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Some coupled fixed point results for set-valued mappings with applications

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Abstract

This paper deals with the study of coupled fixed point theorems for φ-pseudo-contractive set-valued mappings without using the mixed g-monotone property on the closed ball of partial metric spaces. Generalizations of some well-known results concerning existence and location of coupled fixed points are obtained. These coupled fixed point theorems are applied for obtaining the existence results for an elliptic system.

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1. Introduction and preliminaries

In the study of nonlinear differential equations or differential inclusions, the topological methods are used to give us the qualitative information about the existence, localization, stability, and multiplicity of solutions. The topological degree and fixed point theorems are the most topological techniques used, which are closely connected. In the present paper, we are interested in coupled fixed point theorems for set-valued mappings on complete partial metric spaces which have an importance in the last decades.

Recall that the partial metric is an interesting distance function introduced by Matthews [28]. The motivation behind introducing the concept of a partial metric space is to present a version of a Banach contraction principle to solve some problems in computer science.

Recently, Benterki presented a fixed point theorem [8, Theorem 3.2] for φ-pseudo-contractive set-valued mappings in the framework of partial metric spaces about a location of a fixed point with respect to an initial

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value of the set-valued mapping by using Bianchini-Grandolfi gauge functions. Note that a nondecreasing function $\varphi : J \to J$ is called a Bianchini-Grandolfi gauge function or (c)-comparison on J (being a connected interval on $\mathbb{R}^+$ containing 0) if

$$s(t) := \sum_{n=0}^{\infty} \varphi^n(t) \text{ is convergent for all } t \in J,$$

where φ^n denotes the n-th iteration of the function φ and $\varphi^0(t) = t$, i.e.,

$$\varphi^0(t) = t, \quad \varphi^1(t) = \varphi(t), \quad \varphi^2(t) = \varphi(\varphi(t)), \ldots, \varphi^n(t) = \varphi(\varphi^{n-1}(t)).$$

Then the theorem reads as follows:

Theorem 1.1 ([8])**.** *Let (X, p) be a complete partial metric space, and consider a point $\overline{x} \in X$, nonnegative scalar $r > 0$ and a set-valued mapping ϕ from the closed p-ball $\overline{B_p}(\overline{x}, r)$ to the closed subsets of (X, p). Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing and continuous function such that φ is a Bianchini-Grandolfi gauge function on interval J and $\lim_{t \downarrow 0} \varphi(t) = 0$. If there exists $\alpha \in J$ such that the following two conditions hold:*

(a) *$p(\overline{x}, \phi(\overline{x})) < \alpha$, where $s(\alpha) \leqslant p(\overline{x}, \overline{x}) + r$;*
(b) *$\delta_p(\phi(x) \cap \overline{B_p}(\overline{x}, r), \phi(y)) \leqslant \varphi\left(p(x, y)\right), \ \forall x, y \in \overline{B_p}(\overline{x}, r)$,*

then ϕ has a fixed point x^* in $\overline{B_p}(\overline{x}, r)$. If ϕ is a single-valued mapping and $p(\overline{x}, \overline{x}) + 2r \in J$, then x^* is the unique fixed point of ϕ in $\overline{B_p}(\overline{x}, r)$.

This theorem generalizes and extends several known results in the literature (see e.g., [2, 4–6, 13, 14, 18, 22, 25–28, 30, 33]). In the last ten years, several authors studied a coupled fixed point results for single-valued and set-valued mappings on various spaces with or without mixed g-monotone (mixed monotone) property (see e.g., [3, 7, 9, 15, 17, 19–21, 23, 24, 31, 32, 37–39]). Note that the mixed g-monotone property is given by the following:

Definition 1.2. Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to 2^X$ be a set-valued mapping. We say that F has a mixed g-monotone property if for any $x_1, x_2, y_1, y_2 \in X$ we have

$$g(x_1) \preceq g(x_2) \Rightarrow \forall u_1 \in F(x_1, y_1), \exists u_2 \in F(x_2, y_1), u_1 \preceq u_2$$

and

$$g(y_1) \preceq g(y_2) \Rightarrow \forall v_1 \in F(x_1, y_1), \exists v_2 \in F(x_1, y_2), v_1 \succeq v_2.$$

The notion of a coupled fixed point was first initiated by Guo and Lakshmikantham in [16] and then studied by Bhaskar and Lakshmikantham in [15]. They study coupled fixed points for a mappings having the mixed monotone property (i.e., for $g = Id$ the identity function) in a metric space endowed with partial order under contractive conditions and proved the following theorem.

