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Tripled Periodic Boundary Value Problems of Nonlinear Second Order Differential Equations

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Abstract

The present paper proposes a new monotone iteration principle for the existence as well as approximations of the tripled solutions for a tripled periodic boundary value problem of second order ordinary nonlinear differential equations. An algorithm for the tripled solutions is developed and it is shown that the sequences of successive approximations defined in a certain way converge monotonically to the tripled solutions of the related differential equations under some suitable hybrid conditions. A numerical example is also indicated to illustrate the abstract theory developed in the paper.

Keywords: tripled periodic boundary value problems, tripled fixed point theorem, approximate tripled solutions.
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1. Introduction

Given a closed and bounded interval $J = [0, T]$ of a real line $\mathbb{R}$, consider the tripled periodic boundary value problems (in short TPBVPs) of nonlinear second order ordinary nonlinear differential equations (in short DEs) of the form

$$\begin{cases} -x''(t) + \lambda^2 x(t) = f(t, x(t), y(t), z(t)), \\ x(0) = x(T), x'(0) = x'(T), \end{cases} \tag{1.1}$$

$$\begin{cases} -y''(t) + \lambda^2 x(t) = f(t, y(t), x(t), y(t)), \\ y(0) = y(T), y'(0) = y'(T), \end{cases} \tag{1.2}$$

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$$\begin{cases} -z''(t) + \lambda^2 x(t) = f(t, z(t), x(t), y(t)), \\ z(0) = z(T), z'(0) = z'(T), \end{cases} \tag{1.3}$$

for all $t \in J$, where $\lambda \in \mathbb{R}, \lambda > 0$ and $f : J \times C(J, \mathbb{R})^3 \to \mathbb{R}$ is a continuous function. By a coupled solution of the TPBVPs (1.1), (1.2) and (1.3) we mean an ordered pair of differentiable functions $(u, v, w) \in C(J, \mathbb{R}) \times C(J, \mathbb{R}) \times C(J, \mathbb{R})$ that satisfy the DEs (1.1), (1.2) and (1.3), where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

Let $(E, \preceq)$ be a partial ordered set and d be a metric on E such that $(E, \preceq, d)$ becomes a partially ordered metric space. By $E \times E \times E$ we denote a metric space with the metric d^* defined by

$$d^*((x, y, z), (u, v, w)) = d(x, u) + d(y, v) + d(z, w), \tag{1.4}$$

for $(x, y, z), (u, v, w) \in E \times E \times E$. We define a partial order $\preceq$ in $E \times E \times E$ as follows:
Let $(x_1, x_2, x_3), (y_1, y_2, y_3) \in E \times E \times E$. Then,

$$(x_1, x_2, x_3) \preceq (y_1, y_2, y_3) \iff x_1 \preceq y_1, \quad x_2 \succeq y_2 \quad \textit{and} \quad x_3 \preceq y_3. \tag{1.5}$$

Then, the triplet $(E \times E \times E, \preceq, d^*)$ again becomes a partially ordered metric space. Let $\mathcal{F} : E \times E \times E \to E$ and consider the tripled mapping equations,

$$\mathcal{F}(x, y, z) = x, \ \mathcal{F}(y, x, y) = y \quad \textit{and} \quad \mathcal{F}(z, x, y) = z. \tag{1.6}$$

A point $(x^*, y^*, z^*) \in E \times E \times E$ is said to be a tripled solution or tripled fixed point for the coupled mapping equation (1.6) if

$$\mathcal{F}(x^*, y^*, z^*) = x^*, \ \mathcal{F}(y^*, x^*, y^*) = y^* \quad \textit{and} \quad \mathcal{F}(z^*, y^*, x^*) = z^*. \tag{1.7}$$

We need the following definitions in what follows:

Definition 1.1. A partially ordered normed metric space $(E, \preceq, d)$ is called regular, if every nondecreasing (resp. nonincreasing) sequence $\{x_n\}$ converges to x^*, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$.

