Available online at http://vonneumann-publishing.com/cna
Commun. Nonlinear Anal. 3 (2017), 23–33
Research Article

Communications in Nonlinear Analysis

Journal Homepage: www.vonneumann-publishing.com/cna

Some Gamidov like integral inequalities on time scales and applications

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Abstract

In the present paper, we establish some Gamidov like integral inequalities on time scales, the obtained results can be used as tools for the study of certain qualitative properties of solutions for differential and difference equations.

Keywords: Dynamic equations, time scale, integral inequality.
2010 MSC: 26D15, 26D20, 39A12.

1. Introduction

Integral inequalities play a fundamental role in the development of the linear and nonlinear integral equations theory. They are very useful and important tools in the study of the existence, uniqueness, boundedness, stability and other qualitative properties of solutions. One of the best known and widely used is the so-called Gronwall-Bellman integral inequality. In view of their important applications, many and various generalizations, extensions and variants have appeared in literature, we can mention [1, 4, 5, 7, 8, 11, 13, 14, 15, 17, 18, 21] and references cited therein.

Recently, Hilger [10] introduced the theory of time scales, in order to unify continuous and discrete analysis. Many authors have extended some fundamental integral inequalities used in the differential and integral equations theory on time scales for example [2, 3, 9, 12, 16, 19, 20].

In this paper, we investigate some Gamidov like integral inequalities on time scales. The obtained inequalities can be used as tool for the study of certain properties of dynamical equations on time scales.

2. Preliminaries

In this section, we recall without proof some fundamental definitions and results from the calculus on time scales, for more details about time scales, we advise the reader to refer to [6].

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Received 2017-02-13

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The forward jump operator σ on $\mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \in \mathbb{T}$ for all $t \in \mathbb{T}$, C_{rd} denotes the set of rd-continuous functions and the set $\mathbb{T}^{\kappa}$ which is derived from a time scale $\mathbb{T}$ as follows : if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$, otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$. We say that function $f : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t) f(t) \neq 0$, $t \in \mathbb{T}^{\kappa}$. We denote by $\mathcal{R}$ the set of all regressive and rd-continuous functions and $\mathcal{R}^{+} = \{f \in \mathcal{R} : 1 + \mu(t) f(t) > 0\}$. We denote by $[a, b]_{\mathbb{T}}$ the interval in $\mathbb{T}$ which is defined by $\{t \in \mathbb{T} : a \leq t \leq b\}$, where a and b are points in $\mathbb{T}$ with $a < b$.

Definition 2.1 ([6, Definition 1.10]). Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$ there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| < \varepsilon \ |\sigma(t) - s| \quad \text{for all } s \in U.$$

We call $f^{\Delta}(t)$ the delta (or Hilger) derivative of f at t.

Lemma 2.2 ([6, Theorem 1.117]). *Let $a \in \mathbb{T}^{\kappa}$, $b \in \mathbb{T}$ and assume $g : \mathbb{T} \times \mathbb{T}^{\kappa} \to \mathbb{R}$ is continuous at (t, t), where $t \in \mathbb{T}^{\kappa}$ with $t > a$. Also assume that $g^{\Delta}(t, .)$ is rd-continuous on $[a, \sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t, independent of $\tau \in [a, \sigma(t)]$, such that*

$$\left|g(\sigma(t), \tau) - g(s, \tau) - g^{\Delta}(t, \tau)(\sigma(t) - s)\right| < \varepsilon\, |\sigma(t) - s| \quad \textit{for all } s \in U, \tag{2.1}$$

where g^{Δ} denotes the derivative of g with respect to the first variable. Then

$$g(t) := \int_a^t f(t, \tau)\Delta\tau \quad \textit{implies } g^{\Delta}(t) = \int_a^t f^{\Delta}(t, \tau)\Delta\tau + f(\sigma(t), t).$$

Lemma 2.3 ([6, Comparison Theorem]). *Let $u, b \in C_{rd}$ and $a \in \mathcal{R}^{+}$. If*

$$u^{\Delta}(t) \leq a(t)\, u(t) + b(t), \quad \textit{for all } t \in \mathbb{T},$$

then

$$u(t) \leq u(t_0) e_a(t, t_0) + \int_{t_0}^t b(\tau) e_a(t, \sigma(\tau))\Delta\tau, \quad \textit{for all } t \in \mathbb{T}.$$

