Available online at http://vonneumann-publishing.com/cna
Commun. Nonlinear Anal. 4 (2017), 121–129
Research Article

Communications in Nonlinear Analysis

Journal Homepage: www.vonneumann-publishing.com/cna

Explicit bound on some retarded integral inequalities and applications

Badreddine Meftah[a,*], Abdourazek Souahi[b]

[a] Laboratoire des télécommunications, Faculté des Sciences et de la Technologie, University of 8 May 1945 Guelma, P.O. Box 401, 24000 Guelma, Algeria.
[b] University of Badji Mokhtar Annaba, Algeria.

Abstract

In this paper, we establish some new retarded integral inequalities, which can be used as tool in the qualitative study of certain properties of retarded integrodifferential equations.

Keywords: retarded inequality, explicit bound, Gronwall-Bellman inequality.
2010 MSC: 26D15, 26D20.

1. Introduction

Integral inequality provides explicit limits on unknown functions, and is an important tool for studying some qualitative properties for solutions of differential and integral equations. The most famous of these inequalities is that established by Gronwall [6], which can be stated as follows:
If $u(t)$ satisfies the following inequality

$$0 \leq u(t) \leq \int_{\alpha}^{t} [bu(s) + a]\, ds, \ t \in I, \tag{1.1}$$

then $u(t)$ has the following estimate

$$0 \leq u(t) \leq ah \exp(bh), \ t \in I,$$

where a, b are nonnegative constants and $I = [\alpha, \alpha + h]$ is an interval of $\mathbb{R}$.

Since his discovery a lot of efforts have been devoted by many researchers for establishing variants, extensions, and generalizations of (1.1).
Bellman [1], studied the following inequality

$$u(t) \leq k + \int_{a}^{t} g(s)u(s)ds, \tag{1.2}$$

*Corresponding author
Email addresses: badrimeftah@yahoo.fr (Badreddine Meftah), arsouahi@yahoo.fr (Abdourazek Souahi)

Received 2017-10-18

which is a generalization of (1.1), k is a nonnegative constant and J is an interval of $\mathbb{R}$.
He also gave a generalized of his own result by replacing in inequality (1.2) the constant term k by a nonconstant term $n(t)$ as follows (see [2])

$$u(t) \leq n(t) + \int_{\alpha}^{t} g(s)u(s)ds, \ t \in J, \tag{1.3}$$

Lipovan [5], studied the delayed version of inequality (1.2)

$$u(t) \leq k + \int_{\alpha(t_0)}^{\alpha(t)} h(s)u(s)ds, \ t_0 \leq t \leq T, \tag{1.4}$$

where k is a nonnegative constant and $[t_0, T)$ is an interval of $\mathbb{R}$.
Pachpatte [7] investigated the following retarded inequality

$$u(t) \leq k + \int_{a}^{t} g(s)u(s)ds + \int_{a}^{\alpha(t)} h(s)u(s)ds, \forall t \in J, \tag{1.5}$$

where k is a constant and J is an interval of $\mathbb{R}$.
Rashid [8], gave a generalization of inequality (1.5)

$$u(t) \leq f(t) + \int_{a}^{t} g(s)u(s)ds + \int_{a}^{\alpha(t)} h(s)u(s)ds, \forall t \in J, \tag{1.6}$$

where $J = [a, b]$ is an interval of $\mathbb{R}$.
Also Kim [4], established a variante of (1.6). Moreover he discussed the following inequalities

$$u(t) \leq f(t) + \int_{a}^{t} g(s)u^p(s)ds + \int_{a}^{\alpha(t)} h(s)u^p(s)ds, \forall t \in J, \tag{1.7}$$

El-Owaidy et al. [3], discussed the following inequalities

$$u^p(t) \leq f(t) + \int_{a}^{t} g(s)u(s)ds + \int_{a}^{\alpha(t)} h(s)u(s)ds, \forall t \in J \tag{1.8}$$

and

$$u^p(t) \leq f^p(t) + \int_{a}^{t} g(s)u^p(s)ds + \int_{a}^{\alpha(t)} h(s)u^q(s)ds, \forall t \in J \tag{1.9}$$

where $J = [a, b]$ is an interval of $\mathbb{R}$.

Motivated by the above results, in this paper we establish some new delayed integral inequalities, which are more general than those of before, and which can be used as tools for the study of some delayed differential and integral equations.

