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On the set of solutions of a nonconvex hyperbolic differential inclusion of third order

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Abstract

We consider a parametrized nonconvex hyperbolic differential inclusion of third order and we prove that the set of its solutions is a retract of a convex set of a Banach space.

Keywords: set-valued map, hyperbolic differential inclusion, retraction.
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1. Introduction

In this paper, we study the following Darboux problem for a third order hyperbolic differential inclusion:

$$u_{xyz}(x,y,z) \in F(x,y,z,u(x,y,z),s), \quad (x,y,z) \in \Pi := [0,1]\times[0,1]\times[0,1], \tag{1.1}$$

with the initial values

$$\begin{aligned} u(x,y,0) &= \varphi(x,y), \quad (x,y) \in \Pi_1 := [0,1]\times[0,1], \\ u(0,y,z) &= \psi(y,z), \quad (y,z) \in \Pi_1, \\ u(x,0,z) &= \chi(x,z), \quad (x,z) \in \Pi_1, \end{aligned} \tag{1.2}$$

where φ, ψ, χ are absolutely continuous functions satisfying:

$$\begin{aligned} u(x,0,0) &= \varphi(x,0) = \chi(x,0) =: v_1(x), \quad x \in [0,1], \\ u(0,y,0) &= \varphi(0,y) = \psi(y,0) =: v_2(y), \quad y \in [0,1], \\ u(0,0,z) &= \psi(0,z) = \chi(0,z) =: v_3(z), \quad z \in [0,1], \\ u(0,0,0) &= v_1(0) = v_2(0) = v_3(0) =: v_0. \end{aligned} \tag{1.3}$$

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S is a separable metric space and $F : \Pi \times \mathbb{R}^n \times S \to \mathcal{P}(\mathbb{R}^n)$ is a set-valued map.

Several existence results for solutions of Darboux problem (1.1)-(1.2) have been obtained by many authors [3, 4, 6, 8–10] etc. As far as we know, there are no papers concerning qualitative properties of the solutions of this problem, excepting [3], where it is proved the arcwise connectedness of the solution set of this problem.

The aim of the present paper is to prove that the solution set of the problem (1.1)-(1.2) is a retract of a convex set of a Banach space. At the same time this result provides the existence of continuous selections of the solution set multifunction. Moreover, we prove that any two continuous selections from the solution map are homotopic.

We essentially use a result of Bressan, Cellina and Fryszkowski [2] concerning the existence of a retraction of a Banach space on the set of the fixed points of a contractive set-valued map, in the same way as this result was used in [5] by De Blasi, Pianigiani and Staicu to obtain similar topological properties for "classical" hyperbolic differential inclusions. The results in this paper may be interpreted as extensions of the results in [5] to hyperbolic differential inclusions of third order.

The paper is organized as follows: in Section 2 we present the notations, definitions and preliminary results to be used in the sequel and in Section 3 we prove the main results.

2. Preliminaries

Let $\mathcal{P}(\mathbb{R}^n)$ be the family of all nonempty subsets of $\mathbb{R}^n$, let $\Pi := [0,1] \times [0,1] \times [0,1]$, $\Pi_1 := [0,1] \times [0,1]$, $I = [0,1]$, denote by $\mathcal{L}(\Pi)$ the σ-algebra of all Lebesgue measurable subsets of Π. Denote by $\mathcal{B}(\mathbb{R}^n)$ the family of all Borel subsets of $\mathbb{R}^n$. If $A \subset \Pi$ then $\chi_A(.) : \Pi \to \{0,1\}$ denotes the characteristic function of A. For any subset $A \subset \mathbb{R}^n$, we denote by $cl(A)$ the closure of A.

In what follows, as usual, we denote by $C(\Pi, \mathbb{R}^n)$ the Banach space of all continuous functions $x(.) : \Pi \to \mathbb{R}^n$ endowed with the norm $|x(.)|_C = \sup_{(x,y,z)\in\Pi} ||x(t)||$ and by $L^1(\Pi, \mathbb{R}^n)$ the Banach space of all integrable functions $x(.) : \Pi \to \mathbb{R}^n$. Given $a : \Pi \to \mathbb{R}^n$ we denote by L^1_a the Banach space of all integrable functions $\sigma(.) : \Pi \to \mathbb{R}^n$ with the norm

$$|\sigma|_1 = \int\int\int_\Pi a(x,y,z)||\sigma(x,y,z)||dxdydz.$$

Let X be a real separable Banach space with the norm $|.|$ and let (S,d) be a separable metric space. A subset $D \subset L^1(\Pi, X)$ is said to be decomposable, if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathcal{L}(\Pi)$ one has $u\chi_A + v\chi_B \in D$, where $B = \Pi\backslash A$. We denote by $\mathcal{D}(\Pi, X)$ the family of all decomposable nonempty closed subsets of $L^1(\Pi, X)$ and by $\mathcal{D}_1(\Pi, X)$ the family of all decomposable non empty closed bounded subsets of $L^1(\Pi, X)$.

