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A mollified solution of a nonlinear inverse heat conduction problem

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Abstract

In this paper a nonlinear inverse heat conduction problem in one dimensional space is considered. This inverse problem reformulate as an auxiliary inverse problem. Ill-posedness is identified as one of the main characteristics of the inverse problems. So, a numerical algorithm based on the combination of discrete mollification and space marching method is applied to conquer ill-posedness of the auxiliary inverse problem. Moreover, a proof of stability and convergence of the aforementioned algorithm is provided. Eventually, the efficiency of this method is illustrated by a numerical example.

Keywords: Nonlinear inverse heat conduction problem, discrete mollification, space marching method, stability, convergence.
2010 MSC: 65N12, 65N20, 65N21.

1. Introduction

The inverse heat conduction problems are widely practiced in many branches of physics, science and engineering. These problems belong to the class of ill-posed problems in the sense that solution (if it exists) does not continuously depend on the data. Thus, it is impossible to solve this problem using the classical numerical methods and requires special techniques to be employed.

In the context of approximation method for this problem, many approaches have been investigated. Such as function specification [1, 2], Newton-Raphson [7], Tikhonov regularization [25, 23, 24], Levenberg-Marquardt (LM) [20], conjugate gradient [14, 20, 21], recursive least squared method [15] and singular value decomposition method [10, 11, 12, 22].

However, most of the essays in this issue have been limited to the linear problems and just a few studies have been done on the nonlinear inverse heat conduction problems. The aim of this paper is to study a nonlinear inverse heat conduction problem in one dimensional space. Our strategy is to obtain the regularized solution of the proposed problem by a stable and convergent algorithm based on discrete mollification and

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space marching method. Discrete mollification is a convolution based filtering procedure that is appropriate for the regularization of ill-posed problems and for the stabilization of explicit schemes for the solution of partial differential equations [1, 18].

Mollification method is recognized as a reliable regularization method that has been widely applied to solve many ill-posed problems [18]. The idea of this method is very simple [13]: if the data of the problem are not clear and only an approximate amount of data is accessible, it is recommended to find out a sequence of mollification operators to map improper data into well-posed classes of the problem (mollify the improper data). Consequently, the intended problem will be a well-posed one.

The remainder of the paper is organized as follows: In the forthcoming section, we will introduce the mathematical formulation of the nonlinear inverse heat conduction problem and reformulate this problem. Section 3 is devoted to review basic facts about the discrete mollification operator. In section 4, we first regularize reformulated problem then we solve regularized problem by space marching method. Stability and convergence analysis of the numerical algorithm is provided in section 5 and the last section include illustrative examples.

2. Problem description

2.1. Direct problem

Let us consider a nonlinear initial boundary value heat conduction problem of determining $u(x,t)$:

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial}{\partial x}\{a(u)\frac{\partial u(x,t)}{\partial x}\} = f(x,t), \quad 0 < x < 1, \quad 0 < t < T, \tag{2.1}$$

$$u(x,0) = \varphi(x), \quad 0 \le x \le 1, \tag{2.2}$$

$$a(u(0,t))\frac{\partial u(0,t)}{\partial x} = g_1(t), \quad 0 \le t \le T, \tag{2.3}$$

$$u(1,t) = \chi(t), \quad 0 \le t \le T, \tag{2.4}$$

where the diffusion coefficient $a(u)$ is a positive and bounded function and $\varphi(x)$, $g_1(t)$ and $\chi(t)$ are continuous known functions. This problem is called a direct heat conduction problem. The existence and uniqueness of the solution of this problem and more applications and background of the problem are discussed in [8, 18].

