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Some results by quasi-contractive mappings in f-orbitally complete metric space

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Abstract

The purpose of this paper is to obtain the fixed point results by quasi-contractive mappings in f-orbitally complete metric space. These results are generalizations of Ćirić fixed point theorems. Also we extend the recent results which are presented in [P. Kumam, N. Van Dung, K. Sitthithakerngkiet, Filomat, **29** (2015), 1549–1556] and [M. Beesyei, Expo. Math., **33** (2015), 517–525].

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1. Introduction

Banach contraction mapping principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory. The fixed point results obtained from various classes of operators that are weaker than contractive without continuous assumption as generalizations of Banach fixed point theorem, have been studied by many authors. One of the most well-known results in generalizations of Banachs contraction principle is Ćirić fixed point theorem.

Theorem 1.1 ([3]). *Let (X, d) be a f-orbitally complete metric space, $q < 1$, and $f : X \to X$ be a mapping satisfying quasi-contraction, i.e.,*

$$d(fx, fy) \leq q. \max\{d(x, y), d(x, fx), d(y, fy), d(y, fx), d(x, fy)\},$$

then we have:

(H1) *f has a fixed point $x^* \in X$;*

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(H2) $\lim_{n\to\infty} f^n(x) = x^*$ *for every* $x \in X$*;*
(H3) $d(f^n x, x^*) \leq \frac{q^n}{1-q} d(x, fx)$ *for every* $x \in X$*.*

Many extensions of these results are obtained by [2, 4, 8]. On the other hand, generalizations of Ćirić fixed point theorem for multi-valued generalized quasi-contraction mapping has been derived [5].

In this paper, we introduce a new generalization of quasi-contraction mappings and prove fixed point theorems for single and multi-valued mappings in f-orbitally complete metric space, assuming bounded orbit and without that. These results generalize many results, such as [1, 5]. In this paper we proof results that generalized Ćirić fixed point theorem and other fixed point theorems such that theorems in [6, 7].

2. Preliminaries

First, we recall some notions and lemma which will be used in what follows.

Definition 2.1 ([3])**.** Let X be a metric space and $f : X \to X$ be a self map such that for each $x \in X$ and for any positive n, put

$$O_f(x, n) = \{x, fx, \dots, f^n x\} \quad \text{and} \quad O_f(x, \infty) = \{x, fx, \dots, f^n x, \dots\}.$$

The set $O_f(x, \infty)$ is called the orbit of f at x and the metric space X is called f-orbitally complete if every Cauchy sequence in $O_f(x, \infty)$ is convergent in X. Also, every complete metric space is f-orbitally complete for all maps $f : X \to X$.

Definition 2.2. Let (X, d) be a metric space. We define

$$\begin{aligned} D(A, B) &= \inf\{d(a, b) : a \in A, b \in B\}, \\ \rho(A, B) &= \sup\{d(a, b); a \in A, b \in B\}, \\ BN(X) &= \{A : \phi \neq A \subset X \quad \text{and} \quad \delta(A) < +\infty\}, \end{aligned}$$

where $\delta(A) = \sup\{d(a, b) : a, b \in A\}$ and $A, B \subseteq X$.

Definition 2.3 ([3])**.** Let $F : X \to BN(X)$ be a multivalued mapping. Let $x_0 \in X$. An orbit of F at x_0 is a sequence

$$\{x_n : x_n \in Fx_{n-1}, n \in \mathbb{N}\}.$$

A space X is called F-orbitally complete if every Cauchy sequence which is subsequent of an orbit of F at x for some $x \in X$, converges in X.

3. Main result

Before proving the main results we present the following lemma.

Lemma 3.1. *If* $\phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$, ψ *be a continuous function, and* ϕ *be an upper semi-continuous function, and* $\psi^{-1}\phi(t) < t$ *and* $\phi(0) = \psi(0) = 0$, *then* $\lim_{n\to\infty} (\psi^{-1}\phi)^n(t) = 0$ *holds for* $t \geq 0$.

