \tag{3.2}$$

and one has

$$|x_{n+1}(t) - x_n(t)| \leq \int_a^b |f_{n+1}(s) - f_n(s)| ds \leq M_1 \int_a^b [l_1(s) + l_2(s) + l_3(s)] |x_n - x_{n-1}|_C ds \leq \\ M_1(|l_1|_1 + |l_2|_1 + |l_3|_1)[\tfrac{m_0}{|S(b,a)|}(|\beta - \beta_1| + |\alpha - \alpha_1|) + |q|_1].$$

Therefore $(x_n(.))_{n\geq 0}$ is a Cauchy sequence in the Banach space $C(I, \mathbf{R})$, so it converges to $x(.) \in C(I, \mathbf{R})$. Since, for almost all $t \in I$, we have

$$|f_{n+1}(t) - f_n(t)| \leq l_1(t)|x_n(t) - x_{n-1}(t)| + l_2(t)|\max_{s\in[a,t]} x_n(t) - \max_{s\in[a,t]} x_{n-1}(t)|+ \\ l_3(t)|\max_{s\in[t,b]} x_n(t) - \max_{s\in[t,b]} x_{n-1}(t)| \leq [l_1(t) + l_2(t) + l_3(t)]|x_n - x_{n-1}|_C,$$

$\{f_n(.)\}$ is a Cauchy sequence in the Banach space $L^1(I,\mathbf{R})$ and thus it converges to $f(.) \in L^1(I,\mathbf{R})$.

We note that one may write

$$|\int_a^b G(t,s)f_n(s)ds - \int_a^b G(t,s)f(s)ds| \le M_1 \int_a^b |f_n(s)-f(s)|ds \le$$
$$M_1 \int_a^b [l_1(s)+l_2(s)+l_3(s)]|x_{n+1}-x|_C ds \le (|l_1|_1+|l_2|_1+|l_3|_1).|x_{n+1}-x|_C.$$

Therefore, one may pass to the limit in (3.2) and we get $x(t) = Q(t) - \int_a^b G(t,s)f(s)ds$. Moreover, since the values of $F(.,.,.,.)$ are closed and $f_{n+1}(t) \in F(t, x_n(t), \max_{s\in[a,t]} x_n(s), \max_{s\in[t,b]} x_n(s))$ a.e. $(I)$ passing to the limit we obtain $f(t) \in F(t, x(t), \max_{s\in[a,t]} x(s), \max_{s\in[t,b]} x(s))$ a.e. $(I)$.

It remains to prove the estimate (3.1). One has

$$|x_n - x_0|_C \le |x_n - x_{n-1}|_C + ... + |x_2 - x_1|_C + |x_1 - x_0|_C \le [M_1(|l_1|_1+|l_2|_1+|l_3|_1)]^{n-1}[\frac{m_0}{|S(b,a)|}(|\beta-\beta_1|+ |\alpha-\alpha_1|)+|q|_1] + ...M_1(|l_1|_1+|l_2|_1+|l_3|_1)[\frac{m_0}{|S(b,a)|}(|\beta-\beta_1|+|\alpha-\alpha_1|)+|q|_1] + [\frac{m_0}{|S(b,a)|}(|\beta-\beta_1|+|\alpha- \alpha_1|)+|q|_1] \le \frac{1}{1-(|l_1|_1+|l_2|_1+|l_3|_1)}[\frac{m_0}{|S(b,a)|}(|\beta-\beta_1|+|\alpha-\alpha_1|)+|q|_1].$$

Passage to the limit in the last inequality completes the proof. □

*Remark* 3.3. A similar existence result for problem (1.1) as in Theorem 3.2 may be found in [4], namely Theorem 3.3. The approach in [4] which uses fixed point techniques, apart from the requirement that the values of $F(.,.)$ are compact, does not provides a priori bounds for solutions as in (3.1).

## References

[1] J.P. Aubin, H. Frankowska, *Set-valued Analysis*, Birkhauser, Basel, (1990). 2

[2] D.D. Bainov, S. Hristova, *Differential equations with maxima*, Chapman and Hall/CRC, Boca Raton, (2011). 1

[3] A. Cernea, *Existence of solutions for a class of functional differential inclusions with "maxima"*, Fixed Point Theory, **19** (2018), 503–514. 1

[4] A. Cernea, *On a Sturm-Liouville type functional differential inclusion with "maxima"*, Adv. Dyn. Syst. Appl., **13** (2018), 101–112. 1, 3.3

[5] A.F. Filippov, *Classical solutions of differential equations with multivalued right hand side*, SIAM J. Control, **5** (1967), 609–621. 1

[6] L. Georgiev, V.G. Angelov, *On the existence and uniqueness of solutions for maximum equations*, Glasnik Mat., **37** (2002), 275–281. 1

[7] P. Gonzalez, M. Pinto, *Convergent solutions of certain nonlinear differential equations with maxima*, Math. Comput. Modelling, **45** (2007), 1–10. 1

[8] E.N. Mahmudov, *Optimization of Mayer problem with Sturm-Liouville type differential inclusion*, J. Optim. Theory Appl., **177** (2018), 345–375. 1

[9] M. Malgorzata, G. Zhang, *On unstable neutral difference equations with "maxima"*, Math. Slovaca, **56** (2006), 451–463. 1

[10] D. Otrocol, *Systems of functional differential equations with maxima, of mixed type*, Electronic J. Qual. Theory Differ. Equations, **2014** (2014), 9 pages.

[11] D. Otrocol, I.A. Rus, *Functional-differential equations with "maxima" via weakly Picard operator theory*, Bull. Math. Soc. Sci. Math. Roumanie, **51(99)** (2008), 253–261. 2

[12] D. Otrocol, I.A. Rus, *Functional-differential equations with "maxima" of mixed type*, Fixed Point Theory, **9** (2008), 207–220.

[13] E. P. Popov, *Automatic regulation and control*, Nauka, Moskow, (1966) (in Russian). 1