Definition 1.2. A mapping $\mathcal{F} : E \times E \times E \to E$ is called partially continuous at a point $(a, b, c) \in E \times E \times E$ if for $\epsilon > 0$, there exists a $\delta > 0$, such that

$$d^*(\mathcal{F}(x, y, z), \mathcal{F}(a, b, c)) < \epsilon,$$

whenever (x, y, z) is comparable to (a, b, c) and

$$d^*((x, y, z), (a, b, c)) < \delta.$$

If $\mathcal{F}$ is partially continuous at every point of $E \times E \times E$, we say that $\mathcal{F}$ is partially continuous on $E \times E \times E$.

Remark 1.3. If $\mathcal{F}$ is partially continuous on $E \times E \times E$, then it is continuous on every totally ordered set or chain in $E \times E \times E$.

Definition 1.4. A mapping $\mathcal{F} : E \times E \times E \to E$ is called partially compact if $\mathcal{F}(C_1 \times C_2 \times C_3)$ is a relatively compact subset of E for all chains C_1, C_2 and C_3 in E.

The details of compact and continuous operators may be found in the monograph by *Heikkilä* and Lakshmikantham [13] and the references therein.

Definition 1.5. A mapping $\mathcal{F}$ is called mixed monotone, if $\mathcal{F}(x, y, z)$ is nondecreasing in x for each $y \in E$ and nonincreasing in y for each $x \in E$ with respect to the order relation $\preceq$ in E.

Remark 1.6. If $\mathcal{F}$ is mixed monotone, then it is a nondecreasing mapping on $E \times E \times E$ with respect to the order relation $\preceq$ defined in $E \times E \times E$.

Definition 1.7. The order relation $\preceq$ and the metric d on a non- empty set E are said to be compatible if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* implies that the whole sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|.\|)$, the order relation $\preceq$ and the norm $\|.\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|.\|$ are compatible.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the metric defined by the absolute value function has this property. Similarly, every finite dimensional Euclidean space $\mathbb{R}_n$ is compatible with respect to usual component wise order relation and the standard norm in it.

Theorem 1.8. *Let $(E, \preceq, d)$ be a regular partially ordered complete metric space such that the metric d and the order relation $\preceq$ are compatible in every compact chain C of E. Let $\mathcal{F}, E \times E \times E \to E$ is a mixed monotone, partially continuous and partially compact mapping. If there exist elements $x_0, y_0, z_0 \in E$ such that $x_0 \preceq \mathcal{F}(x_0, y_0, z_0)$, $y_0 \succeq \mathcal{F}(y_0, x_0, y_0)$ and $y_0 \preceq \mathcal{F}(y_0, x_0, y_0)$, then $\mathcal{F}$ has a tripled fixed point (x^*, y^*, z^*) and the sequences $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ defined by $x_n = \mathcal{F}(x_{n-1}, y_{n-1}, z_{n-1}) = \mathcal{F}^n(x_0, y_0, z_0)$, $y_n = \mathcal{F}(y_{n-1}, x_{n-1}, y_{n-1}) = \mathcal{F}^n(y_0, x_0, y_0)$ and $z_n = \mathcal{F}(z_{n-1}, y_{n-1}, x_{n-1}) = \mathcal{F}^n(z_0, y_0, x_0)$ converge monotonically to x^*, y^* and z^* respectively.*

Proof. Define the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ of points in E as follows. Choose

$$x_1 = F(x_0, y_0, z_0),$$

$$y_1 = F(y_0, x_0, y_0)$$

and

$$z_1 = F(z_0, y_0, x_0).$$

Then, $x_0 \preceq x_1$, $y_1 \preceq y_0$ and $z_1 \preceq z_0$. Again, choose

$$x_2 = \mathcal{F}^2(x_0, y_0, z_0) = \mathcal{F}(x_1, y_1, z_1) = \mathcal{F}(\mathcal{F}(x_0, y_0, z_0), \mathcal{F}(y_0, x_0, y_0), \mathcal{F}(z_0, y_0, x_0)) \succeq F(x_0, y_0, z_0) = x_1.$$