Definition 2.1. Let Y be a Hausdorff topological space. A subspace X of Y is called retract of Y, if there is a continuous map $h : Y \to X$ such that $h(x) = x, \ \forall x \in X$.
Let M, N be metric spaces with distances d_M, resp. d_N. We denote by $\mathcal{K}(M)$ the space of all nonempty closed bounded subsets of M endowed with the Hausdorff metric d_M, given by

$$H_{d_M}(A,B) = \max\{\sup_{y\in B} d_M(y,A), \sup_{x\in A} d_M(x,B)\}, \quad A, B \in \mathcal{K}(M).$$

By $B_M(x,r)$, we denote the open ball in M centred at x with radius $r > 0$.
A multifunction $F : N \to \mathcal{K}(M)$ is called Hausdorff lower (resp. upper) semicontinuous, if $\forall x_0 \in N$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $F(x_0) \subset \{y \in M;\ d_M(y, F(x)) < \varepsilon\}$ (resp. $F(x) \subset \{y \in M;\ d_M(y, F(x_0)) < \varepsilon\}$) for every $x \in B_N(x_0, \delta)$. F is called Hausdorff continuous, if it is Hausdorff lower and upper semicontinuous.

The next result [2] is essential in the proof of our main theorem.

Theorem 2.2. *Let E be a measure space with a finite, positive, nonatomic measure μ and let $L^1 := L^1(E, X)$ be the Banach space of all Bochner integrable functions $u : E \to X$ with the norm $||u||_1 = \int_E |u(t)|d\mu(t)$. We assume that L^1 is separable. Let $a(.,.) : S \times L^1 \to \mathcal{D}_1(E, X)$ be a Hausdorff continuous multifunction, that is a contraction with respect to the second argument. Consider the set of fixed points*

$$\mathcal{F}_s := \{u;\ u \in a(s, u)\}.$$

Then there exists a continuous mapping $g : S \times L^1 \to L^1$ such that

$$g(s, u) \in \mathcal{F}_s \quad \forall u \in L^1, \quad g(s, u) = u \quad \forall u \in \mathcal{F}_s.$$

In what follows by Λ we mean the linear subspace of $C(\Pi, \mathbb{R}^n)$ consisting of all $\lambda \in C(\Pi, \mathbb{R}^n)$ such that there exist continuous functions $\varphi : \Pi_1 \to \mathbb{R}^n$, $\psi : \Pi_1 \to \mathbb{R}^n$, $\chi : \Pi_1 \to \mathbb{R}^n$ satisfying (1.3) with $\lambda(x, y, z) = \varphi(x, y) + \psi(y, z) + \chi(x, z) - \varphi(x, 0) - \varphi(0, y) - \psi(0, z) + \psi(0, 0) = \varphi(x, y) + \psi(y, z) + \chi(x, z) - v_1(x) - v_2(y) - v_3(z) + v_0$, $(x, y, z) \in \Pi$. Observe that Λ, equipped with the norm of $C(\Pi, \mathbb{R}^n)$, is a separable Banach space. In order to study problems (1.1)-(1.3), we introduce the following assumption:

Hypothesis 2.3. i) $F : \Pi \times \mathbb{R}^n \times S \to \mathcal{P}(\mathbb{R}^n)$ is $\mathcal{L}(\Pi) \otimes \mathcal{B}(\mathbb{R}^n \times S)$ measurable with nonempty compact values;

ii) For any $(x, y, z, u) \in \Pi \times \mathbb{R}^n$ the set-valued map $s \to F(x, y, s)$ is (Hausdorff) continuous on S;

iii) There exist $h \in L^1(\Pi, \mathbb{R}), k \in L^1(\Pi, \mathbb{R})$ such that

$$H(F(x, y, z, u, s), \{0\}) \leq h(x, y, z), \quad \forall (x, y, z, u, s) \in \Pi \times \mathbb{R}^n \times S,$$
$$H(F(x, y, z, u_1, s), F(x, y, z, u_2, s)) \leq k(x, y, z)||u_1 - u_2||,$$
$$\forall\ u_1, u_2 \in \mathbb{R}^n, (x, y, z) \in \Pi, s \in S.$$

For $(x, y, z) \in \Pi$ and $\varepsilon > 0$ we define,

$$Q(x, y, z) = [0, x] \times [0, y] \times [0, z], \quad R(x, y, z) = [x, 1] \times [y, 1] \times [z, 1],$$
$$P(x, y, z, \varepsilon) = [x - \varepsilon, x + \varepsilon] \times [y - \varepsilon, y + \varepsilon] \times [z - \varepsilon, z + \varepsilon].$$