2. Main results

In what follows: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}_0^+ = (0, \infty)$ and $I = [a, b]$ is the given subset of $\mathbb{R}$. $C(I, \mathbb{R}^+)$ denotes the set of all continuous functions from I into $\mathbb{R}^+$ and $C^1(I, I)$ denotes the set of all continuously differentiable functions from I into I.

In order to prove our results, we recall that the ν-weighted arithmetic-geometric mean inequality can be says that for $a, b \geq 0$ and $0 \leq \nu \leq 1$

$$a^\nu b^{1-\nu} \leq \nu a + (1-\nu) b.$$

Theorem 2.1. *Let $u(t), g(t), h(t), m(t), n(t) \in C(I, \mathbb{R}^+)$, and let $\alpha \in C^1(I, I)$ be a nondecreasing function with $a \leq \alpha(t) \leq t$ on I and c is a positive constant. If the following inequality*

$$u(t) \leq c + m(t) \int_a^t g(s)u(s)ds + n(t) \int_a^{\alpha(t)} h(s)u(s)ds \tag{2.1}$$

holds then $u(t)$ has the following estimate

$$u(t) \leq c \exp\left(m(t) \int_a^t g(s)ds + n(t) \int_a^{\alpha(t)} h(s)ds \right). \tag{2.2}$$

Proof. Fixing any number $\bar{t} \in I$ such that

$$m(t) \int_a^t g(s)u(s)ds + n(t) \int_a^{\alpha(t)} h(s)u(s)ds \leq m(\bar{t}) \int_a^t g(s)u(s)ds + n(\bar{t}) \int_a^{\alpha(t)} h(s)u(s)ds, \tag{2.3}$$

for all $t \in I$.
From (2.1) and (2.3), we have

$$u(t) \leq c + m(\bar{t}) \int_a^t g(s)u(s)ds + n(\bar{t}) \int_a^{\alpha(t)} h(s)u(s)ds. \tag{2.4}$$

Define a function $z(t)$ by

$$z(t) = c + m(\bar{t}) \int_a^t g(s)u(s)ds + n(\bar{t}) \int_a^{\alpha(t)} h(s)u(s)ds. \tag{2.5}$$

Clearly $z(t)$ is positive and nondecreasing,

$$u(t) \leq z(t), \ u(\alpha(t)) \leq z(\alpha(t)) \tag{2.6}$$

and

$$z(a) = c. \tag{2.7}$$

Differentiating (2.5) with respect to t, and then using (2.6), we obtain

$$\begin{aligned} z'(t) &\leq m(\bar{t}) g(t)z(t) + n(\bar{t}) \alpha'(t)h(\alpha(t))z(\alpha(t)) \\ &\leq \left(m(\bar{t}) g(t) + n(\bar{t}) \alpha'(t)h(\alpha(t))\right) z(t)). \end{aligned} \tag{2.8}$$

Now, Dividing both sides of (2.8) by $z(t)$, letting $t = s$ in the resulting inequality, and then integrating it with respect to s from a to t, we get

$$z(t) \leq z(a) \exp \left(m\left(\bar{t}\right) \int_{a}^{t} g(s)ds + n\left(\bar{t}\right) \int_{a}^{\alpha(t)} h(s)ds \right). \tag{2.9}$$

Substituting (2.7) into (2.9), we obtain

$$z(t) \leq c \exp \left(m\left(\bar{t}\right) \int_{a}^{t} g(s)ds + n\left(\bar{t}\right) \int_{a}^{\alpha(t)} h(s)ds \right). \tag{2.10}$$

Taking $t = \bar{t}$ in inequality (2.10), we obtain

$$z(\bar{t}) \leq c \exp \left(m\left(\bar{t}\right) \int_{a}^{\bar{t}} g(s)ds + n\left(\bar{t}\right) \int_{a}^{\alpha(\bar{t})} h(s)ds \right). \tag{2.11}$$

Since $\bar{t}$ is arbitrary, then (2.11) is hold for all $t \in I$, and we have

$$z(t) \leq c \exp \left(m\left(t\right) \int_{a}^{t} g(s)ds + n\left(t\right) \int_{a}^{\alpha(t)} h(s)ds \right). \tag{2.12}$$

Combining (2.6) and (2.12), we get the desired result. □

Remark 2.2. Theorem 2.1 recapture Theorem from [1] if we take $m\left(t\right) = 1$ and $n\left(t\right) = 0$, and Corollary from [5] if we take $m\left(t\right) = 0$ and $n\left(t\right) = 1$, and Theorem 2.1 from [8] if we choose $m\left(t\right) = n\left(t\right) = 1$.