Similarly we have

$$y_2 = \mathcal{F}^2(y_0, x_0, y_0) = \mathcal{F}(y_1, x_1, y_1) = \mathcal{F}(\mathcal{F}(y_0, x_0, y_0), \mathcal{F}(x_0, y_0, z_0), \mathcal{F}(y_0, x_0, y_0)) \preceq F(y_0, x_0, y_0) = y_1$$

and

$$z_2 = \mathcal{F}^2(z_0, y_0, x_0) = \mathcal{F}(z_1, y_1, x_1) = \mathcal{F}(\mathcal{F}(z_0, y_0, x_0), \mathcal{F}(x_0, y_0, z_0), \mathcal{F}(y_0, x_0, y_0)) \succeq F(z_0, y_0, x_0) = x_1.$$

Proceeding in this way, by induction, define

$$\begin{cases} x_{n+1} = \mathcal{F}(x_n, y_n, z_n) = \mathcal{F}^n(x_0, y_0, z_0), \\ y_{n+1} = \mathcal{F}(y_n, x_n, y_n) = \mathcal{F}^n(y_0, x_0, y_0), \\ z_{n+1} = \mathcal{F}(z_n, y_n, x_n) = \mathcal{F}^n(z_0, y_0, x_0), \end{cases} \tag{1.8}$$

for $n = 0, 1, 2\dots$ so that

$$x_0 \preceq x_1 \preceq x_2 \preceq \dots \preceq x_n \preceq \dots, \tag{1.9}$$

$$y_0 \succeq y_1 \succeq y_2 \succeq \dots \succeq y_n \succeq \dots \tag{1.10}$$

and

$$z_0 \preceq z_1 \preceq z_2 \preceq \dots \preceq z_n \preceq \dots. \tag{1.11}$$

Thus, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are respectively monotone nondecreasing and monotone nonincreasing sequences and so are chains in E. From the construction of $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$, it follows that

$$\{x_n\} \subseteq \mathcal{F}(\{x_n\}, \{y_n\}, \{z_n\}) \subseteq \mathcal{F}(\{x_n\} \times \{y_n\} \times \{z_n\}).$$

Since $\mathcal{F}$ is partially compact on $E \times E \times E$, one has $\mathcal{F}(\{x_n\} \times \{y_n\} \times \{z_n\})$ is a relatively compact subset of E. As a result, $\overline{\mathcal{F}(\{x_n\} \times \{y_n\} \times \{z_n\})}$ is compact and that $\{x_n\}$ has a convergent subsequence converging to a point, say $x^* \in E$. Since d and $\preceq$ are compatible in every compact chain C of E, the whole sequence $\{x_n\}$ converges to x^*. Similarly, the sequences $\{y_n\}$ converges to a point say $y^* \in E$ and $\{z_n\}$ converges to a point say $z^* \in E$. Equivalently, $(x_n, y_n, z_n) \to (x^*, y^*, z^*)$ in the topology of the norm in $E \times E \times E$. As E is a regular, we have that $x_n \preceq x^*$, $y_n \succeq y^*$ and $z_n \preceq z^*$ for all $n \in \mathbb{N}$. Therefore, we obtain $(x_n, y_n, z_n) \preceq (x^*, y^*, z^*)$ for all $n \in \mathbb{N}$. Finally, by the partial continuity of $\mathcal{F}$, we obtain

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \mathcal{F}(x_n, y_n, z_n) = \mathcal{F}(x^*, y^*, z^*),$$

$$y^* = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} \mathcal{F}(y_n, x_n, y_n) = \mathcal{F}(y^*, x^*, y^*)$$

and

$$z^* = \lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} \mathcal{F}(z_n, y_n, x_n) = \mathcal{F}(z^*, y^*, x^*).$$

Thus (x^*, y^*, z^*) is a tripled fixed point of the mapping $\mathcal{F}$ on $E \times E \times E$ into itself. This completes the proof. □

Remark 1.9. The regularity of the partially ordered metric space E may be replaced with a stronger condition of continuity than the partial continuity of the mappings $\mathcal{F}$ on $E \times E \times E$. Again, the condition of compatibility of the order relation $\preceq$ and the norm $\|.\|$ in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|.\|$.