2.2. Inverse problem

Corresponding to the direct problem (2.1)-(2.4), we consider an inverse problem in which the initial condition $\varphi(x)$ and the boundary condition $\chi(t)$ are unknown. The unknown initial and boundary functions should be calculated using known temperatures which are described mathematically in below:

$$u(x,T) = g_2(x), \quad 0 \le x \le 1, \tag{2.5}$$

$$u(0,t) = g_3(t), \quad 0 \le t \le T. \tag{2.6}$$

Now using the transformation

$$v(x,t) = T_a(u(x,t)) = \int_0^{u(x,t)} a(s)ds,$$

which was applied by cannon [4, 5], the proposed inverse problem reduces to

$$\frac{\partial v(x,t)}{\partial t} - A(v)\frac{\partial^2 v(x,t)}{\partial x^2} = A(v)f(x,t), \quad 0 < x < 1, \quad 0 < t < T, \tag{2.7}$$

$$v(x,0) = \varphi_1(x), \quad 0 \le x \le 1, \tag{2.8}$$

$$\frac{\partial v(0,t)}{\partial x} = g_1(t), \quad 0 \le t \le T, \tag{2.9}$$

$$v(1,t) = \chi_1(t), \quad 0 \le t \le T, \tag{2.10}$$

with overspecified conditions

$$v(x,T) = g_4(x), \quad 0 \le x \le 1, \tag{2.11}$$
$$v(0,t) = g_5(t), \quad 0 \le t \le T, \tag{2.12}$$

where

$$A(v) = a(T_a^{-1}(v)), \varphi_1(x) = \int_0^{u(x,0)} a(s)ds, \chi_1(t) = \int_0^{u(1,t)} a(s)ds, g_4(x) = \int_0^{u(x,T)} a(s)ds$$

and

$$g_5(t) = \int_0^{u(0,t)} a(s)ds.$$

Note that $T_a'(x) = a(x) > 0$, so $T_a(x)$ is an invertible function. The problem (2.7)-(2.12) is equivalent to the problem (2.1)-(2.6). The problem (2.7)-(2.12) is somewhat simpler than (2.1)-(2.6), particularly in that the boundary condition (2.9) is now linear. Once $v(x,t)$ is known numerically, the unknown $u(x,t)$ can be calculated through

$$u(x,t) = T_a^{-1}(v(x,t)).$$

Without loss of generality, we suppose that, instead of $g_1(t)$, $g_4(x)$ and $g_5(t)$ we have approximate amounts of these functions presented as $g_1^\varepsilon(t)$, $g_4^\varepsilon(t)$ and $g_5^\varepsilon(t)$ such that

$$\|g_1^\varepsilon(t) - g_1(t)\|_\infty \le \varepsilon,$$
$$\|g_4^\varepsilon(x) - g_4(x)\|_\infty \le \varepsilon,$$
$$\|g_5^\varepsilon(t) - g_5(t)\|_\infty \le \varepsilon.$$

In the following, a stable and convergent numerical algorithm based on discrete mollification and space marching method will be introduced to find the solution of problem (2.7)-(2.12). Because of the presence noise in the problem's data and ill-posedness of the problem (2.7)-(2.12), we first regularize the proposed problem by discrete mollification method.

3. A summary of discrete mollification method

In this section from [6, 16], the basic idea of discrete mollification method is introduced. For more information about this method see [19].
Let $G = \{g(x_j) = g_j\}_{j=1}^M$ be a discrete function defined on $K = \{x_j, j = 1, .., M\} \subset [0,1]$ satisfying

$$0 \le x_1 < x_2 < .. < x_{M-1} < x_M \le 1.$$

Set

$$s_j = \begin{cases} 0, & j = 0, \\ \frac{1}{2}(x_j + x_{j+1}), & j = 1, .., M-1, \\ 1, & j = M. \end{cases}$$

