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$F(\psi, \varphi)$-contractions for α-admissible mappings on metric spaces and related fixed point results

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Abstract

In this paper, we prove the existence and uniqueness of fixed points for certain α-admissible mappings which are $F(\psi, \varphi)$-contractions on metric spaces. Our results generalize and extend some well-known results in the literature.

Keywords: fixed point, C-class function, α-admissible mapping, $F(\psi, \varphi)$-contraction.
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1. Introduction

Geraghty in [2] introduced an interesting class of auxiliary function to refine the Banach contraction mapping principle. Let $\mathcal{F}$ be the functions $\beta : [0, \infty) \to [0, 1)$ which satisfies the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

By using $\beta \in \mathcal{F}$, Geraghty [2] proved the following remarkable theorem.

Theorem 1.1. ([2]) *Let (X, d) be a complete metric space and $T : X \to X$ be an operator. Suppose that there exists $\beta \in \mathcal{F}$, satisfying the condition,*

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0.$$

If T satisfies the following inequality:

$$d(Tx, Ty) \le \beta(d(x, y))d(x, y), \text{ for any } x, y \in X, \tag{1.1}$$

then T has a unique fixed point.

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Recently, Samet *et al.* [4] introduced the class of $\alpha - \psi$ contractive type mappings and obtain a fixed point result for this new class of mappings in the set up of metric space which properly contains several well-known fixed point theorems including Banach contraction principle.

In this work, we introduce the class of $F(\psi, \varphi)$-contractions and investigate the existence and uniqueness of fixed points for the α-admissible mappings on the metric spaces and we will show that the fixed point results in [3] and Theorem 1.1 are immediate corollaries of our results.

2. Preliminaries

Definition 2.1. Let $f : X \to X$ and $\alpha : X \times X \to (-\infty, +\infty)$. We say that f is an α-admissible mapping if $\alpha(x, y) \geq 1$ implies $\alpha(fx, fy) \geq 1$, for all $x, y \in X$.

Definition 2.2. Let Ψ denote all functions $\psi : [0, \infty) \to [0, \infty)$ satisfied:
(i) ψ is strictly increasing and continuous;
(ii) $\psi(t) = 0$ if and only if $t = 0$.

We let Ψ denote the class of the altering distance functions.

Definition 2.3. ([1]) An ultra altering distance function is a continuous, nondecreasing mapping $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) > 0$ for $t > 0$.

We let Φ denote the class of the ultra altering distance functions.

Definition 2.4. ([1]) A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called C-*class* function if it is continuous and satisfies following axioms:

1. $F(s, t) \leq s$.
2. $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

We denote C-class functions by $\mathcal{C}$.

Example 2.5. ([1]) The following functions $F : [0, \infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s, t \in [0, \infty)$:

1. $F(s, t) = s - t$.
2. $F(s, t) = ms$, $0<m<1$.
3. $F(s, t) = \frac{s}{(1+t)^r}$; $r \in (0, \infty)$.
4. $F(s, t) = \log(t + a^s)/(1 + t)$, $a > 1$.
5. $F(s, t) = \ln(1 + a^s)/2$, $a > e$.
6. $F(s, t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1, r \in (0, \infty)$.

3. Main Result

We start this section with the following theorem.

Theorem 3.1. *let (X, d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Suppose that the following condition is satisfied:*

$$(\psi(d(Tx, Ty) + l))^{\alpha(x,Tx)\alpha(y,Ty)} \leq F(\psi(d(x, y)), \varphi(d(x, y)) + l \tag{3.1}$$

for all $x, y \in X$ and $l \geq 1$, where $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$. Suppose that either,
(a) T is continuous;
or
(b) if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x, Tx) \geq 1$.
If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since T is an α-admissible mapping and $\alpha(x_0, Tx_0) \geq 1$, we deduce that $\alpha(x_1, x_2) = \alpha(Tx_0, T^2x_0) \geq 1$. By continuing this process, we get $\alpha(x_n, Tx_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. By the inequality (3.1) we have