The simple fact concerning the compactibility of the order relation and the norm mentioned in Remark 1.9 has been used in formulating the main results of this paper. In the following section we prove the main existence and approximation results for the TBVP (1.1), (1.2) and (1.3) defined on J.

2. Existence and Approximations Results

We place our considerations of the TBVPs (1.1), (1.2) and (1.3) in the function space $C(J, \mathbb{R})$. We define a norm $\|.\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)| \tag{2.1}$$

and

$$x \leq y \iff x(t) \leq y(t), \tag{2.2}$$

for all $t \in J$. Clearly, $(C(J, \mathbb{R}), \|.\|, \leq)$ is a partially ordered complete normed linear space and has compatibility property with respect to the norm $\|.\|$ and the order relation $\leq$ in certain subsets of it. The following lemma in this connection is useful in what follows:

Lemma 2.1. *Let $(C(J, \mathbb{R}), \leq, \|.\|)$ be a partially ordered Banach space with the norm $\|.\|$ and the order relation $\leq$ defined by (2.1) and (2.2). Then $\|.\|$ and $\leq$ are compatible in every partially compact subset S of $C(J, \mathbb{R})$.*

Proof. Let S be a partially compact subset of $C(J, \mathbb{R})$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t) \leq \ldots . \tag{2.3}$$

for each $t \in J$.

Suppose that a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}_{k\in\mathbb{N}}$ of the monotone real sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent. By monotone characterization, the whole sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent and converges to a point $x(t)$ in S for each $t \in J$.

This shows that the sequence$\{x_n(t)\}_{n\in\mathbb{N}}$ converges to $x(t)$ point-wise on J. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence by Arzel*á*-Ascoli theorem. Hence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges uniformly to x. As a result, $\|.\|$ and $\leq$ are compatible in S. This completes the proof. □

We need the following definition in the sequel.

Definition 2.2. An ordered pair of differentiable functions $(u, v, w) \in C(J,\mathbb{R}) \times C(J,\mathbb{R}) \times C(J,\mathbb{R})$ is said to be a tripled lower solution of the TPBVPs of coupled differential equations (1.1), (1.2) and (1.3) if

$$\begin{cases} -u''(t) + \lambda^2 u(t) = f(t, u(t), v(t), w(t)), \\ u(0) = u(T), u'(0) = u'(T), \end{cases}$$
$$\begin{cases} -v''(t) + \lambda^2 v(t) = f(t, v(t), u(t), v(t)), \\ v(0) = v(T), v'(0) = v'(T), \end{cases}$$
$$\begin{cases} -w''(t) + \lambda^2 w(t) = f(t, w(t), v(t), u(t)), \\ w(0) = w(T), w'(0) = w'(T), \end{cases}$$

for all $t \in J$. Similarly, an ordered pair of differentiable functions $(p, q, r) \in C(J,\mathbb{R}) \times C(J,\mathbb{R}) \times C(J,\mathbb{R})$ is said to be a tripled upper solution of the TPBVPs (1.1), (1.2) and (1.3) if the above inequalities are satisfied with reverse sign.

We consider the following set of hypotheses in what follows:

(H1) f is bounded on $J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with bound M.
(H2) The function $f(t, x, y, z)$ is nondecreasing in x, nonincreasing in y and nondecreasing in z for each $t \in J$.
(H3) The TPBVPs (1.1), (1.2) and (1.3) have a lower coupled solution $(u, v, w) \in C(J,\mathbb{R})\times C(J,\mathbb{R})\times C(J,\mathbb{R})$.
(H4) The TPBVPs (1.1), (1.2) and (1.3) have a lower coupled solution $(p, q, r) \in C(J,\mathbb{R})\times C(J,\mathbb{R})\times C(J,\mathbb{R})$.