Theorem 2.3. *Let $u(t), g(t), h(t), m(t), n\left(t\right) \in C\left(I, \mathbb{R}^{+}\right)$ and $f(t) \in C\left(I, \mathbb{R}_{0}^{+}\right)$ be a nondecreasing function, and let $\alpha \in C^{1}\left(I, I\right)$ be a nondecreasing function with $a \leq \alpha(t) \leq t$ on I. If the following inequality*

$$u(t) \leq f(t) + m\left(t\right) \int_{a}^{t} g(s)u(s)ds + n\left(t\right) \int_{a}^{\alpha(t)} h(s)u(s)ds, \tag{2.13}$$

then $u(t)$ has the following estimate

$$u(t) \leq f\left(t\right) \exp \left(m\left(t\right) \int_{a}^{t} g(s)ds + n\left(t\right) \int_{a}^{\alpha(t)} h(s)ds \right). \tag{2.14}$$

Proof. Since $f(t)$ is positive and monotonic nondecreasing, we can restate (2.13) as follows

$$\frac{u(t)}{f(t)} \leq 1 + m\left(t\right) \int_{a}^{t} g(s)\frac{u(s)}{f(s)}ds + n\left(t\right) \int_{a}^{\alpha(t)} h(s)\frac{u(s)}{f(s)}ds. \tag{2.15}$$

Let

$$v(t) = \frac{u(t)}{f(t)}. \tag{2.16}$$

Using (2.16) in (2.15) we obtain

$$v(t) \leq 1 + m(t) \int_a^t g(s)v(s)ds + n(t) \int_a^{\alpha(t)} h(s)v(s)ds. \tag{2.17}$$

Inequality (2.17) is similar to inequality (2.1) with $c = 1$. Applying Theorem 2.1 for (2.17), we get

$$v(t) \leq \exp\left(m(t) \int_a^t g(s)ds + n(t) \int_a^{\alpha(t)} h(s)ds \right). \tag{2.18}$$

Combining (2.16) and (2.18), we obtain the desired result. □

Remark 2.4. Theorem 2.3 will be reduced to Theorem 2.1 from [8], if we choose $m(t) = n(t) = 1$.

Theorem 2.5. *Let $u(t), g(t), h(t), m(t), n(t), f(t) \in C(I, \mathbb{R}^+)$ such that $f(t)$ is a positive and nondecreasing function, and $\alpha(t) \in C^1(I, I)$ is a nondecreasing function with $a \leq \alpha(t) \leq t$ on I. And let p, q, r, ϵ be a positives numbers with $p \geq q$, and $p \geq r$.*
If the following inequality

$$u^p(t) \leq f^p(t) + m(t) \int_a^t g(s)u^q(s)ds + n(t) \int_a^{\alpha(t)} h(s)u^r(s)ds, \tag{2.19}$$

holds then $u(t)$ has the following estimate

$$u(t) \leq f(t) \{A(t) \exp B(t)\}^{\frac{1}{p}}, \tag{2.20}$$

where

$$A(t) = 1 + \tfrac{p-q}{p} m(t) \int_a^t g(s)f^{q-p}(s)ds + \tfrac{p-r}{p} n(t) \int_a^{\alpha(t)} h(s)f^{r-p}(s)ds \tag{2.21}$$

and

$$B(t) = \tfrac{q}{p} m(t) \int_a^t g(s)f^{q-p}(s)ds + \tfrac{r}{p} n(t) \int_a^{\alpha(t)} h(s)f^{r-p}(s)ds. \tag{2.22}$$

Proof. Using the positivity and the monotonicity of $f(t)$, (2.19) can be restated as

$$\left(\tfrac{u(t)}{f(t)}\right)^p \leq 1 + m(t) \int_a^t g(s)f^{q-p}(s) \left(\tfrac{u(s)}{f(s)}\right)^q ds + n(t) \int_a^{\alpha(t)} h(s)f^{r-p}(s) \left(\tfrac{u(s)}{f(s)}\right)^r ds. \tag{2.23}$$

Let

$$v(t) = \frac{u(t)}{f(t)}. \tag{2.24}$$

Substituting (2.23) in (2.24), we obtain

$$v^p(t) \leq 1 + m(t) \int_a^t g(s)f^{q-p}(s)v^q(s)ds + n(t) \int_a^{\alpha(t)} h(s)f^{r-p}(s)v^r(s)ds. \tag{2.25}$$