Lemma 2.4 ([12, Lemma 2]). *Assume that $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$, for any $\xi > 0$, we have*

$$a^{\frac{q}{p}} \leq \tfrac{q}{p} \xi^{\frac{q-p}{p}} a + \tfrac{p-q}{p} \xi^{\frac{q}{p}}.$$

3. Main results

Theorem 3.1. *Let $u(t), f(t), g(t), w(t), h(t) \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R}^{+})$, where $h(t) > 0$ and nondecreasing function for all $t \in [a, b]_{\mathbb{T}}$. Assume that*

$$\int_a^b w(t) e_Q(t, a)\Delta t < 1. \tag{3.1}$$

If the following inequality

$$u(t) \leq k(t) + h(t) \int_a^t f(s) \left(u(s) + \int_a^s g(\tau) u(\tau)\Delta\tau\right) \Delta s + \int_a^b w(t) u(t)\Delta t, \tag{3.2}$$

holds, then $u(t)$ has the following estimate

$$u(t) \le k(t) + h(t)\left(\phi(t) + \frac{e_Q(t,a)\int_a^b w(t)\phi(t)\Delta t}{1-\int_a^b w(t)e_Q(t,a)\Delta t}\right), \tag{3.3}$$

where

$$L(t) = f(t)k(t) + f(t)\int_a^t g(\tau)k(\tau)\Delta\tau, \tag{3.4}$$

$$Q(t) = f(t)h(t) + f(t)\int_a^t g(\tau)h(\tau)\Delta\tau, \tag{3.5}$$

and

$$\phi(t) = \int_a^t L(\tau)e_Q(t,\sigma(\tau))\Delta\tau + e_Q(t,a)\int_a^b w(t)h^{-1}(t)k(t)\Delta t. \tag{3.6}$$

Proof. Since $h(t)$ is a nondecreasing function, we can restate (3.2) as follows

$$u(t) \le k(t) + h(t)\left(\int_a^t f(s)\left(u(s) + \int_a^s g(\tau)u(\tau)\Delta\tau\right)\Delta s + \int_a^b w(t)h^{-1}(t)u(t)\Delta t\right).$$

Denoting by $z(t)$

$$z(t) = \int_a^t f(s)\left(u(s) + \int_a^s g(\tau)u(\tau)\Delta\tau\right)\Delta s + \int_a^b w(t)h^{-1}(t)u(t)\Delta t. \tag{3.7}$$

Clearly $z(t)$ is nonnegative, nondecreasing,

$$u(t) \le k(t) + h(t)z(t), \tag{3.8}$$

and

$$z(a) = \int_a^b w(t)h^{-1}(t)u(t)\Delta t. \tag{3.9}$$

Differentiating (3.7) with respect to t we get

$$z^{\Delta t}(t) = f(t)\left(u(t) + \int_a^t g(\tau)u(\tau)\Delta\tau\right). \tag{3.10}$$

Substituting (3.8) into (3.10), we get

$$z^{\Delta t}(t) \le f(t)k(t) + f(t)\int_a^t g(\tau)k(\tau)\Delta\tau + f(t)h(t)z(t) + f(t)\int_a^t g(\tau)h(\tau)z(\tau)\Delta\tau. \tag{3.11}$$

Since $z(t)$ is monotonic nondecreasing, (3.11) gives

$$z^{\Delta t}(t) \le L(t) + Q(t)z(t), \tag{3.12}$$

where $L(t)$ and $Q(t)$ are given by (3.4) and (3.5) respectively. Now, using Lemma 2.3 for (3.12), we obtain

$$z(t) \le z(a)e_Q(t,a) + \int_a^t L(\tau)e_Q(t,\sigma(\tau))\Delta\tau. \tag{3.13}$$

Substituting (3.9) into (3.13), we get

$$z(t) \le \int_a^t L(\tau)e_Q(t,\sigma(\tau))\Delta\tau + e_Q(t,a)\int_a^b w(t)h^{-1}(t)u(t)\Delta t. \tag{3.14}$$

Using (3.8) in (3.14), we have

$$z(t) \leq \phi(t) + e_Q(t,a)\int_a^b w(t)z(t)\Delta t, \tag{3.15}$$

where $\phi(t)$ is defined as in (3.6). Since $\int_a^b w(t)z(t)\Delta t$ is a constant, multiplying both sides of (3.15) by $w(t)$ then integrating the result with respect to t from a to b, we get

$$\int_a^b w(t)z(t)\Delta t \leq \int_a^b w(t)\phi(t)\Delta t + \left(\int_a^b w(t)e_Q(t,a)\Delta t\right)\int_a^b w(t)z(t)\Delta t. \tag{3.16}$$