Theorem 1.3 ([15])**.** *Let $(X, d, \preceq)$ be a partially ordered complete metric space and let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property. Assume that there exists a $\lambda \in [0, 1)$ such that*

$$d(F(x, y), F(u, v)) \leqslant \frac{\lambda}{2} \left(d(x, u) + d(y, v)\right), \quad \forall x, y, u, v \in X$$

with $x \preceq u$ and $y \succeq v$. If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then there exist $x^*, y^* \in X$ such that $x^* = F(x^*, y^*)$ and $y^* = F(y^*, x^*)$, i.e., F has a coupled fixed point.

The aim of this paper is to present the local version of coupled fixed point results for φ-pseudo-contractive set-valued mappings on complete partial metric spaces without using the mixed g-monotone property. It yields that several results are obtained as special cases. As an application, we establish the existence of solutions (not necessary positives) for the following nonlinear elliptic system

$$\begin{cases} -u''(t) = f(t, u(t), v(t)) - \lambda. & t \in (0,1), \\ -v''(t) = f(t, v(t), u(t)) - \lambda, & t \in (0,1), \\ u(0) = u(1) = 0 = v(0) = v(1), \end{cases} \tag{1.1}$$

where f be a continuous real function and λ is a nonnegative real constant.

The paper is organized as follows. In this Section we recall some preliminary facts that we need in the sequel, in Section 2 we prove our results and give some related corollaries, and in Section 3, we establish the existence of solutions for nonlinear elliptic system (1.1).

First, we recall some basic concepts about partial metric space.

Definition 1.4. Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X and then the pair (X, p) is called a partial metric space if for any $x, y, z \in X$, the following conditions hold:

(P_1) $p(x,x) = p(y,y) = p(x,y) \Leftrightarrow x = y$;
(P_2) $p(x,x) \leqslant p(x,y)$;
(P_3) $p(x,y) = p(y,x)$;
(P_4) $p(x,y) + p(z,z) \leqslant p(x,z) + p(z,y)$.

Thus, a metric space is precisely a partial metric space for which each self distance $p(x,x) = 0$.

Example 1.5. The following functions p_i $(i \in \{1,2\})$ define a partial metric for each X

$$p_1(x,y) = \max\{x,y\}, \quad x, y \in X = \mathbb{R}^+,$$
$$p_2(x,y) = |x-y| + c, \quad x, y \in X = \mathbb{R} \text{ and } c \geqslant 0.$$

Let (X, p) be a partial metric space. Then

- the closed p-ball of radius r centered at x is denoted by $\overline{B_p}(x,r)$, where
$$\overline{B_p}(x,r) = \{y \in X : p(x,y) \leqslant p(x,x) + r\};$$
- a sequence $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x,x) = \lim\limits_{n\to+\infty} p(x, x_n)$;
- a sequence $\{x_n\}$ is called a Cauchy sequence if $\lim\limits_{n,m\to+\infty} p(x_n, x_m)$ exists (and is finite);
- the partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim\limits_{n,m\to+\infty} p(x_n, x_m)$.