$$\begin{aligned} \psi(d(Tx_{n-1}, Tx_n)) + l &\leq (\psi(d(Tx_{n-1}, Tx_n) + l))^{\alpha(x_{n-1}, Tx_{n-1})\alpha(x_n, Tx_n)} \\ &\leq F(\psi(d(x_{n-1}, x_n)), \varphi(d(x_{n-1}, x_n))) + l, \end{aligned}$$

then we have

$$\psi(d(x_n, x_{n+1})) \leq F(\psi(d(x_{n-1}, x_n)), \varphi(d(x_{n-1}, x_n))) \leq \psi(d(x_{n-1}, x_n)). \tag{3.2}$$

Since ψ is strictly-increasing, inequality (3.2) implies that

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n).$$

It follows that the sequence $\{d(x_n, x_{n+1})\}$ is decreasing. So, there exists $r \in \mathbb{R}_+$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

We want to prove that $r = 0$. Suppose to the contrary $r > 0$. From (3.2) we have

$$\limsup_{n\to\infty} \psi(d(x_n, x_{n+1})) \leq \limsup_{n\to\infty} F(\psi(d(x_{n-1}, x_n)), \varphi(d(x_{n-1}, x_n))) \leq \limsup_{n\to\infty} \psi(d(x_{n-1}, x_n)).$$

Hence we get

$$\psi(r) \leq F(\psi(r), \varphi(r)) \leq \psi(r),$$

that means

$$F(\psi(r), \varphi(r)) = \psi(r).$$

By using the property of the functions F, ψ and φ, we obtain that $\psi(r) = 0$, or $\varphi(r) = 0$, then $r = 0$, which is contradiction and therefore

$$d(x_n, x_{n+1}) \to 0 \quad \text{as} \quad \text{n} \to \infty. \tag{3.3}$$

Now we prove that $\{x_n\}$ is Cauchy sequence in (X, d). Suppose that $\{x_n\}$ is not Cauchy sequence, that means $\lim_{n,m\to\infty} d(x_n, x_m) \neq 0$, so there exist $\varepsilon > 0$ and $\{m_k\} \subset \mathbb{N}$ such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon.$$

Suppose that k is the smallest integer which satisfies the above equation such that

$$d(x_{m_k - 1}, x_{n_k}) < \varepsilon.$$

Now we have

$$\varepsilon \leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) < d(x_{m_k}, x_{m_k-1}) + \varepsilon,$$

thus

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon. \tag{3.4}$$

Again we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k})$$

and

$$d(x_{m_k+1}, x_{n_k+1}) \leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k+1}, x_{n_k}).$$

Taking the limit as $k \to +\infty$, together with (3.3) we have

$$\lim_{k\to\infty} d(x_{l_k+1}, x_{n_k+1}) = \varepsilon. \tag{3.5}$$

Now by (3.1), (3.4) and (3.5) we have

$$\begin{aligned}\psi(d(x_{m_k+1},x_{n_k+1}))+l &\leq (\psi(d(x_{m_k+1},x_{n_k+1}))+l)^{\alpha(x_{m_k},Tx_{m_k})\alpha(x_{n_k},Tx_{n_k})}\\ &= (\psi(d(Tx_{m_k},Tx_{n_k})+l))^{\alpha(x_{m_k},Tx_{m_k})\alpha(x_{n_k},Tx_{n_k})}\\ &\leq F(\psi(d(x_{m_k},x_{n_k})),\varphi(d(x_{m_k},x_{n_k})))+l\\ &\leq \psi(d(x_{m_k},x_{n_k}))+l.\end{aligned}$$

Therefore we get

$$\psi(d(x_{m_k+1},x_{n_k+1}))\leq F(\psi(d(x_{m_k},x_{n_k})),\varphi(d(x_{m_k},x_{n_k})))\leq \psi(d(x_{m_k},x_{n_k})).$$