From (3.1) and (3.16) we have

$$\int_a^b w(t)z(t)\Delta t \leq \frac{\int_a^b w(t)\phi(t)\Delta t}{1-\int_a^b w(t)e_Q(t,a)\Delta t}. \tag{3.17}$$

Using (3.17) in (3.15), we obtain

$$z(t) \leq \phi(t) + \frac{e_Q(t,a)\int_a^b w(t)\phi(t)\Delta t}{1-\int_a^b w(t)e_Q(t,a)\Delta t}. \tag{3.18}$$

Substituting (3.18) into (3.8), we obtain the desired result. □

Theorem 3.2. *Suppose that all the assumptions of Theorem 3.1 are satisfied. And let $p,q,m,r \in \mathbb{R}_0^+$ such that $p \geq q > 0$, $p \geq r > 0$, $p \geq m > 0$ and $\xi > 0$. Assume that*

$$\frac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(t)e_Q(t,a)\Delta t < 1. \tag{3.19}$$

If the following inequality

$$u^p(t) \leq k(t) + h(t)\int_a^t f(s)\left(u^q(s) + \int_a^s g(\tau)u^m(\tau)\Delta\tau\right)\Delta s + \int_a^b w(s)u^r(s)\Delta s, \tag{3.20}$$

holds, then $u(t)$ has the following estimate

$$u(t) \leq \left\{k(t) + h(t)\left(G(t) + \frac{\frac{r}{p}\xi^{\frac{r-p}{p}}e_Q(t,a)\int_a^b w(s)G(s)\Delta s}{1-\frac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(s)e_Q(s,a)\Delta s}\right)\right\}^{\frac{1}{p}}, \tag{3.21}$$

where

$$L(t) = f(t)\left(\frac{q}{p}\xi^{\frac{q-p}{p}}k(t) + \frac{p-q}{p}\xi^{\frac{q}{p}}\right) + \int_a^t\left(\frac{m}{p}\xi^{\frac{m-p}{p}}k(\tau) + \frac{p-m}{p}\xi^{\frac{m}{p}}\right)g(\tau)\Delta\tau, \tag{3.22}$$

$$Q(t) = \frac{q}{p}\xi^{\frac{q-p}{p}}f(t)h(t) + \frac{m}{p}\xi^{\frac{m-p}{p}}\int_a^t g(\tau)h(\tau)\Delta\tau, \tag{3.23}$$

and

$$G(t) = e_Q(t,a)\int_a^b w(s)h^{-1}(s)\left(\frac{r}{p}\xi^{\frac{r-p}{p}}k(s) + \frac{p-r}{p}\xi^{\frac{r}{p}}\right)\Delta s + \int_a^t L(\tau)e_Q(t,\sigma(\tau))\Delta\tau. \tag{3.24}$$

Proof. Denoting by $z(t)$

$$z(t) = \int_a^t f(s)\left(u^q(s) + \int_a^s g(\tau)u^m(\tau)\Delta\tau\right)\Delta s + \int_a^b w(s)h^{-1}(s)u^r(s)\Delta s. \tag{3.25}$$

Clearly, $z(t)$ is nonnegative, nondecreasing,

$$u(t) \leq \{k(t) + h(t)z(t)\}^{\frac{1}{p}}, \tag{3.26}$$

and

$$z(a) = \int_a^b w(s)h^{-1}(s)u^r(s)\Delta s. \tag{3.27}$$

Differentiating (3.25) with respect to t, then using (3.26), we obtain

$$z^{\Delta t}(t) \leq \; f(t)\left\{k(t) + h(t)z(t)\right\}^{\frac{q}{p}} + \int_a^t g(\tau)\left\{k(\tau) + h(\tau)z(\tau)\right\}^{\frac{m}{p}} \Delta\tau. \tag{3.28}$$