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Lemma 1.6. *Let (X, p) be a partial metric space.*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).*
(b) *A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete.*

Let (X, p) be a partial metric space and let $C^p(X)$ be the family of all nonempty and closed subsets of the partial metric space (X, p), induced by the partial metric p. For $x \in X$ and $A, B \in C^p(X)$, we define

$$p(x, A) = \inf\{p(x,a), a \in A\},$$

and

$$\delta_p(A,B) = \sup\{p(a,B), a \in A\},$$

with the convention

$$p(x, \emptyset) = +\infty, \quad \delta_p(\emptyset, B) = 0. \tag{1.2}$$

Definition 1.7. Let (X,p) be a partial metric space, $B \subseteq X$ be a subset and $\phi : B\times B \to C^p$ be a set-valued mapping. An element $(x^*, y^*) \in B \times B$ is called a coupled fixed point of ϕ if

$$\begin{cases} x^* \in \phi(x^*, y^*) \cap B, \\ y^* \in \phi(y^*, x^*) \cap B. \end{cases}$$

A point $(x^*, x^*) \in B \times B$ is called a fixed point of ϕ if $x^* \in \phi(x^*, x^*)$.

Note that if (x^*, y^*) is a coupled fixed point of ϕ, then (y^*, x^*) is coupled fixed point too.

2. The main results

In this section, we state and prove our main result.

Theorem 2.1. *Let (X,p) be a complete partial metric space, $\overline{x} \in X$, and $r > 0$ be a nonnegative scalar. We consider a set-valued mapping $\phi : \overline{B_p}(\overline{x}, r) \times \overline{B_p}(\overline{x}, r) \to C^p(X)$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a increasing and continuous function such that φ is a Bianchini-Grandolfi gauge function on interval J and $\lim_{t\downarrow 0} \varphi(t) = 0$. If there exists $\alpha \in J$ such that the following conditions hold:*

(a) $p(\overline{x}, \phi(\overline{x}, \overline{x})) < \alpha$, *where* $s(\alpha) \leqslant p(\overline{x}, \overline{x}) + r$*;*
(b) $\delta_p(\phi(x, y) \cap \overline{B_p}(\overline{x}, r), \phi(u, v)) \leqslant \varphi(\max\{p(x, u), p(y, v)\}), \quad \forall x, y, u, v \in \overline{B_p}(\overline{x}, r)$,

then ϕ has a coupled fixed point (x^*, y^*) in $\overline{B_p}(\overline{x}, r) \times \overline{B_p}(\overline{x}, r)$. If ϕ is a single-valued mapping and $p(\overline{x}, \overline{x}) + 2r \in J$, then (x^*, y^*) is the unique coupled fixed point of ϕ in $\overline{B_p}(\overline{x}, r) \times \overline{B_p}(\overline{x}, r)$.

Proof. If $\overline{x} \in \phi(\overline{x}, \overline{x})$ or $\varphi \equiv 0$ the proof is finished. So we assume that $\overline{x} \notin \phi(\overline{x}, \overline{x})$ and $\varphi \not\equiv 0$. We consider the Cartesian product $X \times X$ endowed with the partial metric

$$\widetilde{p}((x, y), (u, v)) = \max\{p(x, u), p(y, v)\}$$

and then $(X \times X, \widetilde{p})$ is a complete partial metric. We consider a set-valued mapping

$$\widetilde{\phi} \ : \ \overline{B_{\widetilde{p}}}((\overline{x}, \overline{x}), r) \to C^{\widetilde{p}} := C^p \times C^p$$

defined by

$$\widetilde{\phi}(x, y) = (\phi(x, y), \phi(y, x)) \,.$$

Now, we check that $\widetilde{\phi}$ satisfies all assumptions of Theorem 1.1 on the closed $\widetilde{p}$-ball $\overline{B_{\widetilde{p}}}((\overline{x}, \overline{x}), r)$. Before starting, we need to prove the following two claims.

Claim 2.2. $\overline{B_p}(\overline{x}, r) \times \overline{B_p}(\overline{x}, r) = \overline{B_{\widetilde{p}}}((\overline{x}, \overline{x}), r)$.

