Available online at http://vonneumann-publishing.com/cna
Commun. Nonlinear Anal. 1 (2016), 79–83
Research Article

Communications in Nonlinear Analysis

Journal Homepage: www.vonneumann-publishing.com/cna

# Best Proximity Point for Prešić Type Mappings on Metric Spaces

Mehdi Omidvari

*Department of Mathematics, Abarkouh Branch, Islamic Azad University, Abarkouh, Iran.*

**Abstract**

In this paper, we define the best proximity point for Prešić type non-self mappings and prove some best proximity point theorems in complete metric spaces. 

*Keywords:* Best proximity point, weak $P$-property, Prešić type mappings.
*2010 MSC:* 47H10, 54H25, 90C26.

## 1. Introduction

There are several generalizations of the Banach contraction principle. One such generalization is given by Prešić [5, 6] in 1965. Ćirić and Prešić [1] generalized the Prešić type mappings in 2007 and proved some fixed point theorems. Now let us assume that $A, B$ be two nonempty subsets of a metric space and $T : A \longrightarrow B$. Clearly $T(A) \cap A \neq \emptyset$ is a necessary condition for the existence of a fixed point of $T$. Now if $T(A) \cap A = \emptyset$, then to find an element $x \in A$ such that $d(x, Tx) = d(A, B)$ which called best proximity point, is the idea of best proximity point theorems. The existence and convergence of best proximity points has generalized by several authors such as Prolla[7], Reich[8], Sadiq Basha[9, 10], Vertivel *et al.*[11] and Omidvari *et al.*[2, 3, 4] in many directions. In this paper we prove some best proximity point theorems for Prešić type non-self mappings in complete metric spaces.

## 2. Preliminaries

Let $A, B$ be two non-empty subsets of a metric space $(X, d)$. The following notations will be used throughout this paper:

$d(y, A) := inf\{d(x, y) : x \in A\}$,

$d(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\}$,

*Email address:* mehdi.omidvari@gmail.com (Mehdi Omidvari)

*Received 2016-03-06*

$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$

$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$

We recall that $x \in A$ is a best proximity point of the non-self mapping $T : A \longrightarrow B$ if $d(x, Tx) = dist(A, B)$. It can be observed that a best proximity reduces to a fixed point, if the underlying mapping is a self-mapping.

**Definition 2.1.** [12] Let $(A, B)$ be a pair of nonempty subsets of a metric space $X$ with $A \neq \emptyset$. Then the pair $(A, B)$ is said to have the weak P-property, if and only if

$$\left.\begin{array}{l} d(x_1, y_1) = d(A, B), \\ d(x_2, y_2) = d(A, B), \end{array}\right\} \Longrightarrow d(x_1, x_2) \leq d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It is clear that, for any nonempty subset $A$ of $X$, the pair $(A, A)$ has the weak P-property.
Prešić[5, 6] introduced a kind of mappings called Prešić type mappings and proved the following theorem.

**Theorem 2.2.** *[5, 6] Let $(X, d)$ be a complete metric space, $k$ a positive integer and $T : X^k \longrightarrow X$ a mapping the following contractive type condition:*

$$d\big(T(x_1, x_2, x_3, \dots, x_k), T(x_2, x_3, \dots, x_k, x_{k+1})\big) \leq \sum_{i=1}^{k} q_i d(x_i, x_{i+1}), \tag{2.1}$$

*for every $x_1, \dots, x_{k+1}$ in $X$, where $q_1, q_2, \dots, q_k$ are non-negative constants such that $\sum_{i=1}^{k} q_i < 1$. Then there exists a uniqe point $x$ in $X$ such that $T(x, x, \dots, x) = x$. Moreover, if $x_1, x_2, \dots, x_k$ are arbitrary points in $X$ and for $n \in \mathbb{N}$,*

$$x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1}),$$

*then the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and*

$$\lim x_n = T(\lim x_n, \lim x_n, \dots, \lim x_n).$$

In 2007, Ćirić and Prešić[1] generalized the Prešić type mappings and proved the following theorem.

**Theorem 2.3.** *[1] Let $(X, d)$ be a complete metric space, $k$ a positive integer and $T : X^k \longrightarrow X$ a mapping the following contractive type condition:*

$$d\big(T(x_1, x_2, x_3, \dots, x_k), T(x_2, x_3, \dots, x_k, x_{k+1})\big) \leq \lambda \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\} \tag{2.2}$$

*where $\lambda \in (0, 1)$ is a constant and $x_1, \dots, x_{k+1}$ are arbitrary elements in $X$. Then there exists a point $x$ in $X$ such that $T(x, x, \dots, x) = x$. Moreover, if $x_1, x_2, \dots, x_k$ are arbitrary points in $X$ and for $n \in \mathbb{N}$,*

$$x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1}),$$

*then the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and*

$$\lim x_n = T(\lim x_n, \lim x_n, \dots, \lim x_n).$$

*If in addition we suppose that on diagonal $\Delta \subset X^k$,*

$$d\big(T(u, \dots, u), T(v, \dots, v)\big) < d(u, v), \tag{2.3}$$

*holds for all $u, v \in X$, with $u \neq v$, then $x$ is the unique point in $X$ with $T(x, x, \dots, x) = x$.*

## 3. Main Result

We first define a new kind of the best proximity point.