From Lemma 2.4 and monotonicity of $z\,(t)$, (3.28) gives

$$\begin{aligned} z^{\Delta t}(t) \;\; &\leq \;\; f(t)\left(\tfrac{q}{p}\xi^{\frac{q-p}{p}}k(t) + \tfrac{p-q}{p}\xi^{\frac{q}{p}}\right) + \int_a^t \left(\tfrac{m}{p}\xi^{\frac{m-p}{p}}k(\tau) + \tfrac{p-m}{p}\xi^{\frac{m}{p}}\right) g(\tau)\Delta\tau \\ &\quad + \left(\tfrac{q}{p}\xi^{\frac{q-p}{p}}f(t)h(t) + \tfrac{m}{p}\xi^{\frac{m-p}{p}}\int_a^t g(\tau)h(\tau)\Delta\tau\right) z(t) \\ &= \;\; L(t) + Q(t)z(t), \end{aligned} \tag{3.29}$$

where $L(t)$ and $Q(t)$ are given by (3.22) and (3.23) respectively. Apply Lemma 2.3 for (3.29), we obtain

$$z(t) \leq z(a)e_Q(t,a) + \int_a^t L(\tau)e_Q(t,\sigma(\tau))\Delta\tau. \tag{3.30}$$

Substituting (3.27) into (3.30), then using (3.26) in the resultant inequality, we get

$$z(t) \leq e_Q(t,a)\int_a^b w(s)h^{-1}(s)\left\{k(s) + h(s)z(s)\right\}^{\frac{r}{p}}\Delta s + \int_a^t L(\tau)e_Q(t,\sigma(\tau))\Delta\tau. \tag{3.31}$$

According to Lemma 2.4, (3.31) gives

$$z(t) \leq G(t) + \tfrac{r}{p}\xi^{\frac{r-p}{p}}e_Q(t,a)\int_a^b w(s)z(s)\Delta s, \tag{3.32}$$

where $G(t)$ is defined as in (3.24). Now, multiplying both sides of (3.32) by $w(t)$ and then integrating the result with respect to t over $[a,b]$, we get

$$\int_a^b w(t)z(t)\Delta t \leq \int_a^b w(t)G(t)\Delta t + \left(\tfrac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(t)e_Q(t,a)\Delta t\right)\int_a^b w(s)z(s)\Delta s. \tag{3.33}$$

From (3.33) and (3.19), we have

$$\int_a^b w(t)z(t)\Delta t \leq \frac{\int_a^b w(s)G(s)\Delta s}{1-\frac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(s)e_Q(s,a)\Delta s}. \tag{3.34}$$

Substituting (3.34) in (3.32), we get

$$z(t) \leq G(t) + \frac{\frac{r}{p}\xi^{\frac{r-p}{p}}e_Q(t,a)\int_a^b w(s)G(s)\Delta s}{1-\frac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(s)e_Q(s,a)\Delta s}. \tag{3.35}$$

Using (3.35) in (3.26), we get the desired result. □

Theorem 3.3. *Let $u(t), k(t), h(t), w(t) \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}^+)$ where $h(t)$ and $k(t)$ are nondecreasing functions with $h(t) \neq 0$ for all $t \in [a,b]_{\mathbb{T}}$. Let $g(t,s)$ and $f(t,s)$ be defined as in Lemma 2.2 such that $g^{\Delta}(t,s) \geq 0$ and $f^{\Delta}(t,s) \geq 0$ for all $t \geq s$ satisfying (2.1), and assume that*

$$\int_a^b w(t) e_Q(t,a) \Delta t < 1. \tag{3.36}$$

If the following inequality

$$u(t) \leq k(t) + h(t) \int_a^t f(t,s) \left(u(s) + \int_a^s g(s,\tau) u(\tau) \Delta\tau \right) \Delta s + \int_a^b w(t) u(t) \Delta t \tag{3.37}$$

holds, then $u(t)$ has the following estimate

$$u(t) \leq k(t) + h(t) \left\{ G(t) + \frac{e_Q(t,a) \int_a^b w(t) G(t) \Delta t}{1 - \int_a^b w(t) e_Q(t,a) \Delta t} \right\}, \tag{3.38}$$

where

$$\begin{aligned} L(t) = {} & \int_a^t f^{\Delta t}(t,s) \left(k(s) + \int_a^s g(s,\tau) k(\tau) \Delta\tau \right) \Delta s \\ & + f(\sigma(t), t) \left(k(t) + \int_a^t g(t,\tau) k(\tau) \Delta\tau \right), \end{aligned} \tag{3.39}$$

$$\begin{aligned} Q(t) = {} & \int_a^t f^{\Delta t}(t,s) \left(h(s) + \int_a^s g(s,\tau) h(\tau) \Delta\tau \right) \Delta s \\ & + f(\sigma(t), t) \left(h(t) + \int_a^t g(t,\tau) h(\tau) \Delta\tau \right), \end{aligned} \tag{3.40}$$

and

$$G(t) = e_Q(t,a) \int_a^b w(t) h^{-1}(t) k(t) \Delta t + \int_a^t L(\tau) e_Q(t, \sigma(\tau)) \Delta\tau. \tag{3.41}$$