Proof. Clearly $\lim_{n\to\infty} (\psi^{-1}\phi)^n(0) = 0$ holds. Fix $t > 0$, if $\psi^{-1}\phi(t) = 0$, then $(\psi^{-1}\phi)^n(t) = 0$ $for\ all\ n \in N$; if $\psi^{-1}\phi(t) \neq 0$, then $(\psi^{-1}\phi)^2(t) < \psi^{-1}\phi(t)$ that follows after iterating the inequality $\psi^{-1}\phi(t) < t$. We get that the sequence $((\psi^{-1}\phi)^n)$ is decreasing for $t \geq 0$. On the other hand it is bounded from below. Therefore, the pointwise limit function $f = \lim_{n\to\infty} (\psi^{-1}\phi)^n$ exists on $\mathbb{R}^+$ and takes nonnegative values. Assume that $f(t) > 0$ for some $t > 0$, then we have

$$f(t) = \lim_{n\to\infty} (\psi^{-1}\phi)^{n+1}(t) = \limsup_{n\to\infty} (\psi^{-1}\phi)((\psi^{-1}\phi)^n) \leq (\psi^{-1}\phi)(f(t)) < f(t).$$

This contradiction proves the lemma. $\square$

We denote sets Φ and Ψ as bellow

$$\Phi = \{\varphi : [0,\infty) \to [0,\infty) \text{ such that } \phi \text{ is nondecreasing and upper semi continuous}\},$$
$$\Psi = \{\psi : [0,\infty) \to [0,\infty) \text{ is continuous, strictly increasing, and surjection}\}.$$

Now we present our main results.

Theorem 3.2. *Let (X,d) be an f-orbitally complete metric space and $f : X \to X$ be a self mapping. Suppose that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

(H1) $\phi(t) < \psi(t)$ *for all* $t > 0$;
(H2) $\phi(0) = \psi(0) = 0$,

and we have

$$\psi(d(fx, fy)) \leq \phi(m(x,y)), \tag{3.1}$$

where

$$\begin{aligned} m(x,y) = \max\Big\{ & d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx), d(x,f^2(x)),\\ & d(fx,f^2(x)), d(y,f^2(x)), d(fx,f^2(y))\Big\}, \end{aligned} \tag{3.2}$$

if there exists a point $x \in X$ with bounded orbit, then f has a unique fixed point.

Proof. For $x \in X$ with bounded orbit and $1 \leq i \leq n-1$ and $1 \leq j \leq n$, we have

$$\begin{aligned} \psi(d(f^i x, f^j x)) &\leq \phi(m(f^{i-1}x, f^{j-1}x))\\ &= \phi\Big(\max\Big\{d(f^{i-1}x, f^{j-1}x), d(f^{i-1}x, f^i x), d(f^{j-1}(x), f^j x), d(f^{i-1}x, f^j x),\\ &\qquad d(f^{j-1}x, f^i x), d(f^{i-1}x, f^{i+1}(x)), d(f^i x, f^{i+1}(x)),\\ &\qquad d(f^{j-1}(x), f^{i+1}(x)), d(f^{i+1}(x), f^j(x))\Big\}\Big)\\ &\leq (\phi(\delta[O_f(x,n)]), \end{aligned}$$

where $\delta[O_f(x,n)] = \max\{d(f^i x, f^j x) : 0 \leq i,j \leq n\}$. Therefore,

$$d(f^i x, f^j x) \leq (\psi^{-1}\phi)\delta[O_f(x,n)]. \tag{3.3}$$

On the other hand, there exists $k_n x \leq$ such that

$$d(x, f^{k_n}x) \leq \delta[O_f(x,n)]. \tag{3.4}$$

Now, by using (3.3) and (3.4), for every $n, m \leq 1$ and $n < m$,

$$\begin{aligned} d(f^n x, f^m x) &\leq (\psi^{-1}\phi)(\delta[O(f^{n-1}x, m-n+1)]),\\ (\exists k_1 \quad \text{s.t.} \quad 0 \leq k_1 \leq m-n+1) &= (\psi^{-1}\phi)(d(f^{n-1}x, f^{n-1+k_1}x)), \end{aligned}$$

thus

$$(\exists k_2 \quad \text{s.t.} \quad 0 \leq k_2 \leq m-n+2) \leq (\psi^{-1}\phi)^2(d(f^{n-2}x, f^{n-2+k_2}x) \leq \cdots \leq (\psi^{-1}\phi)^n(\delta[O(x,n)]).$$

Using Lemma 3.1, we have $\lim_{n\to\infty}(\psi^{-1}\phi)^n(t) = 0$, so $\lim_{n\to\infty} d(f^n t, f^m t) = 0$, thus $\{f^n x\}$ is a Cauchy sequence and since X is f-orbitally complete, so we have $\lim_{n\to\infty} f^n x = z$.

Now we show that z is a fixed point of f.