Fixing any number $\bar{t} \in I$ such that

$$\begin{aligned} & 1+m\left(t\right)\int_{a}^{t}g(s)f^{q-p}(s)v^{q}(s)ds+n\left(t\right)\int_{a}^{\alpha(t)}h(s)f^{r-p}(s)v^{r}(s)ds \\ & \leq 1+m\left(\bar{t}\right)\int_{a}^{t}g(s)f^{q-p}(s)v^{q}(s)ds+n\left(\bar{t}\right)\int_{a}^{\alpha(t)}h(s)f^{r-p}(s)v^{r}(s)ds. \end{aligned} \tag{2.26}$$

From (2.25) and (2.26) we have

$$v^{p}(t)\leq 1+m\left(\bar{t}\right)\int_{a}^{t}g(s)f^{q-p}(s)v^{q}(s)ds+n\left(\bar{t}\right)\int_{a}^{\alpha(t)}h(s)f^{r-p}(s)v^{r}(s)ds.$$

Define a function $z\left(t\right)$ by

$$z(t)=1+m\left(\bar{t}\right)\int_{a}^{t}g(s)f^{q-p}(s)v^{q}(s)ds+n\left(\bar{t}\right)\int_{a}^{\alpha(t)}h(s)f^{r-p}(s)v^{r}(s)ds. \tag{2.27}$$

Clearly, $z\left(t\right)$ is positive, nondecreasing,

$$v(t)\leq z^{\frac{1}{p}}(t),v(\alpha(t))\leq z^{\frac{1}{p}}(\alpha(t)) \tag{2.28}$$

and

$$z(a)=1. \tag{2.29}$$

Differentiating (2.27) with respect to t, we get

$$z'(t)=m\left(\bar{t}\right)g(t)f^{q-p}(t)v^{q}(t)+n\left(\bar{t}\right)\alpha'(t)h(\alpha(t))f^{r-p}(\alpha(t))v^{r}(\alpha(t)). \tag{2.30}$$

Using (2.28) in (2.30), we obtain

$$z'(t)\leq m\left(\bar{t}\right)g(t)f^{q-p}(t)z^{\frac{q}{p}}(t)+n\left(\bar{t}\right)\alpha'(t)h(\alpha(t))f^{r-p}(\alpha(t))z^{\frac{r}{p}}(\alpha(t)). \tag{2.31}$$

Now, applying the A-G inequality for (2.31), and then using the fact that $z(\alpha(t))\leq z(t)$, we get

$$\begin{aligned} z'(t) & \leq m\left(\bar{t}\right)g(t)f^{q-p}(t)(\tfrac{q}{p}z(t)+\tfrac{p-q}{p}) \\ & \quad +n\left(\bar{t}\right)\alpha'(t)h(\alpha(t))f^{r-p}(\alpha(t))(\tfrac{r}{p}z(\alpha(t))+\tfrac{p-r}{p}) \\ & =\left(\tfrac{q}{p}m\left(\bar{t}\right)g(t)f^{q-p}(t)+\tfrac{r}{p}n\left(\bar{t}\right)\alpha'(t)h(\alpha(t))f^{r-p}(\alpha(t))\right)z(t) \\ & \quad +\tfrac{p-q}{p}m\left(\bar{t}\right)g(t)f^{q-p}(t)+\tfrac{p-r}{p}n\left(\bar{t}\right)\alpha'(t)h(\alpha(t))f^{r-p}(\alpha(t)). \end{aligned} \tag{2.32}$$

Setting $t=s$ in (2.32), integrating the result over $[a,t]$, and using (2.29), we get

$$\begin{aligned} z(t) & \leq \int_{a}^{t}\left(\tfrac{q}{p}m\left(\bar{t}\right)g(s)f^{q-p}(s)+\tfrac{r}{p}n\left(\bar{t}\right)\alpha'(s)h(\alpha(s))f^{r-p}(\alpha(s))\right)z\left(s\right)ds \\ & \quad +1+\int_{a}^{t}\left(\tfrac{p-q}{p}m\left(\bar{t}\right)g(s)f^{q-p}(s)+\tfrac{p-r}{p}n\left(\bar{t}\right)\alpha'(s)h(\alpha(s))f^{r-p}(\alpha(s))\right)ds. \end{aligned} \tag{2.33}$$