Proof. Define a function $z(t)$ as follows

$$z(t) = \int_a^t f(t,s) \left(u(s) + \int_a^s g(s,\tau) u(\tau) \Delta\tau \right) \Delta s + \int_a^b w(t) h^{-1}(t) u(t) \Delta t. \tag{3.42}$$

Clearly $z(t)$ is nonnegative, nondecreasing,

$$u(t) \leq k(t) + h(t) z(t), \tag{3.43}$$

and

$$z(a) = \int_a^b w(t) h^{-1}(t) u(t) \Delta t. \tag{3.44}$$

Differentiating (3.42) with respect to t, we obtain

$$\begin{aligned} z^{\Delta t}(t) = {} & f(\sigma(t), t) \left(u(t) + \int_a^t g(t,\tau) u(\tau) \Delta\tau \right) \\ & + \int_a^t f^{\Delta t}(t,s) \left(u(s) + \int_a^s g(s,\tau) u(\tau) \Delta\tau \right) \Delta s. \end{aligned} \tag{3.45}$$

Substituting (3.43) into (3.45), we get

$$
\begin{aligned}
z^{\Delta t}(t) \leq\; & f(\sigma(t), t)k(t) + f(\sigma(t), t)\int_a^t g(t,\tau)k(\tau)\Delta\tau \\
& + \int_a^t f^{\Delta t}(t,s)k(s)\Delta s + \int_a^t f^{\Delta t}(t,s)\int_a^s g(s,\tau)k(\tau)\Delta\tau\Delta s \\
& + f(\sigma(t), t)h(t)z(t) + f(\sigma(t), t)\int_a^t g(t,\tau)h(\tau)z(\tau)\Delta\tau \\
& + \int_a^t f^{\Delta t}(t,s)h(s)z(s)\Delta s + \int_a^t f^{\Delta t}(t,s)\int_a^s g(s,\tau)h(\tau)z(\tau)\Delta\tau\Delta s.
\end{aligned}
\tag{3.46}
$$

Since $z(t)$ is monotonic nondecreasing, we can restate (3.46) as follows

$$
z^{\Delta t}(t) \leq L(t) + Q(t)z(t), \tag{3.47}
$$

where $L(t)$ and $Q(t)$ are defined by (3.39) and (3.40) respectively. An application of Lemma 2.3 for (3.47), gives

$$
z(t) \leq z(a)e_Q(t,a) + \int_a^t L(\tau)e_Q(t,\sigma(\tau))\Delta\tau. \tag{3.48}
$$

Using (3.44) in (3.48), we get

$$
z(t) \leq e_Q(t,a)\int_a^b w(t)h^{-1}(t)u(t)\Delta t + \int_a^t L(\tau)e_Q(t,\sigma(\tau))\Delta\tau. \tag{3.49}
$$

Substituting (3.43) in (3.49), we obtain

$$
z(t) \leq G(t) + e_Q(t,a)\int_a^b w(t)z(t)\Delta t, \tag{3.50}
$$

where $G(t)$ is defined by (3.41). Now, multiplying both sides of (3.50) by $w(t)$, then integrating the result with respect to t from a to b, we get

$$
\int_a^b w(t)z(t)\Delta t \leq \int_a^b w(t)G(t)\Delta t + \left(\int_a^b w(t)e_Q(t,a)\Delta t\right)\int_a^b w(t)z(t)\Delta t. \tag{3.51}
$$

From (3.36) and (3.51), we have

$$
\int_a^b w(t)z(t)\Delta t \leq \frac{\int_a^b w(t)G(t)\Delta t}{1 - \int_a^b w(t)e_Q(t,a)\Delta t}. \tag{3.52}
$$

Substituting (3.52) in (3.50), we get

$$
z(t) \leq G(t) + \frac{e_Q(t,a)\int_a^b w(t)G(t)\Delta t}{1 - \int_a^b w(t)e_Q(t,a)\Delta t}. \tag{3.53}
$$