Lemma 2.3. *For any $\sigma \in L^1(J,\mathbb{R})$, x is a solution to the differential equation*

$$\begin{cases} -x''(t) + \lambda^2 x(t) = \sigma(t), t \in J, \\ x(0) = x(T), x'(0) = x'(T), \end{cases} \tag{2.4}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G(t, s)\sigma(s)ds, \tag{2.5}$$

where, $G(t, s)$ is the Green's function associated to the PBVP

$$\begin{cases} -x''(t) + \lambda^2 x(t) = 0, t \in J, \\ x(0) = x(T), x'(0) = x'(T), \end{cases} \tag{2.6}$$

Notice that the Green's function G is continuous and nonnegative on $J\times J\times J$ and therefore, the number $K := \max\{|G(t, s)| : t, s \in [0, T]\}$ exists.

We obtain, an application of above Lemma 2.3 as follows:

Lemma 2.4. *A pair of function $(u, v, w) \in C(J,\mathbb{R}) \times C(J,\mathbb{R}) \times C(J,\mathbb{R})$ is a tripled solution of the TPBVPs (1.1), (1.2) and (1.3) if and only if u, v and w are the solutions of the nonlinear integral equations,*

$$x(t) = \int_0^T G(t, s)f(s, x(s), y(s), z(s))ds, \tag{2.7}$$

$$y(t) = \int_0^T G(t,s)f(s, y(s), x(s), y(s))ds \tag{2.8}$$

and

$$z(t) = \int_0^T G(t,s)f(s, z(s), y(s), x(s))ds, \tag{2.9}$$

for all $t \in J$, where the Green's function $G(t,s)$ is given by (2.6).

Theorem 2.5. *Assume that the hypotheses (H1) through (H3). Then the TPBVPs (1.1), (1.2) and (1.3) have a tripled solution (x^*, y^*, z^*) defined on J and the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ defined by*

$$x_{n+1}(t) = \int_0^T G(t,s)f(s, x_n(s), y_n(s), z_n(s))ds, \tag{2.10}$$

$$y_{n+1}(t) = \int_0^T G(t,s)f(s, y_n(s), x_n(s), y_n(s))ds \tag{2.11}$$

and

$$z_{n+1}(t) = \int_0^T G(t,s)f(s, z_n(s), y_n(s), x_n(s))ds \tag{2.12}$$

for each $t \in J$ converge monotonically to x^*, y^* and z^* respectively.

Proof. Set $E = C(J, \mathbb{R})$. Then, by Lemma 2.1, every compact chain in E possesses the compatibility property with respect to the norm $\|.\|$ and the order relation $\leq$ in E. Consider the mapping $\mathcal{F}$ on $E \times E \times E$ defined as:

$$\mathcal{F}(x,y,z)(t) = \int_0^T G(t,s)f(s, x(s), y(s), z(s))ds, \tag{2.13}$$

$$\mathcal{F}(y,x,y)(t) = \int_0^T G(t,s)f(s, y(s), x(s), y(s))ds \tag{2.14}$$

and

$$\mathcal{F}(z,x,y)(t) = \int_0^T G(t,s)f(s, z(s), y(s), x(s))ds. \tag{2.15}$$

Since Green's function G is continuous on $J \times J \times J$, we have that $\mathcal{F}(x,y,z), \mathcal{F}(y,x,y), \mathcal{F}(z,y,x)) \in E$. As a result, $\mathcal{F}$ defines a mapping $\mathcal{F} : E \times E \times E \to E$. We shall show that $\mathcal{F}$ satisfies the conditions of Theorem 1.8. This will be achieved in a series of following steps.

Step-1. $\mathcal{F}$ is a mixed monotone operator on $E \times E \times E$.
Let $x_1, x_2 \in S$ be such that $x_1 \leq x_2$. Then, by hypothesis (H2),

$$\begin{aligned}
\mathcal{F}(x_1,y,z)(t) &= \int_0^T G(t,s)f(s, x_1(s), y(s), z(t))ds \\
&\leq \int_0^T G(t,s)f(s, x_2(s), y(s), z(t))ds \\
&= \mathcal{F}(x_2,y,z)(t),
\end{aligned}$$

for all $t \in J$. This shows that (x, y, z) is monotone nondecreasing in x for all $t \in J$ and $y, z \in S$. Next, let $y_1, y_2 \in E$ be such that $y_1 \leq y_2$. Then,