Proof. Since $\overline{B_p}(\overline{x}, r) \times \overline{B_p}(\overline{x}, r) \subset \overline{B_{\widetilde{p}}}((\overline{x}, \overline{x}), r)$, by contradiction, we assume that there exists $(a, b) \in \overline{B_{\widetilde{p}}}((\overline{x}, \overline{x}), r) \setminus \overline{B_p}(\overline{x}, r) \times \overline{B_p}(\overline{x}, r)$, i.e.,

$$\widetilde{p}((a, b), (\overline{x}, \overline{x})) \leqslant \widetilde{p}((\overline{x}, \overline{x}), (\overline{x}, \overline{x})) + r \quad \text{ and } \quad \max\{p(a, \overline{x}), p(b, \overline{x})\} > p(\overline{x}, \overline{x}) + r.$$

Then we have

$$\widetilde{p}((\overline{x}, \overline{x}), (\overline{x}, \overline{x})) + r = p(\overline{x}, \overline{x}) + r < \max\{p(a, \overline{x}), p(b, \overline{x})\} = \widetilde{p}((a, b), (\overline{x}, \overline{x})) \leqslant \widetilde{p}((\overline{x}, \overline{x}), (\overline{x}, \overline{x})) + r,$$

which is a contradiction and hence equality holds. □

Claim 2.3. $\widetilde{p}((a, b), \widetilde{\phi}(u, v)) \leqslant \max\{p(a, \phi(u, v)), p(b, \phi(v, u))\}$.

$$\begin{aligned}\widetilde{p}((a,b),\widetilde{\phi}(u,v)) &= \inf_{(c,d)\in\widetilde{\phi}(u,v)} \widetilde{p}((a,b),(c,d))\\ &= \inf_{(c,d)\in(\phi(u,v),\phi(v,u))} \max\{p(a,c),p(b,d)\}\\ &\leqslant \max\{\inf_{c\in\phi(u,v)} p(a,c), \inf_{d\in\phi(v,u)} p(b,d)\}\\ &= \max\{p(a,\phi(u,v)),p(b,\phi(v,u))\}.\end{aligned}$$

First, observe that

$$\widetilde{p}((\overline{x},\overline{x}),\widetilde{\phi}(\overline{x},\overline{x})) \leqslant \max\{p(\overline{x},\phi(\overline{x},\overline{x})),p(\overline{x},\phi(\overline{x},\overline{x}))\} < \alpha$$

and $s(\alpha) \leqslant p(\overline{x},\overline{x}) + r = \widetilde{p}((\overline{x},\overline{x}),(\overline{x},\overline{x})) + r$. That is, condition (a) of Theorem 1.1 holds.

Further, for any $x,y,u,v \in \overline{B_p}(\overline{x},r)$ we obtain

$$\begin{aligned}\delta_{\widetilde{p}}(\widetilde{\phi}(x,y)\cap\overline{B_{\widetilde{p}}}((\overline{x},\overline{x}),r),\widetilde{\phi}(u,v)) &= \sup\{\widetilde{p}((a,b),\widetilde{\phi}(u,v)) : (a,b)\in\widetilde{\phi}(x,y)\cap\overline{B_{\widetilde{p}}}((\overline{x},\overline{x}),r)\}\\ &\leqslant \sup\{\max\{p(a,\phi(u,v)),p(b,\phi(v,u))\} : (a,b)\in\widetilde{\phi}(x,y)\cap\overline{B_{\widetilde{p}}}((\overline{x},\overline{x}),r)\}\\ &\leqslant \begin{cases}\delta_p(\phi(x,y)\cap\overline{B_p}((\overline{x},r),\phi(u,v)), & \text{if } p(a,\phi(u,v)) \geqslant p(b,\phi(v,u)),\\ \delta_p(\phi(y,x)\cap\overline{B_p}((\overline{x},r),\phi(v,u)), & \text{else}\end{cases}\\ &\leqslant \varphi(\max\{p(x,u),p(y,v)\}) = \varphi(\widetilde{p}((x,y),(u,v))),\end{aligned}$$

that is, condition (b) of Theorem 1.1 is fulfilled. Hence, all conditions are satisfied and then there exists a fixed point $(x^*,y^*) \in \widetilde{\phi}(x^*,y^*)\cap\overline{B_{\widetilde{p}}}((\overline{x},\overline{x}),r)$, i.e.,

$$\begin{cases} x^* \in \phi(x^*,y^*)\cap\overline{B_p}(\overline{x},r),\\ y^* \in \phi(y^*,x^*)\cap\overline{B_p}(\overline{x},r).\end{cases}$$