Combining (3.53) and (3.43), we obtain the desired result. □

Theorem 3.4. *Suppose that all the assumptions of Theorem 3.3 are satisfied, let $p, q, m, r \in \mathbb{R}_0^+$ such that $p \geq q > 0$, $p \geq r > 0$, $p \geq m > 0$ and $\xi > 0$. Assume that*

$$
\tfrac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(t)e_Q(t,a)\Delta t < 1. \tag{3.54}
$$

If the following inequality

$$u^p(t) \leq k(t) + h(t) \int_a^t f(t,s) \left(u^q(s) + \int_a^s g(s,\tau) u^m(\tau) \Delta\tau \right) \Delta s + \int_a^b w(t) u^r(t) \Delta t \tag{3.55}$$

holds, then $u(t)$ has the following estimate

$$u(t) \leq \left\{ k(t) + h(t) \left(G(t) + \frac{\frac{r}{p} \xi e_Q(t,a) \int_a^b w(t) G(t) \Delta t}{1 - \frac{r}{p} \xi^{\frac{r-p}{p}} \int_a^b w(t) e_Q(t,a) \Delta t} \right) \right\}^{\frac{1}{p}}, \tag{3.56}$$

where

$$\begin{aligned} L(t) &= f(\sigma(t), t) \left(\tfrac{q}{p} \xi^{\frac{q-p}{p}} k(t) + \tfrac{p-q}{p} \xi^{\frac{q}{p}} \right) \\ &\quad + f(\sigma(t), t) \int_a^t g(t,\tau) \left(\tfrac{m}{p} \xi^{\frac{m-p}{p}} k(\tau) + \tfrac{p-m}{p} \xi^{\frac{m}{p}} \right) \Delta\tau \\ &\quad + \int_a^t f^{\Delta t}(t,s) \left(\tfrac{q}{p} \xi^{\frac{q-p}{p}} k(s) + \tfrac{p-q}{p} \xi^{\frac{q}{p}} \right) \Delta s \\ &\quad + \int_a^t f^{\Delta t}(t,s) \left(\int_a^s g(s,\tau) \left(\tfrac{m}{p} \xi^{\frac{m-p}{p}} k(\tau) + \tfrac{p-m}{p} \xi^{\frac{m}{p}} \right) \Delta\tau \right) \Delta s, \end{aligned} \tag{3.57}$$

$$\begin{aligned} Q(t) &= \tfrac{q}{p} \xi^{\frac{q-p}{p}} f(\sigma(t), t) h(t) + \tfrac{q}{p} \xi^{\frac{q-p}{p}} \int_a^t f^{\Delta t}(t,s) h(s) \Delta s \\ &\quad + f(\sigma(t), t) \tfrac{m}{p} \xi^{\frac{m-p}{p}} \int_a^t g(t,\tau) h(\tau) \Delta\tau \\ &\quad + \tfrac{m}{p} \xi^{\frac{m-p}{p}} \int_a^t f^{\Delta t}(t,s) \int_a^s g(s,\tau) h(\tau) \Delta\tau \Delta s, \end{aligned} \tag{3.58}$$

and

$$G(t) = \int_a^t L(\tau) e_Q(t, \sigma(\tau)) \Delta\tau + e_Q(t,a) \int_a^b w(t) h^{-1}(t) \left(\tfrac{r}{p} \xi^{\frac{r-p}{p}} k(t) + \tfrac{p-r}{p} \xi^{\frac{r}{p}} \right) \Delta t. \tag{3.59}$$

Proof. Denoting by $z(t)$

$$z(t) = \int_a^t f(t,s) \left(u^q(s) + \int_a^s g(s,\tau) u^m(\tau) \Delta\tau \right) \Delta s + \int_a^b w(t) h^{-1}(t) u^r(t) \Delta t. \tag{3.60}$$

It is clear that $z(t)$ is nonnegative, nondecreasing,

$$u(t) \leq \{k(t) + h(t) z(t)\}^{\frac{1}{p}}, \tag{3.61}$$

and

$$z(a) = \int_a^b w(t) h^{-1}(t) u^r(t) \Delta t. \tag{3.62}$$

Differentiating (3.60) with respect to t, we have

$$\begin{aligned} z^{\Delta t}(t) &= f(\sigma(t), t) \left(u^q(t) + \int_a^t g(t,\tau) u^m(\tau) \Delta\tau \right) \\ &\quad + \int_a^t f^{\Delta t}(t,s) \left(u^q(s) + \int_a^s g(s,\tau) u^m(\tau) \Delta\tau \right) \Delta s. \end{aligned} \tag{3.63}$$