Letting $k\to\infty$ in the above inequality, we get

$$\psi(\varepsilon)\leq F(\psi(\varepsilon),\varphi(\varepsilon))\leq\psi(\varepsilon).$$

that means

$$F(\psi(\varepsilon),\varphi(\varepsilon))=\psi(\varepsilon),$$

by using the property of the functions F, ψ and φ, we obtain that $\psi(\varepsilon)=0$ or $\varphi(\varepsilon)=0$, then $\varepsilon=0$, which is contradiction and therefore $\{x_n\}$ is a Cauchy sequence. Now by completeness of X, $x_n\to x$, for some $x\in X$, that means

$$\lim_{n\to\infty} d(x_n,x)=0.$$

First, we suppose that T is continuous, then we have

$$Tx=\lim_{n\to\infty} Tx_n=\lim_{n\to\infty} x_{n+1}=x.$$

So x is a fixed point of T. Now we suppose that (b) holds, then $\alpha(x,Tx)\geq 1$. Now by (3.1) we have

$$\begin{aligned}\psi(d(Tx_n,Tx))+l &\leq (\psi(d(Tx_n,Tx))+l)^{\alpha(x_n,Tx_n)\alpha(x,Tx)}\\ &\leq F(\psi(d(x_n,x)),\varphi(d(x_n,x)))+l\\ &\leq \psi(d(x_n,x))+l,\end{aligned}$$

that is $d(Tx_n,Tx)\leq d(x_n,x)$ and so we get

$$0\leq d(Tx,x)\leq d(Tx,x_{n+1})+d(x,x_{n+1})\leq d(x,x_n)+d(x,x_{n+1}).$$

Letting $n\to\infty$ in the above inequality, we get $d(Tx,x)=0$, that is, $Tx=x$. □

Theorem 3.2. *Let (X,d) be a complete metric space and $T:X\to X$ be an α-admissible mapping. Suppose that the following condition is satisfied:*

$$(\alpha(x,Tx)\alpha(y,Ty)+1)^{\psi(d(Tx,Ty))}\leq 2^{F(\psi(d(x,y)),\varphi(d(x,y)))} \tag{3.6}$$

for all $x,y\in X$, where $\psi\in\Psi$, $\varphi\in\Phi$ and $F\in\mathcal{C}$. Suppose that either,
(a) T is continuous;
or
(b) if $\{x_n\}$ is a sequence in X such that $x_n\to x$, $\alpha(x_n,x_{n+1})\geq 1$ for all n, then $\alpha(x,Tx)\geq 1$.
If there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$. Then T has a fixed point.

Proof. Let $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$. Define a sequence $\{x_n\}$ in X by $x_n=T^nx_0=Tx_{n-1}$ for all $n\in\mathbb{N}$. Since T is an α-admissible mapping and $\alpha(x_0,Tx_0)\geq 1$, we deduce that $\alpha(x_1,x_2)=\alpha(Tx_0,T^2x_0)\geq 1$. By continuing this process, we get $\alpha(x_n,Tx_n)\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$. By the inequality (3.6) we have

$$\begin{aligned}2^{\psi(d(Tx_{n-1},Tx_n))} &\leq (\alpha(x_{n-1},Tx_{n-1})\alpha(x_n,Tx_n)+1)^{\psi(d(Tx_{n-1},Tx_n))}\\ &\leq 2^{F(\psi(d(x_{n-1},x_n)),\varphi(d(x_{n-1},x_n)))},\end{aligned}$$

then we have

$$\psi(d(x_n, x_{n+1})) \leq F(\psi(d(x_{n-1}, x_n)), \varphi(d(x_{n-1}, x_n))) \leq \psi(d(x_{n-1}, x_n)). \tag{3.7}$$

Now similar to the proof of Theorem 3.1 we get

$$d(x_n, x_{n+1}) \to 0 \quad \text{as} \quad \text{n} \to \infty. \tag{3.8}$$

Now we shall prove that $\{x_n\}$ is Cauchy sequence in (X, d). Suppose that $\{x_n\}$ is not Cauchy sequence, that means $\lim_{n,m\to\infty} d(x_n, x_m) \neq 0$, so there exist $\varepsilon > 0$ and $\{m_k\} \subset \mathbb{N}$ such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon.$$