**Definition 3.1.** Let $A$ and $B$ be two non-empty subsets of a metric space $(X,d)$. Let $k$ be a positive integer and $T : A^k \longrightarrow B$ a non-self mapping. $x \in A$ is said to be a best proximity point of $T$ if,

$$d\big(x, T(x,\dots,x)\big) = d(A,B).$$

**Theorem 3.2.** *Let $A$ and $B$ be two non-empty closed subsets of a complete metric space $(X,d)$ such that $A_0 \neq \varnothing$ and $(A,B)$ has the weak $P$-property. Let $k$ be a positive integer and $T : A^k \longrightarrow B$ a non-self mapping satisfies the following condition:*

*(a)* $T(A_0^k) \subseteq B_0$.

*(b) There exists $\lambda \in (0,1)$ such that*

$$d\big(T(x_1,x_2,\dots,x_k), T(x_2,\dots,x_k,x_{k+1})\big) \leq \lambda \max\{d(x_i,x_{i+1}) : 1 \leq i \leq k\}, \tag{3.1}$$

*where $x_1, x_2, \dots, x_{k+1}$ are arbitrary elements in $A$.*

*Then $T$ has a best proximity point in $A$. Moreover, if on diagonal $\Delta \subset A^k$,*

$$d\big(T(u,\dots,u), T(v,\dots,v)\big) < d(u,v), \tag{3.2}$$

*holds for all $u, v \in A$, with $u \neq v$, then $T$ has an unique best proximity point in $A$.*

*Proof.* Choose $(x_1,x_2,\dots,x_k) \in A_0^k$. Since $T(A_0^k) \subseteq B_0$, there exists $x_{k+1} \in A_0$ such that

$$d\big(x_{k+1}, T(x_1,x_2,\dots,x_k)\big) = d(A,B).$$

Again since $(x_2,\dots,x_k,x_{k+1}) \in A_0^k$, there exists $x_{k+2} \in A_0$ such that

$$d\big(x_{k+2}, T(x_1,\dots,x_k,x_{k+1})\big) = d(A,B).$$

Continuing this process, we can find a sequence $\{x_n\}$ in $A_0$ such that

$$d\big(x_{n+k}, T(x_n,x_{n+1},\dots,x_{n+k-1})\big) = d(A,B) \quad \textit{for all } n \in \mathbb{N}. \tag{3.3}$$

We will prove the convergence of sequence $\{x_n\}$ in $A$. $(A,B)$ satisfies the weak $P$-property, therefore from (3.3) we obtain for all $n \in \mathbb{N}$,

$$d(x_{n+k},x_{n+k+1}) \leq d\big(T(x_n,x_{n+1},\dots,x_{n+k-1}), T(x_{n+1},\dots,x_{n+k-1},x_{n+k})\big). \tag{3.4}$$

Put $\alpha_n = d(x_n,x_{n+1})$. We will prove by induction the following inequality;

$$\alpha_n \leq K\theta^n \quad \textit{for all } n \in \mathbb{N}, \tag{3.5}$$

where $\theta = \lambda^{\frac{1}{k}}$ and $K = \max\{\frac{\alpha_1}{\theta}, \frac{\alpha_2}{\theta^2}, \dots, \frac{\alpha_k}{\theta^k}\}$.
Obviously, $0 \leq \theta < 1$ and (3.5) is true for $n = 1, ..., k$. Now let

$$\alpha_n \leq K\theta^n, \alpha_{n+1} \leq K\theta^{n+1}, \dots, \alpha_{n+k-1} \leq K\theta^{n+k-1},$$

be the induction hypotheses. Then by the definition of $T$ and the induction hypotheses and using (3.4), we have

$$\begin{aligned}\alpha_{n+k} &= d(x_{n+k}, x_{n+k+1}) \\ &\leq d\big(T(x_n, x_{n+1}, \dots, x_{n+k-1}), T(x_{n+1}, \dots, x_{n+k-1}, x_{n+k})\big) \\ &\leq \lambda \max\{\alpha_i : i = n, \dots, n+k-1\} \\ &\leq \lambda \max\{K\theta^i : i = n, \dots, n+k-1\} \\ &= \lambda K\theta^n = K\theta^{n+k},\end{aligned} \tag{3.6}$$