If ϕ is a single-valued mapping, then $\widetilde{\phi}$ is also a single-valued mapping and since $p(\overline{x},\overline{x}) + 2r \in J$, i.e., $\widetilde{p}((\overline{x},\overline{x}),(\overline{x},\overline{x})) + 2r \in J$, then (x^*,y^*) is the unique fixed point of $\widetilde{\phi}$ and we have

$$\begin{cases} x^* = \phi(x^*,y^*)\cap\overline{B_p}(\overline{x},r),\\ y^* = \phi(y^*,x^*)\cap\overline{B_p}(\overline{x},r),\end{cases}$$

and the proof is completed. □

Remark 2.4.

1. If p is a partial metric on X, then

$$\widetilde{p_1}((x,y),(u,v)) = \max\{p(x,u),p(y,v)\} \quad \text{and} \quad \widetilde{p_2}((x,y),(u,v)) = p(x,u)+p(y,v)$$

are two equivalent partial metrics on $X\times X$ such that $\widetilde{p_1} \leqslant \widetilde{p_2} \leqslant 2\widetilde{p_1}$.
2. If $\varphi(a+b) = \varphi(a)+\varphi(b)$ for each $a,b\in\mathbb{R}^+$, then the condition (b) from Theorem 2.1 is equivalent to:
   (b′) $\delta_p(\phi(x,y)\cap\overline{B_p}(\overline{x},r),\phi(u,v)) + \delta_p(\phi(y',x')\cap\overline{B_p}(\overline{x},r),\phi(v',u')) \leqslant \varphi(\max\{p(x,u),p(y,v)\}+\max\{p(x',u'),p(y',v')\})$.

Then from Theorem 2.1 and Remark 2.4, we can obtain some corollaries.

Corollary 2.5. *Let (X,p) be a complete partial metric space, $\overline{x}\in X$, and $r>0$ be a nonnegative scalar. We consider a set-valued mapping $\phi : \overline{B_p}(\overline{x},r)\times\overline{B_p}(\overline{x},r) \to C^p(X)$. Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be a increasing and continuous function such that φ is a Bianchini-Grandolfi gauge function on interval J and $\lim_{t\downarrow 0}\varphi(t) = 0$. If there exists $\alpha\in J$ such that the following conditions hold:*

(a) $p(\overline{x}, \phi(\overline{x}, \overline{x})) < \alpha$, *where* $s(\alpha) \leqslant p(\overline{x}, \overline{x}) + r$;
(b) $\delta_p(\phi(x, y) \cap \overline{B_p}(\overline{x}, r), \phi(u, v)) \leqslant \varphi(\frac{1}{2}(p(x, u) + p(y, v)))$, $\forall x, y, u, v \in \overline{B_p}(\overline{x}, r)$,

then ϕ *has a coupled fixed point* (x^*, y^*) *in* $\overline{B_p}(\overline{x}, r) \times \overline{B_p}(\overline{x}, r)$. *If* ϕ *is a single-valued mapping and* $p(\overline{x}, \overline{x}) + 2r \in J$, *then* (x^*, y^*) *is the unique coupled fixed point of* ϕ *in* $\overline{B_p}(\overline{x}, r) \times \overline{B_p}(\overline{x}, r)$.