Using inequality (3.1), if we consider $d(z, fz) > 0$, then

$$\psi(d(f^{n+1}x, fz)) \leq \phi(m(f^n x, z)) = \phi\Big(\max\Big\{d(f^n x, z), d(f^n x, f^{n+1}x), d(f^n(x), fz), d(zx, fz), d(z, f^{n+1}x), d(f^n x, f^{n+2}(x)), d(z, f^{n+2}(x)), d(f^{m-1}(x), f^{n+1}(x)), d(f^{n+1}(x), f^m(x))\Big\}\Big).$$

Since ϕ is l.s.c and ψ is continuous, we have

$$\psi(d(z, fz)) \leq \phi(d(z, fz)) < \psi(d(z, fz)),$$

and this is a contradiction. Thus $z = fz$ and z is a fixed point of f.

Finally we will show that z is a unique fixed point. For contradiction we suppose that z and w are two fixed points of f, then by inequality (3.1) we have

$$\psi(d(z, w) = \phi(d(fz, fw)) \leq \phi(m(z, w)),$$

then

$$\psi(d(z, w) \leq \phi(d(z, w)) < \psi(d(z, w)),$$

so $d(z, w) = 0$ and $z = w$, thus z is an unique fixed point of f. □

As a result of Theorem 3.2 we have the following theorem.

Theorem 3.3. *Let (X, d) be a f-orbitally complete metric space and $\psi \in \Psi$ and $\phi \in \Phi$ be two mappings which satisfy in following conditions:*

(H1) $\phi(t) < \psi(t)$ *for all* $t > 0$*;*

(H2) $(\psi - \phi)$ *be a strictly increasing, surjective function;*

(H3) $\phi(0) = \psi(0) = 0$.

If $f : X \to X$ be a self-mapping such that for all $x, y \in X$ we have:

$$\psi(d(fx, fy)) \leq \phi(m(x, y)), \tag{3.5}$$

where

$$m(x, y) = \max\Big\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx), d(x, f^2(x)), d(fx, f^2(x)), d(y, f^2(x)), d(fy, f^2(x))\Big\},$$

then f has a unique fixed point.

Proof. Let $x_0 \in X$ and n be any positive integer, then since ψ is increasing, there exists $C > 0$ (we can put $C = \psi(d(x_0, f(x_0)) + d(fx_0, f^k(x_0))) - \psi(d(x_0, f^k(x_0))) + 1)$ such that

$$\begin{aligned} \psi(d(x_0, f^k(x_0))) &\leq \psi(d(x_0, f(x_0)) + d(fx_0, f^k(x_0))) \\ &\leq \psi(d(fx_0, f^k(x_0))) + C \\ &\leq \phi(\delta[O(x_0, n)]) + C \quad \text{(by (3.5))}, \end{aligned}$$

so we have

$$(\psi - \phi)[\delta(O(x_0, n))] \leq C,$$

thus

$$\delta[O(x_0, n)] \leq (\psi - \phi)^{-1}(C), \tag{3.6}$$

therefore, by using (3.6) and since n is arbitrary, $\delta[O(x, n)]$ is bounded, therefore, by using Theorem 3.2, the proof is complete. $\square$

Corollary 3.4 ([1]). *Let (X, d) be a f-orbitally complete metric space. Suppose that $f : X \to X$ is a mapping on metric space X, and ϕ is upper semicontinuous, increasing comparison function such that $(id - \phi)$ is a strictly increasing surjection, and f for all $x, y \in X$, satisfies*

$$d(fx, fy) \leq \phi(\max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}).$$

Then f has a unique fixed point x^* in X.

Proof. We put $\psi(t) = t$ and then we apply Theorem 3.3. $\square$

Here we present a corollary of our new results.

Corollary 3.5 ([5]). *Let (X, d) be a metric space. Suppose that $f : X \to X$ is a generalized quasi-contraction, i.e., there exists $q \in [0, 1)$ such that for all $x, y \in X$,*

$$d(fx, fy) \leq qm(x, y),$$

where $m(x, y)$ is as in (3.2), and X is f-orbitally complete. Then we have

(1) *f has a unique fixed point $x^* \in X$;*
(2) *$\lim_{n\to\infty} f^n x = x^*$ for all $x \in X$.*

Proof. It is sufficient that in Theorem 3.3, we put $\psi(t) = t$ and $\varphi(t) = qt$. $\square$

Example 3.6. Let $X = \{1, 2, 3, 4, 5\}$ and define d as

$$\begin{aligned} &d(x, y) = 0 \iff x = y, \\ &d(1, 4) = d(4, 1) = d(1, 5) = d(5, 1) = 3, \\ &d(x, y) = 1 \quad \text{in other points.} \end{aligned}$$

We define $f : X \to X$ by

$$f1 = f2 = f3 = 1, f4 = 2, f5 = 3,$$

$\psi(t) = 2^t - 1$, and

$$\varphi(t) = \begin{cases} 0, & t \in [0, 1], \\ \sqrt{t}, & t > 1. \end{cases}$$

Easily we can show that f is not quasi-contraction for $x = 4$ and $y = 5$, since

$$d(4, 5) = d(4, f4) = d(5, f5) = d(4, f5) = d(5, f4) = 1, \text{ and } \psi(1) = 1,$$
$$d(f^2 4, 4) = d(f2, 4) = d(1, 4) = d(f^2 5, 5) = d(f3, 5) = d(1, 5) = 3, \text{ and } \varphi(3) = \sqrt{3}.$$

But f satisfies in Theorem 3.2 and hence it has a unique fixed point.