Let $p > 0$ is given. Then for any $x \in I_\delta = [p\delta, 1 - p\delta]$ we define discrete mollification of G as follows:

$$J_\delta G(x) = \sum_{j=1}^{M} (\int_{s_{j-1}}^{s_j} \rho_{\delta,p}(x-s)ds) g_j,$$

where

$$\rho_{\delta,p}(x) = \begin{cases} A_p \delta^{-1} \exp(-\frac{x^2}{\delta^2}), & |x| \le p\delta, \\ 0, & |x| > p\delta, \end{cases}$$

such that $A_p = (\int_{-p}^{p} \exp(-s^2)ds)^{-1}$. We usually take p=3 and the radius of mollification, δ is selected automatically by the GCV method (see more [19]). We note that

$$\sum_{j=1}^{M} \int_{s_{j-1}}^{s_j} \rho_{\delta,p}(x-s)ds = \int_{-p\delta}^{p\delta} \rho_{\delta,p}(s)ds = 1.$$

Set

$$\Delta x = \max_{1\le j\le M-1} |x_{j+1} - x_j| \,.$$

In sequence, we will introduce the main properties relating discrete mollification method (see more [19]).

Theorem 3.1 ([9]). *1. Let $g(x) \in C^{0,1}(R^1)$ and $G = \{g(x_j) = g_j\}_{j=1}^M$ be the discrete version of g and let $G^\varepsilon = \{g_j^\varepsilon\}_{j=1}^M$ be the perturbed discrete version of g satisfying $\|G - G^\varepsilon\|_{\infty,K} \le \varepsilon$. Then there exists a constant C, independent of δ, such that*

$$\|J_\delta G^\varepsilon - J_\delta g\|_\infty \le C(\varepsilon + \Delta x).$$

2. If $g'(x) \in C^{0,1}(R^1)$, let $G = \{g(x_j) = g_j\}_{j=1}^M$ and $G^\varepsilon = \{g_j^\varepsilon\}_{j=1}^M$ satisfying $\|G - G^\varepsilon\|_{\infty,K} \le \varepsilon$, then

$$\left\|D(J_\delta G^\varepsilon) - (J_\delta g)'\right\|_\infty \le \frac{C}{\delta}(\varepsilon + \Delta x) + C_\delta (\Delta x)^2.$$

3. Suppose that $G = \{g(x_j) = g_j\}_{j=1}^M$ be the discrete function defined on K, and D_0^δ be a differentiation operator defined by $D_0^\delta(G) = D(J_\delta G)(x)$ then

$$\left\|D_0^\delta(G)\right\|_{\infty,K} \le \frac{C}{\delta} \|G\|_{\infty,K}.$$

3.1. Extension of data

In order to compute $J_\delta G(x)$ throughout the domain $[0,1]$, we have to extend discrete data function g to a bigger interval $I_{\delta'} = [-p\delta, 1+p\delta]$ or confine this function to the $I_\delta = [p\delta, 1-p\delta]$. In this essay, the first approach described in [19] is applied. An optimizing process is practiced to calculate the extension function of g in the intervals of $[-p\delta, 0]$ and $[1, 1+p\delta]$. This process is introduced by Mejia in [17].

4. The regularized Problem and space marching algorithm

To solve proposed inverse problem we consider the following regularized problem which is formulate as

$$\frac{\partial w(x,t)}{\partial t} - A(w)\frac{\partial^2 w(x,t)}{\partial x^2} = A(w)f(x,t), \quad 0 < x < 1, \quad 0 < t < T, \tag{4.1}$$

$$w(x,T) = J_{\delta_3} g_4(x), \quad 0 \le x \le 1, \tag{4.2}$$

$$\frac{\partial w(0,t)}{\partial x} = J_{\delta_1^0} g_1(t), \quad 0 \le t \le 1, \tag{4.3}$$

$$w(0,t) = J_{\delta_2} g_5(t), \quad 0 \le t \le T. \tag{4.4}$$

Now, we solve regularized problem to determine $w(x,t)$ satisfying (4.1)-(4.4). Let $h = 1/M$ and $k = T/N$ be the space and time discretization parameters. The numerical approximations of functions $w(jh, nk), w_x(jh, nk)$ and $w_t(jh, nk)$ are denoted by U_j^n, Q_j^n and R_j^n, respectively.
The space marching scheme for (4.1)-(4.4) is defined by a system of finite differences