For $\sigma \in L^1_a$, consider the following Darboux problem:

$$\begin{aligned} u_{xyz}(x, y, z) &= \sigma(x, y, z), \\ u(x, y, 0) &= \varphi(x, y), \\ u(0, y, z) &= \psi(y, z), \\ u(x, 0, z) &= \chi(x, z). \end{aligned} \tag{2.1}$$

Definition 2.4. Let $\lambda \in \Lambda$. The function $u \in C(\Pi, \mathbb{R}^n)$ given by

$$u(x, y, z) = \lambda(x, y, z) + \int\int\int_{Q(x,y,z)} \sigma(\xi, \eta, \mu)d\xi d\eta d\mu, \quad (x, y, z) \in \Pi,$$

is said to be a solution of (2.1).
Obviously, problem (2.1) has a unique solution, which will be denoted by $u^{\lambda,\sigma}$.

Definition 2.5. Let Hypothesis 2.3 be satisfied and let $\lambda \in \Lambda$. A function $u \in C(\Pi, \mathbb{R}^n)$ is said to be a solution of problem (1.1)-(1.2), if there exists a function $\sigma \in L^1_a$ such that

$$\sigma(x, y, z) \in F(x, y, z, u(x, y, z), s), \quad a.e.\ (\Pi),$$
$$u(x, y, z) = \lambda(x, y, z) + \int\int\int_{Q(x,y,z)} \sigma(\xi, \eta, \mu)d\xi d\eta d\mu, \quad (x, y, z) \in \Pi.$$

We denote by $\mathcal{S}(\lambda, s)$ the solution set of (1.1)-(1.2).

Proposition 2.6. *Let $k(.,.) \in L^1(\Pi, \mathbb{R}_+)$ and $\alpha \in (0,1)$. Then, there exists a continuous function $a : \Pi \to \mathbb{R}$, $a(x,y,z) > 0$ $\forall (x,y,z) \in \Pi$ such that, for any $(x,y,z) \in \Pi$, one has*

$$\int\int\int_{R(x,y,z)} k(\xi,\eta,\mu)a(\xi,\eta,\mu)d\xi d\eta d\mu = \alpha(a(x,y,z)-1).$$

Proof. For $n \in \mathbf{N}$, consider $x_i = i/n$, $i = 0,1,...,n$. Take $n \in \mathbf{N}$ such that

$$\int\int\int_{[x_{i-1},x_i]\times I\times I} k(\xi,\eta,\mu)d\xi d\eta d\mu < \alpha, \quad 1 = 1,2,...,n.$$

Define $T : C_n \to C_n$ by $(Ta)(x,y,z) = \frac{1}{\alpha}\int\int\int_{R(x,y,z)} k(\xi,\eta,\mu)a(\xi,\eta,\mu)d\xi d\eta d\mu + 1$, where $C_n = C([x_{n-1},x_n]\times I\times I, \mathbb{R})$. Since,

$$|Ta_1 - Ta_2|_C \le \frac{1}{\alpha}\int\int\int_{[x_{n-1},x_n]\times I\times I} k(\xi,\eta,\mu)d\xi d\eta d\mu |a_1-a_2|_C < |a_1-a_2|_C,$$

from Banach fixed point theorem, we deduce the existence of a continuous function $a_n : [x_{n-1},x_n]\times I\times I \to (0,\infty)$ such that

$$\int\int\int_{R(x,y,z)} k(\xi,\eta,\mu)a_n(\xi,\eta,\mu)d\xi d\eta d\mu = \alpha(a_n(x,y,z)-1)$$

for any $(x,y,z) \in [x_{n-1},x_n]\times I\times I$.
By induction, we find $a_i : [x_{i-1},x_i]\times I\times I \to (0,\infty)$, $i = 1,2,...,n-1$, such that

$$\int\int\int_{[x,x_i]\times[y,1]\times[z,1]} k(\xi,\eta,\mu)a_i(\xi,\eta,\mu)d\xi d\eta d\mu = \alpha(a_i(x,y,z) - a_{i+1}(x,y,z))$$

for any $(x,y,z) \in [x_{i-1},x_i]\times I\times I$. Finally, we define $a(.,.) : \Pi \to \mathbb{R}$ by

$$a(x,y,z) := \sum_{i=1}^{n} a_i(x,y,z)\chi_{U_i}(x,y,z),$$

where $U_1 = [x_0,x_1]\times I\times I$ and $U_i = (x_{i-1},x_i]\times I\times I$, $i = 2,...,n$ and the proof is completed. □

Proposition 2.7. *Consider $T : \Lambda \times L^1_a \to C(\Pi, \mathbb{R}^n)$ defined by $T(\lambda,\sigma) = u^{\lambda,\sigma}$, where $u^{\lambda,\sigma}$ is solution of (2.1). Then T is a one-to-one maping.*