Let k be the smallest integer which satisfies above equation such that

$$d(x_{m_k-1}, x_{n_k}) < \varepsilon.$$

Again by the proof of Theorem 3.1, we obtain

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon \tag{3.9}$$

and

$$\lim_{k\to\infty} m(x_{l_k+1}, x_{n_k+1}) = \varepsilon. \tag{3.10}$$

Now by (3.6), (3.9) and (3.10) we have

$$\begin{aligned} 2^{\psi(d(x_{m_k+1}, x_{n_k+1}))} &\leq (\alpha(x_{m_k}, Tx_{m_k})\alpha(x_{n_k}, Tx_{n_k}) + 1)^{\psi(d(x_{m_k+1}, x_{n_k+1}))} \\ &\leq 2^{F(\psi(d(x_{m_k}, x_{n_k})), \varphi(d(x_{m_k}, x_{n_k})))}, \end{aligned}$$

therefore we get

$$\psi(d(x_{m_k+1}, x_{n_k+1})) \leq F(\psi(d(x_{m_k}, x_{n_k})), \varphi(d(x_{m_k}, x_{n_k}))) \leq \psi(d(x_{m_k}, x_{n_k})).$$

Letting $k \to \infty$ in the above inequality, we get

$$\psi(\varepsilon) \leq F(\psi(\varepsilon), \varphi(\varepsilon)) \leq \psi(\varepsilon),$$

that means

$$F(\psi(\varepsilon), \varphi(\varepsilon)) = \psi(\varepsilon).$$

By using the property of the functions F, ψ and φ, we obtain that $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$, then $\varepsilon = 0$, which is a contradiction and therefore $\{x_n\}$ is a Cauchy sequence. Now by completeness of X, $x_n \to x$, for some $x \in X$, that means,

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

First, we suppose that T is continuous, then we have

$$Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x.$$

So x is a fixed point of T. Now we suppose that (b) holds, then $\alpha(x, Tx) \geq 1$. By (3.6) we have

$$\begin{aligned} 2^{\psi(d(Tx_n, Tx))} &\leq (\alpha(x_n, Tx_n)\alpha(x, Tx) + 1)^{\psi(d(Tx_n, Tx))} \\ &\leq 2^{F(\psi(d(x_n, x)), \varphi(d(x_n, x)))}, \end{aligned}$$

that is $d(Tx_n, Tx) \leq d(x_n, x)$ and so we get

$$0 \leq d(Tx, x) \leq d(Tx, x_{n+1}) + d(x, x_{n+1}) \leq d(x, x_n) + d(x, x_{n+1}).$$

Letting $n \to \infty$ in the above inequality, we get $d(Tx, x) = 0$, so $Tx = x$. □

Theorem 3.3. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Suppose that the following condition is satisfied:*

$$\alpha(x, Tx)\alpha(y, Ty)\psi(d(Tx, Ty)) \leq F(\psi(d(x, y)), \varphi(d(x, y))) \tag{3.11}$$

for all $x, y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$, Suppose that either,
(a) T is continuous;
or
(b) if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x, Tx) \geq 1$.
If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since T is an α-admissible mapping and $\alpha(x_0, Tx_0) \geq 1$, we deduce that $\alpha(x_1, x_2) = \alpha(Tx_0, T^2x_0) \geq 1$. By continuing this process, we get $\alpha(x_n, Tx_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. By the inequality (3.11) we have

$$\begin{aligned} \psi(d(Tx_{n-1}, Tx_n)) &\leq \alpha(x_{n-1}, Tx_{n-1})\alpha(x_n, Tx_n)\psi(d(Tx_{n-1}, Tx_n)) \\ &\leq F(\psi(d(x_{n-1}, x_n)), \varphi(d(x_{n-1}, x_n))), \end{aligned}$$

then we have

$$\psi(d(x_n, x_{n+1})) \leq F(\psi(d(x_{n-1}, x_n)), \varphi(d(x_{n-1}, x_n))) \leq \psi(d(x_{n-1}, x_n)). \tag{3.12}$$