$$\begin{aligned}\mathcal{F}(x, y_1, z)(t) &= \int_0^T G(t,s) f(s, x(s), y_1(s), z(t)) ds \\ &\geq \int_0^T G(t,s) f(s, x(s), y_2(s), z(t)) ds \\ &= \mathcal{F}(x, y_2, z)(t),\end{aligned}$$

for all $t \in J$ and $x \in S$. Hence $\mathcal{F}(x, y, z)$ is monotone nonincreasing in y for all $x, z \in E$. Thus $\mathcal{F}$ is a mixed monotone mapping on $E \times E \times E$ and

$$\begin{aligned}\mathcal{F}(x, y, z_1)(t) &= \int_0^T G(t,s) f(s, x(s), y(s), z_1(t)) ds \\ &\leq \int_0^T G(t,s) f(s, x(s), y(s), z_2(t)) ds \\ &= \mathcal{F}(x, y, z_2)(t),\end{aligned}$$

for all $t \in J$. This shows that (x, y, z) is monotone nondecreasing in z for all $t \in J$ and $x, y \in S$.

Step-2. $\mathcal{F}$ is partially continuous mixed monotone operator on $E \times E \times E$.

Let $\{X_n\}_{n\in\mathbb{N}} = \{(x_n, y_n, z_n)\}$ be a monotone nondecreasing sequence in a chain $C = C_1 \times C_2 \times C_3$ of $E \times E \times E$ such that $X_n = (x_n, y_n, z_n) \to (x, y, z) = X$ and $X_n \leq X$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem,

$$\begin{aligned}\lim_{n\to\infty} \mathcal{F}(X_n)(t) &= \int_0^T G(t,s)[\lim_{n\to\infty} f(s, x_n(s), y_n(s), z_n(s))] ds \\ &= \int_0^T G(t,s) f(s, x(s), y(s), z(s)) ds \\ &= F(X)(t),\end{aligned}$$

for all $t \in J$. This shows that $F(X_n)$ converges monotonically to $F(X)$ pointwise on J. Next, we will show that $\{F(X_n)\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary. Then, by hypothesis (B2),

$$\begin{aligned}|F(X_n)(t_2) - F(X_n)(t_1)| &\leq |\int_0^T G(t_2,s) f(s, x_n(s), y_n(s), z_n(s)) - \int_0^T G(t_1,s) f(s, x_n(s), y_n(s), z_n(s))| ds \\ &\leq \int_0^T |G(t_2,s) - G(t_2,s)||f(s, x_n(s), y_n(s), z_n(s))| ds \\ &\leq M_f \int_0^T |G(t_2,s) - G(t_2,s)| ds \to 0 \quad as \quad t_2 - t_1 \to 0\end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{F}(X_n) \to \mathcal{F}(X)$ is uniform and hence $\mathcal{F}$ is a partially continuous on $E \times E \times E$.

Step-3. $\mathcal{F}$ is a partially compact mixed monotone operator on $E \times E \times E$.

Let C_1, C_2 and C_3 be three arbitrary chains in E. We show that $\mathcal{F}(C_1 \times C_2 \times C_3)$ is a relatively compact subset of E. To finish it is enough to prove that $\mathcal{F}(C_1 \times C_2 \times C_3)$ is uniformly bounded and equicontinuous set in E. Let $x \in C_1$, $y \in C_2$ and $z \in C_3$ be arbitrary. Then, by (H1),

$$|\mathcal{F}(x, y, z)(t)| \leq \int_0^T G(t,s)|f(s, x(s), y(s), z(t))| ds \leq M_f K T = r,$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{F}(x, y, z)\| \leq r$ for all $x \in C_1$, $y \in C_2$ and $z \in C_3$. Hence, $\mathcal{F}(C_1 \times C_2 \times C_3)$ is a uniformly bounded subset of E. Next, we show that $\mathcal{F}(C_1 \times C_2 \times C_3)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ be arbitrary. Then, for any $z \in \mathcal{F}(C_1 \times C_2 \times C_3)$, there exist $x \in C_1$, $y \in C_2$ and $z \in C_3$ such that $l = \mathcal{F}(x, y, z)$. Without loss of generality, we may assume that $x(t_1) \geq x(t_2)$ and $y(t_1) \leq y(t_2)$. Therefore, by the definition of $\mathcal{F}$,