Using Gronwall-Bellman lemma, we obtain

$$z(t) \leq \left(1+\tfrac{p-q}{p}m\left(\bar{t}\right)\int_a^t g(s)f^{q-p}(s)ds+\tfrac{p-r}{p}n\left(\bar{t}\right)\int_a^{\alpha(t)} h(s)f^{r-p}(s)ds\right)$$
$$\exp\left(\tfrac{q}{p}m\left(\bar{t}\right)\int_a^t g(s)f^{q-p}(s)ds+\tfrac{r}{p}n\left(\bar{t}\right)\int_a^{\alpha(t)} h(s)f^{r-p}(s)ds\right). \tag{2.34}$$

Taking $t=\bar{t}$ in inequality (2.34), we obtain

$$z(t) \leq \left(1+\tfrac{p-q}{p}m\left(\bar{t}\right)\int_a^{\bar{t}} g(s)f^{q-p}(s)ds+\tfrac{p-r}{p}n\left(\bar{t}\right)\int_a^{\alpha(\bar{t})} h(s)f^{r-p}(s)ds\right)$$
$$\exp\left(\tfrac{q}{p}m\left(\bar{t}\right)\int_a^{\bar{t}} g(s)f^{q-p}(s)ds+\tfrac{r}{p}n\left(\bar{t}\right)\int_a^{\alpha(\bar{t})} h(s)f^{r-p}(s)ds\right). \tag{2.35}$$

Since $\bar{t}$ is arbitrary, then (2.35) is hold for all $t \in I$, and we write

$$z(t) \leq A(t)\exp B(t), \tag{2.36}$$

where A and B are defined as in (2.21) and (2.22) respectively. Combining (2.24), (2.28), and (2.36) we get the desired result. □

3. Applications

In this section we present some applications of our results. Let us consider the following retarded integral equation:

$$u(t)=\Phi\left(t,\int_a^t G(s,u(s))ds,\int_a^{\alpha(t)} H(s,u(s))ds\right), \tag{3.1}$$

for all $t \in [a,b] \subset \mathbb{R}$, where $\Phi : I \times \mathbb{R}^2 \to \mathbb{R}$, $G,H : I \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and $\alpha \in C^1(I,I)$ be nondecreasing functions with $a \leq \alpha(t) \leq t$ on I.

Proposition 3.1. *Assume that*

$$\left|\Phi\left(t,\int_a^t G(s,u(s))ds,\int_a^{\alpha(t)} H(s,u(s))ds\right)\right| \leq |y(t)|+m(t)\left|\int_a^t G(s,u(s))ds\right|+n(t)\left|\int_a^{\alpha(t)} H(s,u(s))ds\right| \tag{3.2}$$

and

$$\begin{cases} |y(t)| \leq c \\ |G(t,u(t))| \leq g(t)\,|u(t)| \\ |H(t,u(t))| \leq h(t)\,|u(t)|, \end{cases} \tag{3.3}$$

where $c \geq 0$ and g,h,m,n satisfy the hypotheses of Theorem 2.1. If $u(t)$ is any solution of (3.1)-(3.3), then $u(t)$ satisfies the following estimate

$$|u(t)| \leq c\exp\left(m(t)\int_a^t g(s)ds+n(t)\int_a^{\alpha(t)} h(s)ds\right). \tag{3.4}$$

Proof. Let $u(t)$ be a solution of (3.1), using the modulus, we obtain

$$|u(t)| = \left|\Phi\left(t, \int_a^t G(s,u(s))ds, \int_a^{\alpha(t)} H(s,u(s))ds\right)\right|. \tag{3.5}$$

From (3.2) and (3.3), we can restate (3.5) as follows

$$|u(t)| \leq c + m(t)\int_a^t g(s)|u(s)|\,ds + n(t)\int_a^{\alpha(t)} h(s)|u(s)|\,ds. \tag{3.6}$$

Now, an application of Theorem 2.1 for (3.6) gives the estimate (3.4). □

Proposition 3.2. *Assume that*

$$\left|\Phi\left(t, \int_a^t G(s,u(s))ds, \int_a^{\alpha(t)} H(s,u(s))ds\right)\right| \leq |y(t)| + m(t)\left|\int_a^t G(s,u(s))ds\right| + n(t)\left|\int_a^{\alpha(t)} H(s,u(s))ds\right| \tag{3.7}$$

and

$$\begin{cases} |y(t)| \leq f(t) \\ |G(t,u(t))| \leq g(t)|u(t)| \\ |H(t,u(t))| \leq h(t)|u(t)|, \end{cases} \tag{3.8}$$

where f, g, h, m and n satisfy the hypotheses of Theorem 2.3. If $u(t)$ is any solution of (3.1),(3.7) and (3.8), then $u(t)$ satisfies the following estimate

$$|u(t)| \leq f(t)\exp\left(m(t)\int_a^t g(s)ds + n(t)\int_a^{\alpha(t)} h(s)ds\right). \tag{3.9}$$