Proof. We use the inequality $\frac{1}{2}\widetilde{p_2} \leqslant \widetilde{p_1}$, Remark 2.4, and the increasingness of φ to complete the proof. □

Corollary 2.6 (Bhaskar and Lakshmikantham version)**.** *Let* (X, p) *be a complete partial metric space, and consider a point* $\overline{x} \in X$, *and nonnegative scalars* $r > 0$ *and* $0 \leqslant \lambda < 1$. *We consider a set-valued mapping* $\phi : \overline{B_p}(\overline{x}, r) \times \overline{B_p}(\overline{x}, r) \to C^p(X)$. *Let the following two conditions hold:*

(a) $p(\overline{x}, \phi(\overline{x}, \overline{x})) < (p(\overline{x}, \overline{x}) + r)(1 - \lambda)$;
(b) $\delta_p(\phi(x, y) \cap \overline{B_p}(\overline{x}, r), \phi(u, v)) \leqslant \frac{\lambda}{2}(p(x, u) + p(y, v))$, $\forall x, y, u, v \in \overline{B_p}(\overline{x}, r)$.

Then ϕ *has a coupled fixed point* (x^*, y^*) *in* $\overline{B_p}(\overline{x}, r) \times \overline{B_p}(\overline{x}, r)$. *If* ϕ *is a single-valued mapping, then* (x^*, y^*) *is the unique coupled fixed point of* ϕ *in* $\overline{B_p}(\overline{x}, r) \times \overline{B_p}(\overline{x}, r)$.

Proof. We apply Corollary 2.5 for $\varphi(t) = \lambda t$ which is a Bianchini-Grandolfi gauge function on $J = [0, +\infty)$ and $s(t) = \dfrac{t}{1 - \lambda}$. Taking $\alpha = (p(\overline{x}, \overline{x}) + r)(1 - \lambda) \in J$, we complete the proof. □

Notice that the condition (b) from Corollary 2.6 can be re-written, by using Remark 2.4, as follows:

(b″) $\delta_p(\phi(x, y) \cap \overline{B_p}(\overline{x}, r), \phi(u, v)) + \delta_p(\phi(y, x) \cap \overline{B_p}(\overline{x}, r), \phi(v, u)) \leqslant \lambda(p(x, u) + p(y, v))$,

and then we get the local version of [3, Theorem 2.1] for set-valued mappings on partial metric spaces as follows.

Corollary 2.7. *Let* (X, p) *be a complete partial metric space, and consider a point* $\overline{x} \in X$, *and nonnegative scalars* $r > 0$ *and* $k, l \geqslant 0$ *such that* $0 \leqslant k + l < 1$. *We consider a set-valued mapping* $\phi : \overline{B_p}(\overline{x}, r) \times \overline{B_p}(\overline{x}, r) \to C^p(X)$. *Let the following two conditions hold:*

(a) $p(\overline{x}, \phi(\overline{x}, \overline{x})) < (p(\overline{x}, \overline{x}) + r)(1 - (k + l))$;
(b) $\delta_p(\phi(x, y) \cap \overline{B_p}(\overline{x}, r), \phi(u, v)) \leqslant lp(x, u) + kp(y, v)$, $\forall x, y, u, v \in \overline{B_p}(\overline{x}, r)$.

Then ϕ *has a coupled fixed point* (x^*, y^*) *in* $\overline{B_p}(\overline{x}, r) \times \overline{B_p}(\overline{x}, r)$. *If* ϕ *is a single-valued mapping, then* (x^*, y^*) *is the unique coupled fixed point of* ϕ *in* $\overline{B_p}(\overline{x}, r) \times \overline{B_p}(\overline{x}, r)$.