and this complete the inductive proof. Now let $m, n \in \mathbb{N}$ such that $m \geq n$. By using (3.5), we receive that

$$\begin{aligned} d(x_n, x_m) &\leq \textstyle\sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \textstyle\sum_{i=n}^{m-1} K\theta^i \\ &\leq K\theta^n \textstyle\sum_{i=1}^{\infty} \theta^i \\ &= \frac{K\theta^n}{1-\theta}.\end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence in $A$. Since $X$ is a complete metric space and $A$ is a closed subset of $X$, there exists $x \in A$ such that $\lim_{n\to\infty} x_n = x$. Then for any $n \in \mathbb{N}$, we have

$$\begin{aligned} d\big(x, T(x, \dots, x)\big) &\leq d(x, x_{n+k}) + d\big(x_{n+k}, T(x_n, x_{n+1}, \dots, x_{n+k-1})\big) + d\big(T(x_n, x_{n+1}, \dots, x_{n+k-1}), T(x, \dots, x)\big) \\ &\leq d(x, x_{n+k}) + d(A, B) + d\big(T(x, \dots, x, x), T(x, \dots, x, x_n)\big) \\ &\quad + d\big(T(x, \dots, x, x_n), T(x, \dots, x_n, x_{n+1})\big) + \dots \\ &\quad + d\big(T(x, x_n, x_{n+1}, \dots, x_{n+k-2}), T(x_n, x_{n+1}, \dots, x_{n+k-1})\big) \\ &\leq d(x, x_{n+k}) + d(A, B) + \lambda d(x, x_n) \\ &\quad + \lambda \max\{d(x, x_n), d(x_n, x_{n+1})\} + \dots + \lambda \max\{d(x, x_n), d(x_n, x_{n+1}), \dots, d(x_{n+k-2}, x_{n+k-1})\}.\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we obtain that $d\big(x, T(x, \dots, x)\big) = d(A, B)$. Also, we received that

$$d\big(\lim x_n, T(\lim x_n, \dots, \lim x_n)\big) = d(A, B).$$

Now suppose (3.2) holds. we show that $x$ is unique. Let $x^* \in A$ be a best proximity point of $T$ such that $x \neq x^*$. Since $(A, B)$ has the weak $P$-property and (3.2) holds,

$$d(x, x^*) \leq d\big(T(x, \dots, x), T(x^*, \dots, x^*)\big) < d(x, x^*),$$

which is a contradiction. Hence $x$ is an unique element in $A$ and this completes the proof of theorem. □

*Remark* 3.3. Theorem 3.2 is a generalization of Theorem 2.3 if $A = B = X$.

The following result is an immediate consequence of Theorem 3.2.

**Corollary 3.4.** *Let $A$ and $B$ be two non-empty closed subsets of a complete metric space $(X, d)$ such that $A_0 \neq \varnothing$ and $(A, B)$ has the weak $P$-property. Let $k$ be a positive integer and $T : A^k \longrightarrow B$ a non-self mapping satisfies the following condition:*

*(a)* $T(A_0^k) \subseteq B_0$.

*(b) There exist non-negative constants* $q_1, q_2, \ldots, q_k$ *such that* $\sum_{i=1}^{k} q_i < 1$ *and*

$$d\big(T(x_1, x_2, x_3, \ldots, x_k), T(x_2, x_3, \ldots, x_k, x_{k+1})\big) \leq \sum_{i=1}^{k} q_i d(x_i, x_{i+1}),$$

*where* $x_1, x_2, \ldots, x_{k+1}$ *are arbitrary elements in* $A$.

*Then* $T$ *has a best proximity point in* $A$. *Moreover, if on diagonal* $\Delta \subset A^k$,

$$d\big(T(u, \ldots, u), T(v, \ldots, v)\big) < d(u, v),$$

*holds for all* $u, v \in A$, *with* $u \neq v$, *then* $T$ *has an unique best proximity point in* $A$.

*Proof.* Put $\lambda = \sum_{i=1}^{k} q_i$. Obviously $T$ satisfies in (3.1) and this completes the proof of corollary. □

**Example 3.5.** Let $X = \mathbb{R}$ with the usual metric. Given $A = \{-2, 2\}, B = \{-1, 1\}$ and $T : A^2 \longrightarrow B$ by

$$T(x, y) = \begin{cases} \dfrac{x+y}{4}, & x = y, \\[2ex] \dfrac{x+y}{4} + 1, & x \neq y, \end{cases}$$

It is clear that, for any $\lambda \in (\frac{1}{2}, 1)$, the non-self mapping $T$ satisfies in the conditions of Theorem 3.2 and $d\big(-2, T(-2, -2)\big) = d\big(2, T(2, 2)\big) = d(A, B)$.

## 4. Acknowledgement

The author would like to thanks to Islamic Azad University, Abarkouh Branch, to support this research.

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