Substituting (3.61) in (3.63), then using Lemma 2.4 for the resultant inequality, we obtain

$$\begin{aligned} z^{\Delta t}(t) &\leq L(t) + \tfrac{q}{p}\xi^{\frac{q-p}{p}} f(\sigma(t), t)h(t)z(t) \\ &+ f(\sigma(t), t)\tfrac{m}{p}\xi^{\frac{m-p}{p}} \int_a^t g(t,\tau)h(\tau)z(\tau)\Delta\tau \\ &+ \tfrac{q}{p}\xi^{\frac{q-p}{p}} \int_a^t f^{\Delta t}(t,s)h(s)z(s)\Delta s \\ &+ \tfrac{m}{p}\xi^{\frac{m-p}{p}} \int_a^t f^{\Delta t}(t,s)\int_a^s g(s,\tau)h(\tau)z(\tau)\Delta\tau\Delta s, \end{aligned}$$

where $L(t)$ is given by (3.57). Since $z(t)$ is monotonic nondecreasing the above inequality can be restated as follows

$$z^{\Delta t}(t) \leq L(t) + Q(t)z(t), \tag{3.64}$$

where $Q(t)$ is defined in (3.58). According to Lemma 2.3, we have

$$z(t) \leq z(a)e_Q(t,a) + \int_a^t L(\tau)e_Q(t,\sigma(\tau))\Delta\tau. \tag{3.65}$$

Substituting (3.62) in (3.65), we get

$$z(t) \leq e_Q(t,a)\int_a^b w(t)h^{-1}(t)u^r(t)\Delta t + \int_a^t L(\tau)e_Q(t,\sigma(\tau))\Delta\tau. \tag{3.66}$$

Using (3.61) in (3.66), then applying Lemma 2.3 for the resultant inequality, we obtain

$$z(t) \leq G(t) + \tfrac{r}{p}\xi^{\frac{r-p}{p}} e_Q(t,a)\int_a^b w(t)z(t)\Delta t, \tag{3.67}$$

where $G(t)$ is defined in (3.59). Now, multiplying both sides of (3.67) by $w(t)$ and then integrating the result with respect to t over $[a,b]$, we get

$$\int_a^b w(t)z(t)\Delta t \;\; \leq \;\; \int_a^b w(t)G(t)\Delta t + \left(\tfrac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(t)e_Q(t,a)\Delta t\right)\int_a^b w(t)z(t)\Delta t. \tag{3.68}$$

Thus, from (3.54) and (3.68), we have

$$\int_a^b w(t)z(t)\Delta t \leq \frac{\int_a^b w(t)G(t)\Delta t}{1-\frac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(t)e_Q(t,a)\Delta t}. \tag{3.69}$$

Substituting (3.69) in (3.67), we obtain

$$z(t) \leq G(t) + \frac{\frac{r}{p}\xi^{\frac{r-p}{p}}e_Q(t,a)\int_a^b w(t)G(t)\Delta t}{1-\frac{r}{p}\xi^{\frac{r-p}{p}}\int_a^b w(t)e_Q(t,a)\Delta t}. \tag{3.70}$$

Combining (3.70) and (3.61), we get the desired result. □

4. Applications

In this section, we present two applications. Let us consider the following dynamic equation on time scales

$$u^p(t) = k(t) + h(t) \int_a^t \varphi\left(t, s, u(s), \int_a^s \chi(s, \tau, u(\tau))\, \Delta\tau\right) \Delta s + \int_a^b \psi(s, u(s))\, \Delta s, \tag{4.1}$$

where p is positive constant, $k, h : \mathbb{T} \to \mathbb{R}^+$ are right-dense continuous functions on $\mathbb{T}$ such as h is a nondecreasing function, $\varphi : \mathbb{T} \times \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ be right-dense continuous on $\mathbb{T} \times \mathbb{T}$ and continuous on $\mathbb{R}^2$, $\chi : \mathbb{T} \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be right-dense continuous on $\mathbb{T} \times \mathbb{T}$ and continuous on $\mathbb{R}$, and $\psi : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be right-dense continuous on $\mathbb{T}$ and continuous on $\mathbb{R}$ for all t such that $a \le \tau \le s \le t \le b$.