$$U_{j+1}^n = U_j^n + hQ_j^n, \tag{4.5}$$

$$Q_{j+1}^n = Q_j^n + h\frac{(R_j^n - A(U_j^n)f_j^n)}{A(U_j^n)}, \tag{4.6}$$

$$R_{j+1}^n = R_j^n + h(D_0)_t(J_{\delta_1^j} Q_j^n), \tag{4.7}$$

where D_0 is the centered difference operator denoting by

$$D_0 f(t) = \frac{f(t+\Delta t) - f(t-\Delta t)}{2\Delta t}.$$

The algorithm of scheme (4.5)-(4.7) is as follows:
1. Choose the radii of mollification, δ_1, δ_2 and δ_3 using GCV method.
2. Put
$U_0^n = J_{\delta_2} g_5^{\varepsilon}(nk), n = 0, .., N,$
$U_j^N = J_{\delta_3} g_4^{\varepsilon}(jh), j = 1, .., M,$
$Q_0^n = J_{\delta_1^0} g_1^{\varepsilon}(nk), n = 0, .., N.$
3. Perform linear exterapolation to compute R_0^0.
4. Put
$R_0^n = D_t(J_{\delta_2} g_5^{\varepsilon}(nk)), n = 1, .., N.$
5. Set $j = 0$ and do while $j \le M - 1$,

$$\begin{aligned}
U_{j+1}^n &= U_j^n + hQ_j^n, \\
Q_{j+1}^n &= Q_j^n + h\frac{(R_j^n - A(U_j^n) f_j^n)}{A(U_j^n)}, \\
R_{j+1}^n &= R_j^n + h(D_0)_t (J_{\delta_1^j} Q_j^n).
\end{aligned}$$

In order to analyze stability and convergence of the numerical scheme, we assume

$$u(x,t) \in C^2([0,1] \times [0,T]).$$

5. Stability and convergence of the algorithm

In this section we establish the stability and convergence of the space marching scheme (4.5)-(4.7). From now on we use the notation

$$|Y_j| = \max_n \{|Y_j^n|\}.$$

Theorem 5.1. *(Stability theorem) There exists constant M_1, such that*

$$\max\{|U_M|, |Q_M|, |R_M|, M_{f_1}\} \le \exp(M_1) \max\{|U_0|, |Q_0|, |R_0|, M_{f_1}\}.$$

Proof. Let $\min_v\{A(v)\} = \xi$, $M_{f_1} = \max_{(x,t)\in[0,1]\times[0,T]} |A(v(x,t)) f(x,t)|$, From (4.5) and (4.6), we have

$$|U_{j+1}^n| \le (1+h) \max\{|U_j^n|, |Q_j^n|\}, \tag{5.1}$$

$$|Q_{j+1}^n| \le (1+h\xi) \max\{|Q_j^n|, |R_j^n|, M_{f_1}\}. \tag{5.2}$$

Applying theorem 3.1 and Eqn. (4.7), one may write

$$|R_{j+1}^n| \le (1 + h\frac{C}{|\delta|_{-\infty}}) \max\{|Q_j^n|, |R_j^n|\}, \tag{5.3}$$

where C is a constant which is independent of δ and

$$|\delta|_{-\infty} = \min_j (\delta_1^j).$$

Following (5.1)-(5.3)

$$\max\{|U_{j+1}|, |Q_{j+1}|, |R_{j+1}|, M_{f_1}\} \le (1 + hM_1) \max\{|U_j|, |Q_j|, |R_j|, M_{f_1}\},$$

where

$$M_1 = \max\{1, \xi, \frac{C}{|\delta|_{-\infty}}\}.$$

After M iteration of the last inequality, we achieve

$$\max\{|U_M|, |Q_M|, |R_M|, M_{f_1}\} \le (1 + hM_1)^M \max\{|U_0|, |Q_0|, |R_0|, M_{f_1}\},$$

which implies

$$\max\{|U_M|, |Q_M|, |R_M|, M_{f_1}\} \le \exp(M_1) \max\{|U_0|, |Q_0|, |R_0|, M_{f_1}\}.$$