Proof. Obviously, T is linear. In order to prove that T is injective, let $(\lambda_i,\sigma_i) \in \Lambda\times L^1_a$, $i = 1,2$ such that $T(\lambda_1,\sigma_1) = T(\lambda_2,\sigma_2)$. It follows $\lambda_1 = \lambda_2$ and if we put $\sigma = \sigma_1 - \sigma_2$, we find

$$\int\int\int_{Q(x,y,z)} \sigma(\xi,\eta,\mu)d\xi d\eta d\mu = 0, \quad \forall (x,y,z) \in \Pi. \tag{2.2}$$

Consider L the set of Lebesgue points of $\sigma(.,.)$ which belongs to the interior of Π; the Lebesgue measure of the set $\Pi\backslash L$ is 0 (e.g., [7] page 217) and for any $(\xi,\eta,\mu) \in L$,

$$\lim_{\varepsilon\to 0}\frac{1}{8\varepsilon^3}\int\int\int_{P(\xi,\eta,\mu,\varepsilon)} (\sigma(\xi,\eta,\mu) - \sigma(x,y,z))dxdydz = 0.$$

It is clear that

$$\sigma(\xi,\eta,\mu) = \frac{1}{8\varepsilon^3}\int\int\int_{P(\xi,\eta,\mu,\varepsilon)} (\sigma(\xi,\eta,\mu) - \sigma(x,y,z))dxdydz + \frac{1}{8\varepsilon^3}\int\int\int_{P(\xi,\eta,\mu,\varepsilon)} \sigma(x,y,z)dxdydz \tag{2.3}$$

and

$$\begin{aligned}\int\int\int_{P(\xi,\eta,\mu,\varepsilon)}\sigma(x,y,z)dxdydz=&\int\int\int_{Q(x+\varepsilon,y+\varepsilon,z+\varepsilon)}\sigma(x,y,z)dxdydz\\&+\int\int\int_{Q(x+\varepsilon,y-\varepsilon,z-\varepsilon)}\sigma(x,y,z)dxdydz\\&+\int\int\int_{Q(x-\varepsilon,y+\varepsilon,z-\varepsilon)}\sigma(x,y,z)dxdydz\\&+\int\int\int_{Q(x-\varepsilon,y-\varepsilon,z+\varepsilon)}\sigma(x,y,z)dxdydz\\&-\int\int\int_{Q(x-\varepsilon,y+\varepsilon,z+\varepsilon)}\sigma(x,y,z)dxdydz\\&-\int\int\int_{Q(x+\varepsilon,y-\varepsilon,z+\varepsilon)}\sigma(x,y,z)dxdydz\\&-\int\int\int_{Q(x+\varepsilon,y+\varepsilon,z-\varepsilon)}\sigma(x,y,z)dxdydz\\&-\int\int\int_{Q(x-\varepsilon,y-\varepsilon,z-\varepsilon)}\sigma(x,y,z)dxdydz.\end{aligned}\tag{2.4}$$

From (2.2), (2.3) and (2.4), passing with $\varepsilon\to 0$, we obtain $\sigma(\xi,\eta,\mu)\equiv 0$ and therefore $\sigma_1=\sigma_2$. □

Let $(\lambda,s,v)\in\Lambda\times S\times L^1_a$, let $u^{\lambda,\sigma}:\Pi\to\mathbb{R}^n$ be solution of (2.1). Define

$$\mathcal{V}(\lambda,s,\sigma):=\{g\in L^1_a; g(x,y,z)\in F(x,y,z,u^{\lambda,\sigma}(x,y,z),s),\quad a.p.t.(\Pi)\}.\tag{2.5}$$

$$\mathcal{F}(\lambda,s):=\{g\in L^1_a; g\in\mathcal{V}(\lambda,s,\sigma)\}.\tag{2.6}$$

$$W=\{u\in C(\Pi,\mathbb{R}^n);\ \exists(\lambda,\sigma)\in\Lambda\times L^1_a \text{ such that } u=u^{\lambda,\sigma}\}.$$

Since $\mathcal{V}(\lambda,s,\sigma)$ is a bounded closed decomposable set, (2.5) defines a set-valued map $\mathcal{V}:\Lambda\times S\times L^1_a\to\mathcal{D}(L^1_a)$. From Proposition 2.7 for any $u\in W$, there exists a unique pair $(\lambda,\sigma)\in\Lambda\times L^1_a$ such that, $u=u^{\lambda,\sigma}$. Therefore, we denote by $u^{\lambda,\sigma}$ any element of W.
Let $k(.,.)\in L^1(\Pi,\mathbb{R})$ as in Hypothesis 2.3 and $a:\Pi\to\mathbb{R}$ the corresponding mapping given by Proposition 2.6. With this choice of a, for $u^{\lambda,\sigma}\in W$ we define,

$$|u^{\lambda,\sigma}|_W=|u^{\lambda,\sigma}|_C+|\sigma|_1.\tag{2.7}$$

Proposition 2.8. *W endowed with the norm defined in (2.7) is a Banach space.*

Proof. Let T the map defined in Proposition 2.7. Since $W=T(\Lambda\times L^1_a)$, W is a linear space. From the injectivity of T, we find that $|.|_W$ is a norm on W. It remains to prove that W with the norm in (2.7) is complete. Let $\{u^{\lambda_n,\sigma_n}\}$ a Cauchy sequence in W. If $m,n\in\mathbb{N}$, we have

$$|u^{\lambda_n,\sigma_n}-u^{\lambda_m,\sigma_m}|_W=|u^{\lambda_n,\sigma_n}-u^{\lambda_m,\sigma_m}|_C+|\sigma_n-\sigma_m|_1.$$