$$\begin{aligned}
|l(t_1) - l(t_2)| &= |\mathcal{F}(x, y, z)(t_1) - \mathcal{F}(x, y, z)(t_2)| \\
&= |\int_0^{t_1} f(s, x(s), y(s), z(s))ds - \int_0^{t_2} f(s, x(s), y(s), z(s))ds| \\
&\leq |\int_{t_2}^{t_1} f(s, x(s), y(s), z(s))ds| \\
&\leq M_f|t_1 - t_2|ds \to 0 \quad as \quad t_1 \to t_2
\end{aligned}$$

uniformly for all $x \in C_1$, $y \in C_2$ and $z \in C_3$. As a result, we have

$$|\mathcal{F}(x, y, z)(t_1) - \mathcal{F}(x, y, z)(t_2)| \to 0 \quad as \quad t_1 \to t_2,$$

uniformly for all $(x, y, z) \in C_1 \times C_2 \times C_3$. Consequently $\mathcal{F}(C_1 \times C_2 \times C_3)$ is an equi-continuous set of E. We apply Arzela-Ascoli theorem and deduce that $\mathcal{F}(C_1 \times C_2 \times C_3)$ is a relatively compact subset of E. Hence $\mathcal{F}$ is partially relatively compact on $E \times E \times E$. Now $\mathcal{F}$ is a partially continuous and partially compact mixed monotone operator on $E \times E \times E$ into E. Again, by hypothesis (H3), there exist elements x_0, y_0 and z_0 in S such that $x_0 \leq \mathcal{F}(x_0, y_0, z_0)$, $y_0 \geq \mathcal{F}(y_0, x_0, y_0)$ and $z_0 \leq \mathcal{F}(z_0, y_0, x_0)$. Thus all the conditions of Theorem 1.8 are satisfied and hence the tripled equations $\mathcal{F}(x, y, z) = x$, $\mathcal{F}(y, x, y) = y$ and $\mathcal{F}(z, y, x) = z$ have a tripled solution (x^*, y^*, z^*) and the sequences $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ defined by (2.14) and (2.15) converge monotonically to x^*, y^* and z^* respectively. This completes the proof. □

Remark 2.6. The conclusion of Theorem 2.5 also remains true if we replace the hypothesis (H3) with (H4). The proof of Theorem 2.5 under this new hypothesis is obtained using similar arguments with appropriate modifications.

Example 2.7. Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the tripled PBVPs,

$$\begin{cases} -x''(t) + x(t) = \tanh x(t) - \tanh y(t) + \tanh z(t), \\ x(0) = x(1), x'(0) = x'(1), \end{cases} \tag{2.16}$$

$$\begin{cases} -y''(t) + y(t) = \tanh y(t) - \tanh x(t) + \tanh y(t), \\ y(0) = y(1), y'(0) = y'(1) \end{cases} \tag{2.17}$$

and

$$\begin{cases} -z''(t) + z(t) = \tanh z(t) - \tanh y(t) + \tanh x(t), \\ z(0) = z(1), z'(0) = z'(1), \end{cases} \tag{2.18}$$

for all $t \in [0, 1]$.
Here, the function f is given by

$$f(t, x, y, z) = \tanh x - \tanh y + \tanh z,$$

for all $t \in [0, 1]$ and $x, y, z \in \mathbb{R}$. Clearly, f is uniformly continuous and bounded on $J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with bound $M_f = 2$. Furthermore, $f(t, x, y, z)$ is nondecreasing in x for each $t \in J$ and $y, z \in \mathbb{R}$, nonincreasing in y for each $t \in J$ and $x, z \in \mathbb{R}$ and is nondecreasing in z for each $t \in J$ and $x, y \in \mathbb{R}$. Finally, there exist functions