Proposition 4.1. *Let $p = 1$, assume that*

$$\begin{aligned} \left|\varphi\left(t, s, u(s), \int_a^s \chi(s, \tau, u(\tau))\, \Delta\tau\right)\right| &\le f(s) \left[|u(s)| + \int_a^s |\chi(s, \tau, u(\tau))|\, \Delta\tau\right] \\ |\chi(s, \tau, u(\tau))| &\le g(\tau)\, |u(\tau)| \\ |\psi(s, u(s))| &\le w(s)\, |u(s)|, \end{aligned} \tag{4.2}$$

where $f, g,$ and w satisfy the hypotheses of Theorem 3.1. If $u(t)$ is any solution of (4.1)-(4.2), then $u(t)$ satisfies the following estimate

$$|u(t)| \le k(t) + h(t) \left(\phi(t) + \frac{e_Q(t,a) \int_a^b w(t)\phi(t)\Delta t}{1 - \int_a^b w(t) e_Q(t,a)\Delta t}\right), \tag{4.3}$$

where Q and ϕ are defined in (3.5) and (3.6) respectively.

Proof. Let $u(t)$ be a solution of (4.1), taking $p = 1$, then using the properties of modulus, (4.1) becomes

$$|u(t)| \le k(t) + h(t) \int_a^t \left|\varphi\left(t, s, u(s), \int_a^s \chi(s, \tau, u(\tau))\, \Delta\tau\right)\right| \Delta s + \int_a^b |\psi(t, u(t))|\, \Delta t. \tag{4.4}$$

Using (4.2) and the hypotheses given on f, g, h and w, we can restate (4.4) as follows

$$|u(t)| \le k(t) + h(t) \int_a^t f(s) \left[|u(s)| + \int_a^s g(\tau)\, |u(\tau)|\, \Delta\tau\right] \Delta s + \int_a^b w(t)\, |u(t)|\, \Delta t. \tag{4.5}$$

Now, an application of Theorem 3.1 for (4.5) gives the estimate (4.3). □

Proposition 4.2. *Assume that*

$$\begin{aligned} \left|\varphi\left(t, s, u(s), \int_a^s \chi(s, \tau, u(\tau))\, \Delta\tau\right)\right| &\le f(t, s) \left(|u(s)|^q + \int_a^t |\chi(s, \tau, u(\tau))|\, \Delta\tau\right) \\ |\chi(s, \tau, u(\tau))| &\le g(s, \tau)\, |u(\tau)|^m \\ |\psi(s, u(s))| &\le w(s)\, |u(s)|^r, \end{aligned} \tag{4.6}$$

where p, q, m, r and $f(t, s), g(t, s), w(s)$ satisfy the hypotheses of Theorem 3.4. If $u(t)$ is any solution of (4.1) and (4.6), then $u(t)$ satisfies the following estimate

$$|u(t)| \le \left\{k(t) + h(t) \left(G(t) + \frac{\frac{r}{p}\xi e_Q(t,a) \int_a^b w(t) G(t)\Delta t}{1 - \frac{r}{p}\xi^{\frac{r-p}{p}} \int_a^b w(t) e_Q(t,a)\Delta t}\right)\right\}^{\frac{1}{p}}, \tag{4.7}$$

where L, Q and G are defined as in (3.57)-(3.59) respectively.

Proof. Let $u(t)$ be a solution of (4.1), using the properties of modulus, we obtain

$$|u(t)|^p \leq k(t) + h(t) \int_a^t \left| \varphi \left(t, s, u(s), \int_a^s \chi(s, \tau, u(\tau)) \Delta\tau \right) \right| \Delta s + \int_a^b |\psi(s, u(s))| \Delta s. \tag{4.8}$$

From (4.6) and the hypotheses given on f, g, h and w, we can restate (4.8) as follows

$$|u(t)|^p \leq f(t) + h(t) \int_a^t f(t,s) \left(|u(t)|^q + \int_a^s g(s,\tau) |u(t)|^m \Delta\tau \right) \Delta s + \int_a^b w(s) |u(s)|^r \Delta s. \tag{4.9}$$

Now, an application of Theorem 3.4 for (4.9) gives the estimate (4.7). □

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