Proof. Let $u(t)$ be a solution of (3.1), using the modulus, we obtain

$$|u(t)| = \left|\Phi\left(t, \int_a^t G(s,u(s))ds, \int_a^{\alpha(t)} H(s,u(s))ds\right)\right|. \tag{3.10}$$

From (3.7) and (3.8), we can restate (3.10) as follows

$$|u(t)| \leq f(t) + m(t)\int_a^t g(s)|u(s)|\,ds + n(t)\int_a^{\alpha(t)} h(s)|u(s)|\,ds. \tag{3.11}$$

Now, an application of Theorem 2.3 for (3.11) gives the estimate (3.9). □

Proposition 3.3. *Assume that*

$$\left|\Phi\left(t, \int_a^t G(s,u(s))ds, \int_a^{\alpha(t)} H(s,u(s))ds\right)\right| \leq \left(|y(t)| + m(t)\left|\int_a^t G(s,u(s))ds\right| + n(t)\left|\int_a^{\alpha(t)} H(s,u(s))ds\right|\right)^{\frac{1}{p}} \tag{3.12}$$

and

$$\left\{ \begin{array}{c} |y(t)| \leq f^p(t) \\ |G(t,u(t))| \leq g(t)\, |u(t)|^q \\ |H(t,u(t))| \leq h(t)\, |u(t)|^r \, , \end{array} \right. \tag{3.13}$$

where f, g, h, m, *and* p, q, r *satisfy the hypotheses of Theorem 2.5. If* $u(t)$ *is any solution of* (3.1),(3.12) *and* (3.13), *then* $u(t)$ *satisfies the following estimate*

$$|u(t)| \leq f(t) \left\{ A(t) \exp B(t) \right\}^{\frac{1}{p}}, \tag{3.14}$$

A *and* B *are defined as in* (2.21) *and* (2.22) *respectively.*

Proof. Let $u(t)$ be a solution of (3.1), using the modulus, we obtain

$$|u(t)| = \left| \Phi \left(t, \int_a^t G(s,u(s))ds, \int_a^{\alpha(t)} H(s,u(s))ds \right) \right|. \tag{3.15}$$

From (3.12) and (3.13), we can restate (3.15) as follows

$$|u(t)|^p \leq f^p(t) + m(t) \int_a^t g(s)\, |u(s)|^s \, ds + n(t) \int_a^{\alpha(t)} h(s)\, |u(s)|^r \, ds. \tag{3.16}$$

Now, an application of Theorem 2.5 for (3.16) gives the estimate (3.14). □

References

[1] R. Bellman, *The stability of solutions of linear differential equations*, Duke Math. J., **10** (1943). 643–647. 1, 2.2

[2] R. Bellman, *Asymptotic series for the solutions of linear differential-difference equations.* Rend. Circ. Mat. Palermo (2) **7** (1958), 261–269. 1

[3] H. El-Owaidy, A. Abdeldaim, A.A. El-Deeb, *On some new nonlinear retarded integral inequalities with iterated integrals and their applications in differential-integral equations*, Math. Sci. Lett., **3** (2014), 157–164. 1

[4] Y. H. Kim, *On some new integral inequalities for functions in one and two variables*, Acta Math. Sin. (Engl. Ser.), **21** (2005), 423–434. 1

[5] O. Lipovan, *A retarded Gronwall-like inequality and its applications*, J. Math. Anal. Appl., **252** (2000), 389–401. 1, 2.2

[6] T. H. Gronwall, *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Ann. of Math. (2), **20** (1919), 292–296. 1

[7] B. G. Pachpatte, *Explicit bounds on certain integral inequalities*, J. Math. Anal. Appl., **267** (2002), 48–61. 1

[8] M. H. M. Rashidi, *Explicit bounds on retarded Gronwall-Bellman inequality*, Tamkang J. Math., **43** (2012), 99–108. 1, 2.2, 2.4