Since $|\lambda_n-\lambda_m|_C\le|u^{\lambda_n,\sigma_n}-u^{\lambda_m,\sigma_m}|_C$ and $\{u^{\lambda_n,\sigma_n}\}$ is Cauchy in $C(I,\mathbb{R}^n)$, it follows that the sequence $\{\lambda_n\}_n$ is convergent to some $\lambda\in\Lambda$. Similarly, we get that $\{\sigma_n\}_n$ converges to some $\sigma\in L^1_a$.

Obviously, if $u^{\lambda,\sigma}$ is solution of problem (2.1), then $u^{\lambda,\sigma} \in W$. Finally, taking into account Proposition 2.6, one may write

$$
\begin{aligned}
||u^{\lambda_n,\sigma_n}(x,y,z) - u^{\lambda,\sigma}(x,y,z)|| &\leq |\lambda_n - \lambda|_C + \int\int\int_{Q(x,y,z)} ||\sigma_n(\xi,\eta,\mu) - \sigma(\xi,\eta,\mu)||d\xi d\eta d\mu \\
&= |\lambda_n - \lambda|_C + \int\int\int_{Q(x,y,z)} \frac{1}{a(\xi,\eta,\mu)} a(\xi,\eta,\mu)||\sigma_n(\xi,\eta,\mu) - \sigma(\xi,\eta,\mu)||d\xi d\eta d\mu \\
&< |\lambda_n - \lambda|_C + \int\int\int_{Q(x,y,z)} a(\xi,\eta,\mu)||\sigma_n(\xi,\eta,\mu) - \sigma(\xi,\eta,\mu)||d\xi d\eta d\mu \\
&\leq |\lambda_n - \lambda|_C + |\sigma_n - \sigma|_1
\end{aligned}
$$

and thus u^{λ_n,σ_n} converges to $u^{\lambda,\sigma}$ in W, i.e. W is completed. $\square$

For $\lambda \in \Lambda$ we define,

$$W(\lambda) = \{u \in W;\ u(x,y,0) = \lambda(x,y,0), \quad u(0,y,z) = \lambda(0,y,z), \quad u(x,0,z) = \lambda(x,0,z), \quad \forall (x,y,z) \in \Pi\}.$$

Clearly, $W(\lambda) \subset W$ is nonempty closed convex set and $\mathcal{S}(\lambda,s) \subset W(\lambda), \quad \forall s \in S$.

3. Main results

We are able now to prove the main result of this paper.

Theorem 3.1. *Assume that Hypothesis 2.3 is satisfied and define,*

$$G = \{(\lambda,s,u) \in \Lambda \times S \times W; \quad (\lambda,s) \in \Lambda \times S, u \in W(\lambda)\}.$$

Then, there exists a continuous mapping $\Phi : G \to W$ such that for any $(\lambda,\mu) \in \Lambda \times S$,

$$\Phi(\lambda,s,u) \in \mathcal{S}(\lambda,s), \quad \forall u \in W(\lambda), \tag{3.1}$$

$$\Phi(\lambda,s,u) = u, \quad \forall u \in \mathcal{S}(\lambda,s). \tag{3.2}$$

Proof. Consider $\mathcal{V} : \Lambda \times S \times L^1_a \to \mathcal{D}(L^1_a)$ the set-valued map defined in 2.5. First, we prove that $\mathcal{V}(.)$ is (Hausdorff) continuous. At the begining, we show that $\mathcal{V}(.)$ is (Hausdorff) lower semicontinuous. By contrary, assume that there exists $\varepsilon > 0$ and a sequence $(\lambda_n, s_n, \sigma_n)_{n\in\mathbf{N}}$ that converges to $(\lambda_0, s_0, \sigma_0)$ in $\Lambda \times S \times L^1_a$ and a sequence $g_n \in L^1_a$ with $g_n \in \mathcal{V}(\lambda_0, s_0, \sigma_0)$ such that

$$d_{L^1_a}(g_n, \mathcal{V}(\lambda_n, s_n, \sigma_n)) \geq \varepsilon, \quad \forall n \in \mathbb{N}. \tag{3.3}$$