Now similar to the proof of Theorem 3.1 we get

$$d(x_n, x_{n+1}) \to 0 \quad \text{as} \quad \text{n} \to \infty. \tag{3.13}$$

Now we shall prove that $\{x_n\}$ is Cauchy sequence in (X, d). Therefore suppose that $\{x_n\}$ is not Cauchy sequence, that means $\lim_{n,m\to\infty} d(x_n, x_m) \neq 0$, so there exist $\varepsilon > 0$ and $\{m_k\} \subset \mathbb{N}$ such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon.$$

Suppose that k is the smallest integer which satisfies above equation such that

$$d(x_{m_k-1}, x_{n_k}) < \varepsilon.$$

Again by the proof of Theorem 3.1 we obtain that

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon \tag{3.14}$$

and

$$\lim_{k\to\infty} m(x_{l_k+1}, x_{n_k+1}) = \varepsilon. \tag{3.15}$$

Now by (3.11), (3.14) and (3.15) we have

$$\begin{aligned} \psi(d(x_{m_k+1}, x_{n_k+1})) &\leq \alpha(x_{m_k}, Tx_{m_k})\alpha(x_{n_k}, Tx_{n_k})\psi(d(x_{m_k+1}, x_{n_k+1})) \\ &\leq F(\psi(d(x_{m_k}, x_{n_k})), \varphi(d(x_{m_k}, x_{n_k}))), \end{aligned}$$

therefore we get

$$\psi(d(x_{m_k+1}, x_{n_k+1})) \leq F(\psi(d(x_{m_k}, x_{n_k})), \varphi(d(x_{m_k}, x_{n_k}))) \leq \psi(d(x_{m_k}, x_{n_k})).$$

Letting $k \to \infty$ in the above inequality, we get

$$\psi(\varepsilon) \leq F(\psi(\varepsilon), \varphi(\varepsilon)) \leq \psi(\varepsilon),$$

that means

$$F(\psi(\varepsilon), \varphi(\varepsilon)) = \psi(\varepsilon).$$

By using the property of the functions F, ψ and φ, we obtain that $\psi(\varepsilon)=0$ or $\varphi(\varepsilon)=0$, then $\varepsilon=0$, which is contradiction and therefore $\{x_n\}$ is a Cauchy sequence. Now by completeness of X, $x_n \to x$, for some $x \in X$, that means,

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

First, we suppose that T is continuous, then we have

$$Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x.$$

So x is a fixed point of T. Now we suppose that (b) holds then $\alpha(x, Tx) \geq 1$. Now by (3.11) we have

$$\begin{aligned} \psi(d(Tx_n, Tx)) &\leq \alpha(x_n, Tx_n)\alpha(x, Tx)\psi(d(Tx_n, Tx)) \\ &\leq F(\psi(d(x_n, x)), \varphi(d(x_n, x))), \end{aligned}$$

that is, $d(Tx_n, Tx) \leq d(x_n, x)$, and so we get

$$0 \leq d(Tx, x) \leq d(Tx, x_{n+1}) + d(x, x_{n+1}) \leq d(x, x_n) + d(x, x_{n+1}).$$

Letting $n \to \infty$ in the above inequality, we get $d(Tx, x) = 0$, that is, $Tx = x$. □

Theorem 3.4. *Assume that all of the hypotheses of Theorems 3.1, or 3.2 or 3.3 hold. Adding the following condition:*
(c) if $Tx = x$ then $\alpha(x, Tx) \geq 1$,
then the fixed point of T is unique.