$$x_0(t) = -\left[\frac{e^2(e^{-t} - e^t)}{(e-1)} + \frac{e(1 - e^{-t})}{(e-1)}\right],$$

$$y_0(t) = -\left[\frac{e^2(e^{-t} - e^t)}{(e-1)} + \frac{e(1 - e^{-t})}{(e-1)}\right]$$

and

$$z_0(t) = -\left[\frac{e^2(e^{-t} - e^t)}{(e-1)} + \frac{e(1 - e^{-t})}{(e-1)}\right],$$

such that

$$\begin{cases} -x_0''(t) + x_0(t) = \tanh x_0(t) - \tanh y_0(t) + \tanh z_0(t), \\ x_0(0) = x_0(1), x_0'(0) = x_0'(1), \end{cases} \tag{2.19}$$

$$\begin{cases} -y_0''(t) + y_0(t) = \tanh y_0(t) - \tanh x_0(t) + \tanh y_0(t), \\ y_0(0) = y_0(1), y_0'(0) = y_0'(1) \end{cases} \tag{2.20}$$

and

$$\begin{cases} -z_0''(t) + z_0(t) = \tanh z_0(t) - \tanh y_0(t) + \tanh x_0(t), \\ z_0(0) = z_0(1), z_0'(0) = z_0'(1), \end{cases} \tag{2.21}$$

for all $t \in J$. Thus, the nonlinearity f satisfies all the hypotheses (H1) through (H3) of Theorem 2.5. Hence, the TPBVPs (2.15) have a tripled solution (x^*, y^*, z^*) defined on $[0, 1]$ and the sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$x_{n+1}(t) = \int_0^1 G(t,s)[\tanh x_n(s) - \tanh y_n(s) + \tanh z_n(s)]ds, \quad t \in [0,1],$$

$$y_{n+1}(t) = \int_0^1 G(t,s)[\tanh y_n(s) - \tanh x_n(s) + \tanh y_n(s)]ds, \quad t \in [0,1]$$

and

$$z_{n+1}(t) = \int_0^1 G(t,s)[\tanh z_n(s) - \tanh y_n(s) + \tanh x_n(s)]ds, \quad t \in [0,1],$$

where $G(t, s)$ is a Green's function associated with the PBVP

$$\begin{cases} -x''(t) + x(t) = 0, \quad t \in J, \\ x(0) = x(1), x'(0) = x'(1), \end{cases} \tag{2.22}$$

given by

$$G(t,s) = \frac{1}{2(e-1)} \begin{cases} e^{1+s-t} + e^{t-s}; 0 \le s \le t \le 1, \\ e^{1+t-s} + e^{s-t}; 0 \le t \le s \le 1, \end{cases} \tag{2.23}$$

converges monotonically to x^*, y^* and z^* respectively.

Remark 2.8. Finally, we mention that Theorem 1.8 may be applied to various nonlinear initial and boundary value problems of ordinary coupled differential equations for proving the existence as well as algorithms for the tripled solutions under suitable mixed monotonic and partial compactness type conditions.

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References

[1] H. Aydi, E. Karapinar, M. Postolache, *Tripled coincidence point theorems for weak $\varphi-$contractions in partially ordered metric spaces*, Fixed Point Theory Appl., **2012** (2012), 12 pages.

[2] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fundamenta Mathematicae, **3** (1922), 133–181.

[3] V. Berinde, M. Borcut, *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Anal., **74** (2011), 4889–4897.

[4] A. Gupta, *An application of Meir-Keeler type tripled fixed point theorem*, Int. J. Adv. Appl. Math. Mech., **2** (2014), 21–38.

[5] S. Heikkilä, V. Lakshmikantham, *Monotone iterative techniques for discontinuous nonlinear differential equations*, Marcel Dekker inc., New York (1994).

[6] A. Meir, E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl., **28** (1969), 326–329.

[7] J. J. Nieto, R. R. Lopez, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sinica Engl. Ser., **23** (2007), 2205–2212.

[8] A. C. M. Ran, M. C. B Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Am. Math. Soc., **132** (2004), 1435–1443.