Proof. For $\lambda = l + k < 1$, we are using Remark 2.4 for any $x, y, u, v \in \overline{B_p}(\overline{x}, r)$, and we get

$$\begin{aligned}\delta_p(\phi(x, y) \cap \overline{B_p}(\overline{x}, r), \phi(u, v)) + \delta_p(\phi(y, x) \cap \overline{B_p}(\overline{x}, r), \phi(v, u)) &\leqslant (l + k)(p(x, u) + p(y, v)) \\ &\leqslant \lambda(p(x, u) + p(y, v)).\end{aligned}$$

We complete the proof by applying Corollary 2.6. □

3. Application

In this section, we consider $X = C([0, 1])$ the space of continuous real functions defined on $I = [0, 1]$ and $d : X \times X \to \mathbb{R}^+$ a metric defined by

$$d(u, v) = \|u - v\| = \sup_{t \in I} |u(t) - v(t)|.$$

We set the partial metric

$$p(u, v) = d(u, v) + c = \sup_{t \in I} |u(t) - v(t)| + c, \quad c \geqslant 0,$$

and since $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) = 2\|x-y\|$, so by Lemma 1.6, (X,p) is complete since the metric space $(X, \|\cdot\|)$ is complete. We apply our main results to study the existence of solutions for the following elliptic system with Dirichlet boundary conditions

$$\begin{cases} -u''(t) = f(t, u(t), v(t)) - \lambda, & t \in (0,1), \\ -v''(t) = f(t, v(t), u(t)) - \lambda, & t \in (0,1), \\ u(0) = u(1) = 0 = v(0) = v(1), \end{cases} \tag{3.1}$$

where $f : (0,1) \times X \times X \to \mathbb{R}$ is a continuous function and $\lambda \geqslant 0$. The existence of solutions for Dirichlet boundary value problems has been studied extensively. For examples, see [1, 10, 34–36] for a single variable and [11, 12, 29] for system of two variables.

Now, we consider the following conditions.

1. There exist a constant $C \geqslant 0$ and $K(\lambda)$ a positive continuous function defined for $\lambda \geqslant C$.
2. There exists an increasing and continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that φ is a Bianchini-Grandolfi gauge function on interval J and $\lim\limits_{t \downarrow 0} \varphi(t) = 0$.
3. There exists $\alpha \in J$ such that $s(\alpha) \leqslant c + K(\lambda)$.
4. $\|f(\cdot, 0, 0) - \lambda\| < 8(\alpha - c)$.
5. $|f(t,a,b) - f(t,a',b')| \leqslant \begin{cases} 8\varphi(\max\{|a-a'|, |b-b'|\} + c) - 8c, & (a,b) \neq (a',b'), \\ 0, & (a,b) = (a',b') \end{cases}$ for all $t \in I$, and $|a|, |a'|, |b|, |b'| \leqslant K(\lambda)$.

Theorem 3.1. *For a fixed $c \geqslant 0$, suppose that conditions* (1)-(5) *hold. Then* (3.1) *has at least one solution* (u^*, v^*) *in* $(C^2((0,1)) \cap C([0,1]))^2$ *such that* $\max\{\|u^*\|, \|v^*\|\} \leqslant K(\lambda)$. *Moreover, if $c + 2K(\lambda) \in J$, then the solution (u^*, v^*) is unique.*

Proof. It is well known that $(u^*, v^*) \in (C^2((0,1)) \cap C([0,1]))^2$ is a solution of system (3.1) if and only if $(u^*, v^*) \in C([0,1]) \times C([0,1])$ is a solution of the following nonlinear integral system

$$\begin{cases} u(t) = \int_0^1 G(t,s)[f(s, u(s), v(s)) - \lambda] ds, & t \in I, \\ v(t) = \int_0^1 G(t,s)[f(s, v(s), u(s)) - \lambda] ds, & t \in I, \end{cases} \tag{3.2}$$

where $G(t,s)$ is the Green function of the second-order Sturm-Liouville boundary value problem

$$\begin{cases} -z''(t) = 0, & t \in (0,1), \\ z(0) = 0, & z(1) = 0. \end{cases} \tag{3.3}$$

It is known that

$$G(t,s) = \begin{cases} t(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(1-t), & 0 \leqslant s \leqslant t \leqslant 1, \end{cases} \tag{3.4}$$

and then for all $t \in I$, we have

$$\int_0^1 G(t,s) ds = \frac{1}{2} t(1-t),$$

which implies that $\sup\limits_{t \in I} \int_0^1 G(t,s) ds = \frac{1}{8}$. Let us define a sample set-valued mapping $A : X \times X \to X$ by

$$A(u,v)(t) = \int_0^1 G(t,s)[f(s, u(s), v(s)) - \lambda] ds \quad \text{for all } u, v \in X.$$

Thus the existence of solutions to system (3.2) is equivalent to find the coupled fixed point to nonlinear operator A.