Proof. Suppose that $u, v \in X$ are two fixed points of T such that $u \neq v$. Then $\alpha(u, Tu) \geq 1$ and $\alpha(v, Tv) \geq 1$. For Theorem 3.1 we have

$$\psi(d(Tu, Tv) + l \leq (\psi(d(Tu, Tv) + l))^{\alpha(u,Tu)\alpha(v,Tv)} \leq F(\psi(d(u, v)), \varphi d(u, v)) + l. \tag{3.16}$$

For Theorem 3.2 we have

$$2^{\psi(d(Tu,Tv))} \leq (\alpha(u, Tu)\alpha(v, Tv) + 1)^{\psi(d(Tu,Tv))} \leq 2^{F(\psi(d(u,v)),\varphi(d(u,v)))}. \tag{3.17}$$

For Theorem 3.3 we have

$$\psi(d(Tu, Tv)) \leq (\alpha(u, Tu)\alpha(v, Tv) + 1)\psi(d(Tu, Tv)) \leq F(\psi(d(u, v)), \varphi(d(u, v))). \tag{3.18}$$

Therefore the equations (3.16), (3.17) and (3.18) imply that

$$F(\psi(d(u, v)), \varphi(d(u, v))) = \psi(d(Tu, Tv))$$

and so by the properties of functions F, ψ and φ we have

$$d(u, v) = 0,$$

thus

$$u = v.$$

□

4. Consequences

By Theorems 3.1, 3.2 and 3.3 we obtain the following corollaries as an extension of several known results in the literature.
If we let $\varphi(t) = \psi(t) = t$, we get the following three corollaries:

Corollary 4.1. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Suppose that the following condition is satisfied:*

$$(d(Tx, Ty) + l))^{\alpha(x,Tx)\alpha(y,Ty)} \leq F(d(x, y)), d(x, y)) + l \tag{4.1}$$

for all $x, y \in X$ and $l \geq 1$, where $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$, Suppose that either,
(a) T is continuous;
or
(b) if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x, Tx) \geq 1$.
If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has a fixed point.

Corollary 4.2. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Suppose that the following conditions are satisfied:*

$$(\alpha(x, Tx)\alpha(y, Ty) + 1)^{d(Tx,Ty)} \leq 2^{F(d(x,y)),d(x,y))} \tag{4.2}$$

for all $x, y \in X$ and $l \geq 1$, where $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$, Suppose that either,
(a) T is continuous;
or
(b) if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x, Tx) \geq 1$.
If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has a fixed point.

Corollary 4.3. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Suppose that the following conditions are satisfied:*

$$\alpha(x, Tx)\alpha(y, Ty)d(Tx, Ty) \leq F(d(x, y), d(x, y)) \tag{4.3}$$

for all $x, y \in X$ and $l \geq 1$, where $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$, Suppose that either,
(a) T is continuous;
or
(b) if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x, Tx) \geq 1$.
If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has a fixed point.

If we let $\beta \in \mathbb{F}$, $\varphi(t) = \psi(t) = t$ and $F(s, t) = \beta(s)s$, then the following results of [3] have derived from our results.

Corollary 4.4 (Theorem 4 in [3])**.** *let (X, m) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Assume that there exists a function $\beta : \mathbb{R}^+ \to [0, 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and*

$$(d(Tx, Ty) + l)^{\alpha(x,Tx)\alpha(y,Ty)} \leq \beta(d(x, y))d(x, y) + l \tag{4.4}$$

for all $x, y \in X$ where $l \geq 1$. Suppose that either,
(a) T is continuous;
or
(b) if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x, Tx) \geq 1$.
If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has a fixed point.

Corollary 4.5 (Theorem 6 in [3]). *let (X, m) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Assume that there exists a function $\beta : \mathbb{R}^+ \to [0, 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and*

$$(\alpha(x, Tx)\alpha(y, Ty))^{d(Tx,Ty)} \leq 2^{\beta(d(x,y))d(x,y)} \tag{4.5}$$

for all $x, y \in X$ where $l \geq 1$. Suppose that either,
(a) T is continuous;
or
(b) if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x, Tx) \geq 1$.
If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has a fixed point.

Corollary 4.6 (Theorem 8 in [3]). *let (X, m) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Assume that there exists a function $\beta : \mathbb{R}^+ \to [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and*

$$(\alpha(x, Tx)\alpha(y, Ty))d(Tx, Ty) \leq \beta(d(x, y))d(x, y) \tag{4.6}$$

for all $x, y \in X$ where $l \geq 1$. Suppose that either,
(a) T is continuous;
or
(b) if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x, Tx) \geq 1$.
If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has a fixed point.

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