For $n \in \mathbb{N}$, we define $M_n : \Pi \to \mathcal{P}(\mathbb{R}^n)$ by

$$M_n(x,y,z) := F(x,y,z,u^{\lambda_n,\sigma_n}(x,y,z),s_n) \cap B(g_n(x,y,z), d(g_n(x,y,z), F(x,y,z,u^{\lambda_n,\sigma_n}(x,y,z),s_n))).$$

Since the set-valued map M_n is measurable, there exists a measurable selection $\overline{g}_n \in \mathcal{V}(\lambda_n, s_n, \sigma_n)$ such that for *a.e.* (Π)

$$||g_n(x,y,z) - \overline{g}_n(x,y,z)|| = d(g_n(x,y,z), F(x,y,z,u^{\lambda_n,\sigma_n}(x,y,z),s_n)).$$

Since $g_n(x,y,z) \in F(x,y,z,u^{\lambda_0,\sigma_0}(x,y,z),s_0)$, one has

$$
\begin{aligned}
&\int\int\int_\Pi a(x,y,z)||g_n(x,y,z) - \overline{g}_n(x,y,z)||dxdydz = \\
&\int\int\int_\Pi a(x,y,z)H(F(x,y,z,u^{\lambda_0,\sigma_0}(x,y,z),s_0), F(x,y,z,u^{\lambda_n,\sigma_n}(x,y,z),s_n))dxdydz \leq \\
&\int\int\int_\Pi a(x,y,z)H(F(x,y,z,u^{\lambda_n,\sigma_n}(x,y,z),s_n), F(x,y,z,u^{\lambda_0,\sigma_0}(x,y,z),s_n))dxdydz + \\
&\int\int\int_\Pi a(x,y,z)H(F(x,y,z,u^{\lambda_0,\sigma_0}(x,y,z),s_n), F(x,y,z,u^{\lambda_0,\sigma_0}(x,y,z),s_0))dxdydz.
\end{aligned}
$$

Let $w_n(.,.)$ be the function under the second integral; using Hypothesis 2.3 we get,

$$|g_n - \overline{g}_n|_1 \leq \int\int\int_\Pi a(x,y,z)k(x,y,z)||u^{\lambda_n,\sigma_n}(x,y,z) - u^{\lambda_0,\sigma_0}(x,y,z)||dxdydz + \int\int\int_\Pi w_n(x,y,z)dxdydz.$$

Letting $n \to \infty$, using Lebesgue's dominated convergence Theorem, we obtain $|g_n - \overline{g}_n|_1 \to 0$ as $n \to \infty$. Therefore, there exists $n_0 \in \mathbb{N}$, such that $|g_n - \overline{g}_n|_1 < \frac{1}{2}\varepsilon$, $\forall n \geq n_0$ and thus $d_{L^1_a}(g_n, \mathcal{V}(\lambda_n, s_n, \sigma_n)) < \frac{\varepsilon}{2}$, $\forall n \geq n_0$, which contradicts (3.3). So, $\mathcal{V}(.)$ is (Hausdorff) lower semicontinuous. Similarly, we can prove that $\mathcal{V}(.)$ is (Hausdorff) upper semicontinuous. Hence $\mathcal{V}(.)$ is (Hausdorff) continuous.
Secondly, we prove that for any $(\lambda, s, \sigma_1), (\lambda, s, \sigma_2) \in \Lambda \times S \times L^1_a$,

$$H(\mathcal{V}(\lambda, s, \sigma_1), \mathcal{V}(\lambda, s, \sigma_2)) \leq \alpha|\sigma_1 - \sigma_2|_1. \tag{3.4}$$

Let $(\lambda, s, \sigma_i) \in \Lambda \times S \times L^1_a$, $i = 1, 2$. For any $(x, y, z) \in \Pi$ we have,

$$||u^{\lambda,\sigma_1}(x,y,z) - u^{\lambda,\sigma_2}(x,y,z)|| \leq \int\int\int_{Q(x,y,z)} ||\sigma_1(\xi,\eta,\mu) - \sigma_2(\xi,\eta,\mu)||d\xi d\eta d\mu. \tag{3.5}$$

Consider an arbitrary $g_1 \in \mathcal{V}(\lambda, s, \sigma_1)$ and take $g_2 \in \mathcal{V}(\lambda, s, \sigma_2)$ such that,

$$||g_1(x,y,z) - g_2(x,y,z)|| = d(g_1(x,y,z), F(x,y,z,u^{\lambda,\sigma_2}(x,y,z), s) \quad a.e.(\Pi).$$