Now, we check that A satisfies all assumptions of Theorem 2.1 on the closed p-ball of radius $K(\lambda)$ centered at 0_X, the null function of X is denoted by $\overline{B_p}(0_X, K(\lambda))$.

First, the use of assumptions (1)-(4) give the following

$$\begin{aligned} p(0_X, A(0_X, 0_X)) &= \sup_{t\in I} |0_X(t) - A(0_X, 0_X)(t)| + c \\ &\leqslant \sup_{t\in I} \left| \int_0^1 G(t,s)[f(s, 0_X(s), 0_X(s)) - \lambda] ds \right| + c \\ &\leqslant \sup_{t\in I} \int_0^1 G(t,s) \left| f(s, 0_X(s), 0_X(s)) - \lambda \right| ds + c \\ &\leqslant \frac{1}{8} \| f(\cdot, 0, 0) - \lambda \| + c < \frac{8(\alpha - c) + 8c}{8} = \alpha \end{aligned}$$

and $s(\alpha) \leqslant c + K(\lambda) = p(0_X, 0_X) + K(\lambda)$. Thus the condition (a) of Theorem 2.1 is satisfied.

Let $x, y, u, v \in \overline{B_p}(0_X, K(\lambda))$, then we have two cases. The first one is if $A(x, y) \notin \overline{B_p}(0_X, K(\lambda))$, then according to convention (1.2) we have

$$\delta_p(A(x,y) \cap \overline{B_p}(0_X, K(\lambda)), A(u,v)) = 0 \leqslant \varphi(\max\{p(x,u), p(y,v)\}).$$

So we assume that $A(x, y) \in \overline{B_p}(0_X, K(\lambda)$. From condition (5), we have

$$\begin{aligned} \delta_p(A(x,y) \cap \overline{B_p}(0_X, K(\lambda)), A(u,v)) &= p(A(x,y), A(u,v)) \\ &= \sup_{t\in I} |A(x,y)(t) - A(u,v)(t)| + c \\ &= \sup_{t\in I} \left| \int_0^1 G(t,s)(f(s, x(s), y(s)) - f(s, u(s), v(s))) ds \right| + c \\ &\leqslant \sup_{t\in I} \int_0^1 G(t,s) \left| f(s, x(s), y(s)) - f(s, u(s), v(s)) \right| ds + c \\ &\leqslant \sup_{t\in I} \int_0^1 G(t,s)(8\varphi(\max\{|x(s) - u(s)|, |y(s) - v(s)|\} + c) - 8c) ds + c \\ &\leqslant \left(\sup_{t\in I} \int_0^1 G(t,s) ds \right) (8\varphi(\max\{\|x - u\|, \|y - v\|\} + c) - 8c) + c \\ &\leqslant \frac{1}{8}(8\varphi(\max\{\|x - u\|, \|y - v\|\} + c) - 8c) + c \\ &\leqslant \varphi(\max\{p(x,u), p(y,v)\}). \end{aligned}$$

Thus all conditions are satisfied and then A has a coupled fixed point (u^*, v^*) in $\overline{B_p}(0_X, K(\lambda)) \times \overline{B_p}(0_X, K(\lambda))$, i.e., $\max\{p(u^*, 0_X), p(v^*, 0_X)\} \leqslant p(0_X, 0_X) + K(\lambda) \Leftrightarrow \max\{\|u^*\|, \|v^*\|\} \leqslant K(\lambda)$. Since A is a single-valued and if $c + 2K(\lambda) \in J$, i.e., $p(0_X, 0_X) + 2K(\lambda) \in J$, then (u^*, v^*) is unique. □

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