So, the space marching scheme (4.5)-(4.7) is stable and proof is complete for fixed M_1. □

Theorem 5.2. *(Convergence theorem) For fixed δ, as h, k and ε tend to zero then the numerical scheme* (4.5)-(4.7) *converge to the mollified exact solution.*

Proof. Set

$$B = \max_v\{A(v)\},\ B_3 = \min_{j,n}\{A(w_j^n)A(U_j^n)\},\ B_4 = \max_{(x,t)\in[0,1]\times[0,T]}\{|w_t(x,t)|\}.$$

We begin with the definition of the discrete error functions

$$\begin{aligned}\Delta U_j^n &= U_j^n - w(jh, nk),\\ \Delta Q_j^n &= Q_j^n - w_x(jh, nk),\\ \Delta R_j^n &= R_j^n - w_t(jh, nk),\end{aligned}$$

then, we obtain

$$\begin{aligned}\Delta U_{j+1}^n &= U_{j+1}^n - w((j+1)h, nk)\\ &= \Delta U_j^n + (U_{j+1}^n - U_j^n) - (w((j+1)h, nk) - w(jh, nk))\\ &= \Delta U_j^n + h(Q_j^n - w_x(jh, nk)) + O(h^2)\\ &= \Delta U_j^n + h\Delta Q_j^n + O(h^2),\end{aligned} \tag{5.4}$$

$$\begin{aligned}\Delta Q_{j+1}^n &= Q_{j+1}^n - w_x((j+1)h, nk)\\ &= \Delta Q_j^n + (Q_{j+1}^n - Q_j^n) - (w_x((j+1)h, nk) - w_x(jh, nk))\\ &= \Delta Q_j^n + h(\frac{(R_j^n - A(U_j^n)f_j^n)}{A(U_j^n)} - \frac{w_t(jh, nk) - A(w(jh, nk))f(jh, nk)}{A(w_j^n)}) + O(h^2)\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\Delta R_{j+1}^n &= R_{j+1}^n - w_t((j+1)h, nk)\\ &= \Delta R_j^n + (R_{j+1}^n - R_j^n) - (w_t((j+1)h, nk) - w_t(jh, nk))\\ &= \Delta R_j^n + h(D_0(J_{\delta_1^j} Q_j^n) - w_{xt}(jh, nk)) + O(h^2).\end{aligned} \tag{5.6}$$

Then, from (5.4) and (5.5), it is obtained that

$$\left|\Delta U_{j+1}^n\right| \le \left|\Delta U_j^n\right| + h\left|\Delta Q_j^n\right| + O(h^2), \tag{5.7}$$

$$\left|\Delta Q_{j+1}^n\right| \le \left|\Delta Q_j^n\right| + hB_3\{B\left|\Delta R_j^n\right| + 2BB_4\} + O(h^2). \tag{5.8}$$

Due to the Theorem 3.1 and equation (5.6), we have

$$\left|\Delta R_{j+1}^n\right| \le \left|\Delta R_j^n\right| + h(C\frac{\left|\Delta Q_j^n\right| + k}{|\delta|_{-\infty}} + C_\delta k^2) + O(h^2), \tag{5.9}$$

which C and C_δ are constants.