From the last equality, the fact that $g_1(x,y,z) \in F(x,y,z,u^{\lambda,\sigma_1}(x,y,z), s)$, (3.5) and Proposition 2.6 one has

$$\begin{aligned}
|g_1 - g_2|_1 &\leq \int\int\int_\Pi a(x,y,z)H(F(x,y,z,u^{\lambda,\sigma_1}(x,y,z),s), F(x,y,z,u^{\lambda,\sigma_2}(x,y,z),s))dxdydz \\
&\leq \int\int\int_\Pi a(x,y,z)k(x,y,z)||u^{\lambda,\sigma_1}(x,y,z) - u^{\lambda,\sigma_2}(x,y,z)||dxdydz \\
&\leq \int\int\int_\Pi a(x,y,z)k(x,y,z)(\int\int\int_{Q(x,y,z)} ||\sigma_1(\xi,\eta,\mu) - \sigma_2(\xi,\eta,\mu)||d\xi d\eta d\mu)dxdydz \\
&= \int\int\int_\Pi ||\sigma_1(\xi,\eta,\mu) - \sigma_2(\xi,\eta,\mu)||(\int\int\int_{R(\xi,\eta,\mu)} a(x,y,z)k(x,y,z)dxdydz)d\xi d\eta d\mu) \\
&\leq \alpha \int\int\int_\Pi ||\sigma_1(\xi,\eta,\mu) - \sigma_2(\xi,\eta,\mu)||a(\xi,\eta,\mu)d\xi d\eta d\mu \\
&= \alpha|\sigma_1 - \sigma_2|_1.
\end{aligned}$$

It follows that $d_{L^1_a}(g_1, \mathcal{V}(\lambda, s, \sigma_2)) \leq \alpha|\sigma_1 - \sigma_2|_1$ and since $g_1 \in \mathcal{V}(\lambda, s, \sigma_1)$ is arbitrary, we deduce that $\sup_{g_1 \in \mathcal{V}(\lambda,s,\sigma_1)} d_{L^1_a}(g_1, \mathcal{V}(\lambda, s, \sigma_2)) \leq \alpha|\sigma_1 - \sigma_2|_1$. From the last inequality and the similar inequality obtained by interchanging g_1 with g_2 we find (3.4).

In the next step of the proof, we apply Theorem 2.2 and we find a continuous function $\phi : \Lambda \times S \times L^1_a \to L^1_a$ such that, for any $(\lambda, s) \in \Lambda \times S$,

$$\phi(\lambda, s, \sigma) \in \mathcal{F}(\lambda, s), \quad \forall \sigma \in L^1_a, \tag{3.6}$$

$$\phi(\lambda, s, \sigma) = \sigma, \quad \forall \sigma \in \mathcal{F}(\lambda, s). \tag{3.7}$$

Let $(\lambda, s, \sigma) \in G$. Since $u \in W(\lambda)$, there exists $\sigma \in L^1_a$ such that $u = u^{\lambda,\sigma}$ with $u^{\lambda,\sigma}$ solution of (2.1). Thus $(\lambda, s, \sigma) = (\lambda, s, u^{\lambda,\sigma})$. Consider $\Phi(\lambda, s, u^{\lambda,\sigma}) : \Pi \to \mathbb{R}^n$ defined by

$$\Phi(\lambda, s, u^{\lambda,\sigma})(x,y,z) = \lambda(x,y,z) + \int\int\int_{Q(x,y,z)} \phi(\lambda, s, \sigma)(\xi,\eta,\mu)d\xi d\eta d\mu. \tag{3.8}$$

Since $\Phi(\lambda, s, u^{\lambda,\sigma}) = u^{\lambda,\phi(\lambda,s,\sigma)}$ the last equality defines a set-valued map $\Phi : G \to \mathcal{P}(W)$. Next, we show that Φ is continuous and satisfies (3.1) and (3.2). Take $\varepsilon > 0$, for $(\lambda_0, s_0, u^{\lambda_0,\sigma_0}), (\lambda, s, u^{\lambda,\sigma}) \in G$ we have

$$\begin{aligned} |\Phi(\lambda, s, u^{\lambda,\sigma}) - \Phi(\lambda_0, s_0, u^{\lambda_0,\sigma_0})|_W &= |u^{\lambda,\phi(\lambda,s,\sigma)} - u^{\lambda_0,\phi(\lambda_0,s_0,\sigma_0)}|_C + |\phi(\lambda, s, \sigma) - \phi(\lambda_0, s_0, \sigma_0)|_1 \\ &\le |\lambda - \lambda_0|_C + (1 + \frac{1}{m}) + |\phi(\lambda, s, \sigma) - \phi(\lambda_0, s_0, \sigma_0)|_1, \end{aligned} \tag{3.9}$$

where $m = \min_{(x,y,z)\in\Pi} a(x, y, z) \ge 1$ (see Proposition 2.6).