Here we prove same results for multi-valued functions.

Definition 3.7. Let $F : X \to BN(X)$ be a multi-valued mapping. For all $x, y \in X$, $\psi \in \Psi$ *and* $\varphi \in \Phi$, we define (ψ, ϕ)-quasi contraction by

$$\psi(\rho(Fx, Fy)) \leq \phi(m(x, y)),$$

which

$$m(x,y) = \max\Big\{d(x,y), \rho(x,Fx), D(x,Fy), D(y,Fx), D(x,F^2(x)), \\ D(Fx,F^2(x)), D(y,F^2(x)), D(Fy,F^2(x))\Big\},$$

and also ψ and ϕ satisfy

(H1) $\phi(t) < \psi(t)$ for all $t > 0$;

(H2) $\phi(0) = \psi(0) = 0$.

The following theorem presents the fixed point theorem for (ψ, ϕ)-quasi contraction mapping.

Theorem 3.8. *Let (X, d) be a metric space and $F : X \to BN(X)$ be a multi-valued map. Suppose that F is a (ψ, ϕ)-quasi contraction and X is an f-orbitally complete metric space. If there exists a point $x \in X$ with bounded orbit, then F has a unique fixed point x^* in X and $Fx^* = \{x^*\}$.*

Proof. Given $\alpha \in (0, 1)$ and defined a single valued mapping $T : X \to X$ by the following statement:

$$\forall x \in X \quad T(x) \in F(x) \quad \text{satisfies} \quad \phi(d(x, Tx)) \geq \alpha.\phi(\rho(x, Fx)).$$

By Definition 3.7 and the condition of F, we have:

$$\begin{aligned}
\psi(d(Tx,Ty)) &\leq \psi(\rho(Fx,Fy)) \\
&\leq \phi\Big[\max\Big\{d(x,y), \rho(x,Fy), D(x,Fy), D(y,Fx), \\
&\qquad D(x,F^2(x)), D(Fx,F^2(x)), D(y,F^2(x)), D(Fy,F^2(x)),\Big\}\Big] \\
&= \max\Big\{\phi(d(x,y)), \phi(\rho(x,Fy)), \phi(D(y,Fy)), \phi(D(y,Fx)), \\
&\qquad \phi(D(x,F^2x)), \phi(D(Fx,F^2(x))), \phi(D(y,F^2(x))), \phi(D(Fy,F^2(x)))\Big\} \\
&= \frac{1}{\alpha}\Big[\max\Big\{\alpha.\phi(d(x,y)), \alpha.\phi(\rho(x,Fy)), \alpha.\phi(D(y,Fy)), \alpha.\phi(D(y,Fx)), \\
&\qquad \alpha.\phi(D(x,F^2x)), \alpha.\phi(D(Fx,F^2(x))), \alpha.\phi(D(y,F^2(x))), \alpha.\phi(D(Fy,F^2(x)))\Big\}\Big]. \\
&\leq \frac{1}{\alpha}\Big[\max\Big\{\phi(d(x,y)), \phi(d(x,Ty)), \phi(d(y,Ty)), \phi(d(y,Tx)), \\
&\qquad \phi(d(x,T^2x)), \phi(d(Tx,T^2(x))), \phi(d(y,T^2(x))), \phi(d(Ty,T^2(x)))\Big\}\Big] \\
&= \frac{1}{\alpha}\phi\Big[\max\Big\{(d(x,y)), d(x,Ty), d(y,Ty), d(y,Tx), \\
&\qquad d(x,T^2x), d(Tx,T^2(x)), d(y,T^2(x)), d(Ty,T^2(x))\Big\}\Big]
\end{aligned}$$

for every $x, y \in X$.

We put $\phi^* = \frac{1}{\alpha}\phi$, since $\phi^* \in \phi$, then by Theorem 3.2 we conclude that T has a fixed point in x^*. Then $d(x^*, fx^*) = 0$ implies that $D(x^*, fx^*) = 0$, so x^* is a fixed point of F and $F(x^*) = \{x^*\}$. □

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