Set

$$\Delta_j = \max\{\left|\Delta U_j^n\right|, \left|\Delta Q_j^n\right|, \left|\Delta R_j^n\right|\},\ C_0 = \max\{1, BB_3, \tfrac{C}{|\delta|_{-\infty}}\},\ C_1 = \tfrac{Ck}{|\delta|_{-\infty}} + C_\delta k^2 + 2BB_3B_4.$$

Then it is concluded that

$$\Delta_{j+1} \le (1 + hC_0)\Delta_j + hC_1 + O(h^2).$$

Therefore, after M iteration, we derive

$$\Delta_M \le (1 + hC_0)^M \Delta_0 + h(1 + hC_0)^{M-1} C_1 + ... + h(1 + hC_0)C_1 + hC_1. \tag{5.10}$$

Theorem 3.1 is directed to the following inequalities

$$\begin{aligned}|\Delta U_0^n| \le& C(\varepsilon + k),\\ |\Delta Q_0^n| \le& C(\varepsilon + k),\\ |\Delta R_0^n| \le& \frac{C}{|\delta|_{-\infty}}(\varepsilon + k) + C_\delta k^2,\end{aligned}$$

from these inequalities we see that when ε, h and k tend to zero, Δ_0 and the right hand side of inequality (5.10) tend to zero and so does Δ_M and the proof is complete. □

6. Numerical experiment

In this section, we present a numerical example to illustrate the effectiveness and stability of our proposed method. Stability of the method with respect to noise in the data is investigated using noisy data. The noisy discrete data functions are generated by adding a random perturbation to the exact data functions. For example, for the boundary function $g(t)$ we simulate noisy discrete data function as follows:

$$g^\varepsilon(t_n) = g(t_n) + r\varepsilon, \quad n = 0, 1, .., N,$$

where r is a random number in $[-1, 1]$ and ε is the noise level. The radii of mollification are chosen automatically by the GCV method. For checking the accuracy of our algorithm, we use weighted l^2-norm which for $u(x, t)$ is formulate as

$$\sqrt{\frac{1}{MN}\sum_{n=1}^{N}\sum_{j=1}^{M}\left|u(x_j, t_n) - U_j^n\right|^2}.$$

In this example, we take $T = 1$. This example implemented using Mathematica 10.3.1 software.

Example 6.1. It is easy that the function $a(u) = (u + 1)^2$, satisfies in problem (2.1)-(2.6) with $\varphi(x) = \sin(x)(1 - \cos(1))^2$, $g_1(t) = (1 - \cos(t - 1))^2$, $\chi(t) = \sin(1)(1 - \cos(t - 1))^2$, $g_2(x) = 0$, $g_3(t) = 0$ and $f(x, t)$ is taken so that the exact solution is

$$u(x, t) = \sin(x)(1 - \cos(t - 1))^2.$$

Using the transformation

$$v(x, t) = T_a(u(x, t)) = \int_0^{u(x,t)} a(s)ds,$$

we have

$$v(x, t) = \frac{(1 + u(x, t))^3 - 1}{3} = \frac{(1 + \sin(x)(1 - \cos(t - 1))^2)^3 - 1}{3}$$

and

$$A(v) = (3v + 1)^{\frac{2}{3}},$$

Table 1: Weighted l^2 errors.

ε	M	N	$u(x,t)$	$u(x,0)$	$u(1,t)$
0.001	10	10	0.00398225	0.00891594	0.00613761
0.001	20	20	0.00325090	0.00820382	0.00567592
0.001	30	30	0.00300591	0.00770168	0.00561988
0.01	10	10	0.00622965	0.01128690	0.00880938
0.01	20	20	0.00550470	0.00965564	0.00884753
0.01	30	30	0.00522123	0.00950996	0.00819372