Taking into account that ϕ is continuous we find $\delta \in (0, \varepsilon)$ such that if $|\lambda - \lambda_0|_C < \delta$, $d(s, s_0) < \delta$, $|\sigma - \sigma_0|_1 < \delta$, then $|\phi(\lambda, s, \sigma) - \phi(\lambda_0, s_0, \sigma_0)|_1 < \varepsilon$. Take $(\lambda, s, u^{\lambda,\sigma}) \in G$ such that $|\lambda - \lambda_0|_C < \delta$, $d(s, s_0) < \delta$, $|u^{\lambda,\sigma} - u^{\lambda_0,\sigma_0}|_W < \delta$. Since $|\sigma - \sigma_0|_1 \le |u^{\lambda,\sigma} - u^{\lambda_0,\sigma_0}|_W < \delta$ it follows $|\phi(\lambda, s, \sigma) - \phi(\lambda_0, s_0, \sigma_0)|_1 < \varepsilon$. Therefore, from (3.9), we deduce $|\Phi(\lambda, s, u^{\lambda,\sigma}) - \Phi(\lambda_0, s_0, u^{\lambda_0,\sigma_0})|_W < \delta + (1 + \frac{1}{m})\varepsilon < (2 + \frac{1}{m})\varepsilon$ i.e., Φ is continuous.

Let, now $(\lambda, s) \in \Lambda \times S$. If $u \in W(\lambda)$ there exists $\sigma \in L^1_a$ such that $u = u^{\lambda,\sigma}$. From (3.6), $\phi(\lambda, s, \sigma) \in \mathcal{F}(\lambda, s)$ and thus $u^{\lambda,\phi(\lambda,s,\sigma)} \in \mathcal{S}(\lambda, s)$. But $\Phi(\lambda, s, u^{\lambda,\sigma}) = u^{\lambda,\phi(\lambda,s,\sigma)}$ and $u^{\lambda,\sigma} = u$, then $\Phi(\lambda, s, \sigma) \in \mathcal{S}(\lambda, s)$ which proves (3.1)

Consider, again, $(\lambda, s) \in \Lambda \times S$ and $u \in \mathcal{S}(\lambda, s)$. There exists $\sigma \in \mathcal{V}(\lambda, s, \sigma)$ such that $u^{\lambda,\sigma} = u$. So, $\sigma \in \mathcal{F}(\lambda, s)$ and by (3.7) $\phi(\lambda, s, \sigma) = \sigma$. From the last equality and (3.8) we get $\Phi(\lambda, s, \sigma) = \Phi(\lambda, s, u^{\lambda,\sigma}) = u^{\lambda,\phi(\lambda,s,\sigma)} = u^{\lambda,\sigma} = u$, which proves (3.2). □

Remark 3.2. From Theorem 3.1, $\mathcal{S}(\lambda, s) \subset W(\lambda)$ is a retract. Since $W(\lambda) \subset W$ is convex, from [1], page 85, $\mathcal{S}(\lambda, s) \subset W(\lambda)$ is an absolute retract. Therefore, $\mathcal{S}(\lambda, s)$ is a closed contractible subspace of W.

Corollary 3.3. *Assume that Hypothesis 2.3 is satisfied. Then, there exists a continuous function $\tau : \Lambda \times S \to W$ such that*

$$\tau(\lambda, s) \in \mathcal{S}(\lambda, s), \quad \forall(\lambda, s) \in \Lambda \times S. \tag{3.10}$$

Proof. For $\lambda \in \Lambda$ we set $u(\lambda) = u^{\lambda,0}$ solution of problem (2.1) (with $\sigma = 0$). Define $\tau : \Lambda \times S \to W$ by $\tau(\lambda, s) := \Phi(\lambda, s, u(\lambda))$, where $\Phi(.)$ is the mapping constructed in Theorem 3.1. Since $u(\lambda) \in W(\lambda)$, the map τ is well defined. On the other hand, since $|u(\lambda) - u(\lambda_0)|_W = |\lambda - \lambda_0|_C$, $\forall \lambda, \lambda_0 \in \Lambda$, τ is continuous and from (3.1), $\tau(., .)$ satisfies (3.10). □

Corollary 3.4. *Assume that Hypothesis 2.3 is satisfied. For $i = 1, 2$, consider $\tau_i : \Lambda \times S \to W$ continuous functions such that*

$$\tau_i(\lambda, s) \in \mathcal{S}(\lambda, s), \quad \forall(\lambda, s) \in \Lambda \times S.$$

Then, there exists a continuous function $h : \Lambda \times S \times I \to W$ such that

i) $h(\lambda, s, 0) = \tau_1(\lambda, s)$, $h(\lambda, s, 1) = \tau_2(\lambda, s)$, $\forall(\lambda, s) \in \Lambda \times S$;

ii) $h(\lambda, s, x) \in \mathcal{S}(\lambda, s)$, $\forall(\lambda, s, x) \in \Lambda \times S \times I$.

Proof. Define $h : \Lambda \times S \times I \to W$, by

$$h(\lambda, s, x) := \Phi(\lambda, s, (1 - x)\tau_1(\lambda, s) + x\tau_2(\lambda, s)),$$

where $\Phi(.)$ is the mapping constructed in Theorem 3.1. □

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