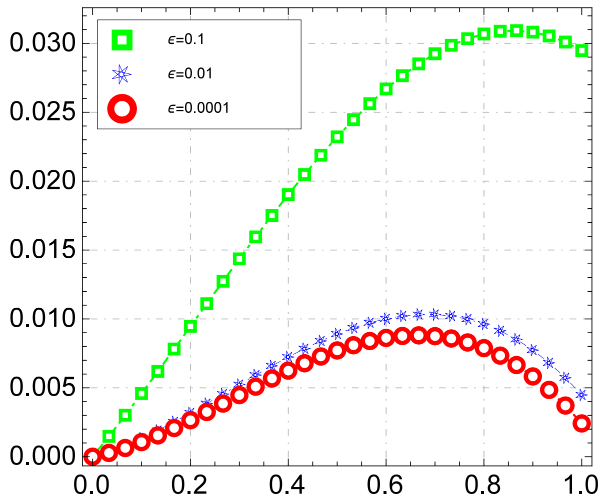

Figure 1: The absolute errors for $u(x,0)$ for three levels of noise with $\varepsilon = 0.0001, 0.01, 0.1$ and $M = N = 30$.

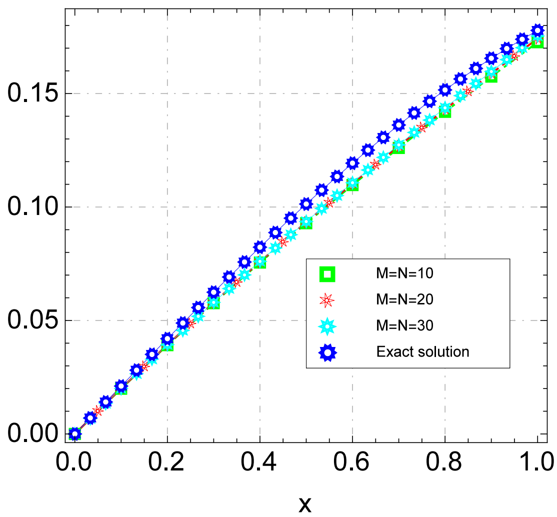

Figure 2: The exact and regularized solutions for $u(x,0)$ with noise level $\varepsilon = 0.001$.

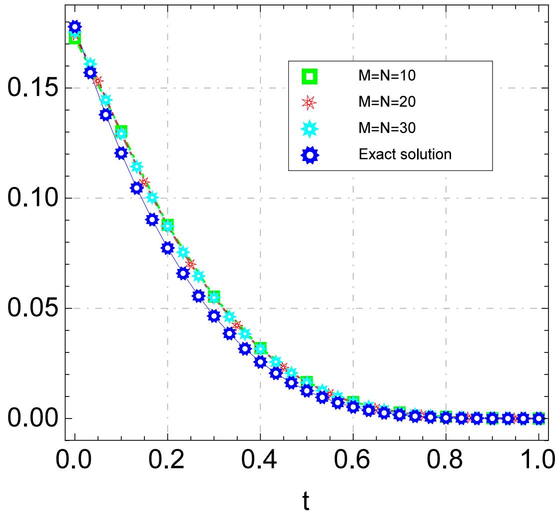

Figure 3: The exact and regularized solutions for $u(1,t)$ with noise level $\varepsilon = 0.001$.

where $v(x,t)$ is exact solution of (2.7)-(2.12). After solving problem (2.7)-(2.12) by the proposed method the unknown $u(x,t)$ can be calculated through

$$u(x,t) = (3v(x,t) + 1)^{\frac{1}{3}} - 1.$$

Table 1 highlights the Weighted l^2 errors in $u(x,0)$, $u(1,t)$ and $u(x,t)$ with two noise levels $\varepsilon = 0.01$ and 0.001. This Table and Figs. 1-3 show that at fix noise level ε with decreasing h and k the accuracy of our algorithm will be increased. To investigate the dependence of errors of the solutions on the noise levels, the absolute errors between the exact and computed $u(x,0)$ for three levels of noise $\varepsilon = 0.0001, 0.01$ and 0.1 with $M = N = 30$ is shown in Fig. 1.
Holistically, by this Fig. we can see as ε declines, the accuracy of approximated solutions will enhance. Our approximate solutions are demonstrate the efficiency of the method computationally.

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